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Sets of prime power order generators of finite groups*

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ABSTRACT. A subset X of prime power order elements of a finite group G is called pp-independent if there is no proper subset Y of X such that $\langle Y, \Phi(G)\rangle = \langle X, \Phi(G)\rangle$, where $\Phi(G)$ is the Frattini subgroup of G. A group G has property $\mathcal{B}_{pp}$ if all pp-independent generating sets of G have the same size. G has the pp-basis exchange property if for any pp-independent generating sets B_1, B_2 of G and $x \in B_1$ there exists $y \in B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ is a pp-independent generating set of G. In this paper we describe all finite solvable groups with property $\mathcal{B}_{pp}$ and all finite solvable groups with the pp-basis exchange property.

1. Introduction

Throughout this paper, all groups are finite. Let G be a group. We denote by $\Phi(G)$ the Frattini subgroup of G and we call a group with the trivial Frattini subgroup a Frattini-free group. For other notation, terminology and results one can consult for example [3, 4].

In this paper, our purpose is to extend the famous theorem of Burnside known as Burnside basis theorem. This theorem provides that the Frattini quotient of every p-group is an elementary abelian p-group. Hence it can

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be view as a vector space over the field of order p. So generating sets of p-groups share properties with generating sets of vector spaces. However generating sets outside the class of p-groups do not have such properties, even generating sets of cyclic groups whose order is divisible by at least two different primes.

Obviously all elements of p-groups have prime power orders. So also in arbitrary groups we want to consider sets of prime power order generators. In this purpose we introduce the concept of a pp-element which simplifies our considerations. So we say that an element $g \in G$ is a *pp-element* if it has prime power order, while by p-element, as usual, we mean an element of order being a power of a prime p. Many authors have studied similar problems concerning sets of not only pp-generators, see for instance [1, 8, 9, 11] and the reference therein. In particular in [1] groups in which all minimal generating sets have the same size are classified.

A subset X of pp-elements of a group G will be called *pp-independent* if $\langle Y, \Phi(G)\rangle \neq \langle X, \Phi(G)\rangle$ for every $Y \subset X$ and a *pp-base* of G if X is a pp-independent generating set of G. We say that a finite group G

- has *property* $\mathcal{B}_{pp}$ (is a $\mathcal{B}_{pp}$-group for short) if all pp-bases of G have the same size;
- has the *pp-embedding property* if every pp-independent set of G can be embedded to a pp-base of G;
- has the *pp-basis exchange property* if for any two pp-basis B_1, B_2 and $x \in B_1$ there exists $y \in B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ is a pp-base of G.
- is a *pp-matroid* group if G has property $\mathcal{B}_{pp}$ and the pp-embedding property.

In view of the above definitions, Burnside basis theorem provides that all finite p-groups are pp-matroid and have the pp-basis exchange property. Another example, outside the class of p-groups, is a group called a scalar extension. After [6] we say that G is a *scalar extension* if $G = P \rtimes Q$, where P is an elementary abelian p-group, Q is a non-trivial cyclic q-group for distinct primes $p \neq q$ such that Q acts faithfully on P and the $\mathbb{F}_p[Q]$-module P is a direct sum of isomorphic copies of one simple module. This construction will be constantly use in our further considerations. A scalar extension is not always a pp-matroid group (only if Q has prime order, see [11]) but every scalar extension is a $\mathcal{B}_{pp}$-group (see [8]).

Our focus of interest is to study the structure of groups which have one of the properties listed above. Solvable groups with the pp-embedding property were studied in [10, 11]. In [7] all pp-matroid groups were described. Moreover in [7] it was proved that pp-matroid groups have the pp-basis exchange property.

The properties of pp-matroid groups imply that every maximal pp-independent set of a pp-matroid group G is a pp-base of G. Let $\mathcal{I}$ be the family of all pp-independent sets of G. Then the pair $(\mathcal{I}, G)$ forms a matroid where every pp-base of G is a base of a matroid $(\mathcal{I}, G)$ (see [12]). Hence pp-matroid groups can be view as a generalization of p-groups in the sense of generating sets. Thus the aim of this paper is to describe the structure of solvable groups with property $\mathcal{B}_{pp}$ and the structure of solvable groups with the pp-basis exchange property.

By [6, Theorem 4.2], we know that every pp-independent set (pp-base) of $G/\Phi(G)$ may be lifted to a pp-independent set (pp-base) of G. Hence using properties of the Frattini subgroup we obtain the following

Theorem 1.1. *A group G has property $\mathcal{B}_{pp}$, the pp-embedding property, the pp-basis exchange property if and only if $G/\Phi(G)$ has, respectively, property $\mathcal{B}_{pp}$, the pp-embedding property and the pp-basis exchange property. In particular G is pp-matroid if and only if $G/\Phi(G)$ is pp-matroid.*

Based on the above theorem we may restrict our consideration to Frattini-free groups. The structure of the paper is as follows. We present our concepts and main results in Sections 1. In Section 2 we present the classification of all solvable groups with property $\mathcal{B}_{pp}$. The proof of Theorem 1.2 is presented in Section 3.

Theorem 1.2. *Let G be a Frattini-free solvable group. Then G has property $\mathcal{B}_{pp}$ if and only if it is one of the following groups:*

1) *an elementary abelian p-group;*
2) *a scalar extension;*
3) *a direct product of groups given in (1) and (2) with pairwise coprime orders.*

Using the above theorem we describe in Section 3 solvable groups with the pp-basis exchange property. The proof of Theorem 1.3 is presented in Section 4.

Theorem 1.3. *Let G be a Frattini-free solvable groups. Then G has the pp-basis exchange property if and only if G is a $\mathcal{B}_{pp}$-group such that all pp-elements of G have prime orders.*

2. Groups with property $\mathcal{B}_{pp}$

In this section we present the classification of groups with property $\mathcal{B}_{pp}$. First results concerning a $\mathcal{B}_{pp}$-groups appear in [6,8]. We recall some of these results which we will apply in further proofs.

Theorem 2.1 ([6]). *Let $G = P \rtimes Q$ be a non-trivial semidirect product, where P is a p-group and Q is a cyclic q-group, for distinct primes $p \neq q$. Then the following conditions are equivalent:*

1) *G is a $\mathcal{B}_{pp}$-group.*
2) *$G/\Phi(G)$ is a scalar extension.*

Furthermore, suppose that the above conditions hold. Then all minimal generating sets of G have the same size.

Theorem 2.2 ([8]). *Let G be a group and $G/\Phi(G)$ be a scalar extension. Then*

1) *G has a unique Sylow p-subgroup P;*
2) *$G = P \rtimes Q$ for a Sylow q-subgroup Q and all Sylow q-subgroups of G are cyclic;*
3) *$\Phi(G) = \Phi(P) \times \langle x^{q^m} \rangle$, where x is a generator of Q. Moreover, x^{q^m} centralizes P.*

Theorem 2.3 ([6]). *If G is a $\mathcal{B}_{pp}$-group, then every homomorphic image of G is also a $\mathcal{B}_{pp}$-group.*

Theorem 2.4 ([6]). *Let G_1 and G_2 be groups with coprime orders. Then G_1 and G_2 are $\mathcal{B}_{pp}$-groups if and only if $G_1 \times G_2$ is a $\mathcal{B}_{pp}$-group.*

3. Solvable $\mathcal{B}_{pp}$-groups

In this section we investigate the finite solvable $\mathcal{B}_{pp}$-groups. The following lemmas will be needed for proving Theorem 1.2.

Remark 3.1. Let G be a solvable group and $G = P \rtimes H$, where P is a minimal normal subgroup of G and $C_H(P) = 1$. Assume that $d \geqslant 2$ is a size of a minimal generating set of G. Theorem 7 of [2] follows that there exists a minimal generating set $\{h_1, \dots, h_d\}$ of H such that $\langle h_1^{x_1}, \dots, h_d^{x_d} \rangle = G$ for some $x_1, \dots, x_d \in P$. We say then, after the authors, that (P, H) does not satisfy the strong complement property.

Lemma 3.2. *Let G be a solvable group with a minimal normal p-subgroup P, and let H be a complement to P in G, where p is a prime and p does not divide $|H|$. Assume that H is a non-cyclic q-group for some prime q*

or $H/\Phi(H)$ is a scalar extension. If H acts non-trivially on P, then G is not a $\mathcal{B}_{pp}$-group.

Proof. By assumption, H is a q-group or $H/\Phi(H)$ is a scalar extension. Hence, by Theorem 2.1, all minimal generating sets of H have the same size, say d. Assume $d \geqslant 2$. Since H acts non-trivially on P, by Remark 3.1 there exists a minimal generating set $\{h_1, \ldots, h_d\}$ of H such that $\langle h_1^{x_1}, \ldots, h_d^{x_d} \rangle = G$ for some $x_1, \ldots, x_d \in P$. Observe that $\{h_1^{x_1}, \ldots, h_d^{x_d}\}$ is a generating set of pp-elements of G. Hence there exists a pp-base $B' \subseteq \{h_1^{x_1}, \ldots, h_d^{x_d}\}$ of G such that $|B'| \leqslant n$.

On the other hand $P = \langle a \rangle^H$, for every $1 \neq a \in P$. Hence $\{a, h_1, \ldots, h_n\}$ is a pp-base of G. Thus G is not a $\mathcal{B}_{pp}$-group. □

Lemma 3.3. *Let G be a solvable group with a minimal normal p-subgroup P, and let H be a nilpotent complement to P in G, where p is a prime and p does not divide $|H|$. If G is an indecomposable $\mathcal{B}_{pp}$-group, then H is a cyclic q-group for some prime divisor q of $|H|$.*

Proof. Assume that $H = P_1 \times \ldots \times P_n$, where P_i is a Sylow p_i-subgroup of H and $[P, P_i] \neq 1$, for $i = 1, \ldots, n$. Let $1 \neq a \in P$ and B_i be a pp-base of P_i, for $i = 1, \ldots, n$. Then $\{a\} \cup B_1 \cup \ldots \cup B_n$ is a pp-base of G. Moreover assume that $B_1 = \{x_1, \ldots, x_k\}$ and $B_2 = \{y_1, \ldots, y_l\}$. Hence there exist $c_1, \ldots, c_k \in P$ such that $x_1^a = c_1 x_1, \ldots, x_k^a = c_k x_k$. Observe that $(x_i^a)^{y_j}(x_i^a)^{-1} = (c_i x_i)^{y_j}(c_i x_i)^{-1} = c_i^{y_j} c_i^{-1} \neq 1$ for at least one $j \in \{1, \ldots, l\}$. So $\{x_1^a, \ldots, x_k^a, y_1, \ldots y_l\} \cup B_3 \cup \ldots \cup B_k$ is a pp-base of G, a contradiction. Hence only one P_i acts non-trivially on P. Without loss of generality we can set $i = 1$. Then $G = (P \rtimes P_1) \times (P_2 \times \ldots \times P_n)$, in contradiction to our assumption. This contradiction implies that $H = P_1$. Thus, by Lemma 3.2, H is a cyclic q-group with $q = p_1$. □

Lemma 3.4. *Let G be a solvable group with a minimal normal p-subgroup P_1, where p is a prime and let H be a complement to P in G. Assume that $H = Q \rtimes P_2$, where Q is a q-group for some prime $q \neq p$ and P_2 is a cyclic p-group such that $H/\Phi(H)$ is a scalar extension. Then G is not a $\mathcal{B}_{pp}$-group.*

Proof. Since P_1 is a minimal normal subgroup of G and G is solvable, P_1 is elementary abelian and $\langle g \rangle^H = P_1$ for all $1 \neq g \in P_1$. Let $P_2 = \langle y \rangle$ and $\{x_1, \ldots, x_n\} \subseteq Q$ be a minimal set such that $\langle x_1, \ldots, x_n \rangle^{P_2} = Q$. Hence, by the assumption, $\{g, x_1, \ldots, x_n, y\}$ is a pp-base of G. We need to consider the following cases:

1. $[P_1, Q] \neq 1, [P_1, P_2] \neq 1$;

2. $[P_1, Q] = 1, [P_1, P_2] \neq 1$;
3. $[P_1, Q] \neq 1, [P_1, P_2] = 1$;
4. $[P_1, Q] = 1, [P_1, P_2] = 1$.

1. In this case there exists $a \in P_1$ such that $Q^a \neq Q$. From [4, Theorem 2.3], we know that $P_1 = C_{P_1}(Q) \times [P_1, Q]$. Assume first that $C_{P_1}(Q) \neq 1$. Since P_1 is a minimal normal subgroup of G, there exists $a \in C_{P_1}(Q)$ such that $a^{y^{-1}} \notin C_{P_1}(Q)$. Set $b = a^{y^{-1}}$. Then $b^y = a \in C_{P_1}(Q)$ and $b^{-1}b^{y^{-1}} \notin C_{P_1}(Q)$. It follows that $Q^b \neq Q^{b^{y^{-1}}}$. Since $H/\Phi(H)$ is a scalar extension,

$$Q/\Phi(Q) = Q_1\Phi(Q)/\Phi(Q) \times \ldots \times Q_n\Phi(Q)/\Phi(Q),$$

where $Q_i\Phi(Q)/\Phi(Q)$ is a simple $\mathbb{F}_q[Q]$-module. Hence $\langle x_i \rangle^Q = Q_i$, for every $x_i \in Q_i \setminus \Phi(Q)$. Moreover $Q = Q_1 \cdot \ldots \cdot Q_n$. It follows that there exists Q_i such that $Q_i^b \neq Q_i$, for some $i = 1, \ldots, n$. Thus at least one element, say x_i, satisfies $x_i^b \notin Q_i$. Consider the set $X = \{y, x_1^b, \ldots, x_n^b\}$. Observe that

$$x_i^{by} = x^{yy^{-1}by} = (x_i^y)^{b^y}.$$

Since $x_i^y \in Q$ and $b^y \in C_{P_1}(Q)$, we have $x_i^{by} = x_i^y \in \langle X \rangle$. Hence $x_i \in \langle X \rangle$ and further $c = x_i^{-1}x_i^b \in P_1 \cap \langle X \rangle$, where $c \neq 1$. It follows that $G = \langle X \rangle$ and X is a generating set of pp-elements of G. So there exists a pp-base $B \subseteq X$ of G such that $|B| < n + 2 = |\{g, x_1, \ldots, x_n, y\}|$.

Assume now that $C_{P_1}(Q) = 1$ and $C_{P_1}(P_2) \neq 1$. So $P_1 = [P_1, Q]$ and hence there exists $c \in C_{P_1}(P_2)$, where $c = [a, x_1^{-1}]$, for some $a \in P_1$ and $x_1 \in Q \setminus \Phi(Q)$. Thus there exist $x_2, \ldots, x_n \in Q$ such that $\langle x_1, \ldots, x_n \rangle^{P_2} = Q$. Let $X = \{x_1^a, \ldots, x_n^a, y\}$. Since $x_1^a = [a, x_1^{-1}]x_1 = cx_1$ for some $1 \neq c \in P_1$, we have $(x_1^a)^{-1}(x_1^a)^y = x_1^{-1}x_1^y$. Moreover $1 \neq x_1^{-1}x_1^y \in \langle X \rangle$. Hence $\langle x_1 \rangle^{P_2} \subseteq \langle X \rangle$ and $x_1 \in \langle X \rangle$. So $1 \neq x_1^{-1}x_1^a \in \langle X \rangle \cap P_1$ and $G = \langle X \rangle$. It follows that $\{a, x_1, \ldots, x_n, y\}$ and $\{y, x_1^a, \ldots, x_n^a\}$ are a pp-base of G.

2. Now there exists $a \in P_1$ and at least one x_i, say x_1 such that $[a, y] \neq 1 \neq [x_1, y]$. Then $y^a(y^{x_1})^{-1} = a^{-1}a^{y^{-1}}x_1^{y^{-1}}x_1 \neq 1$. It follows that $\{y^a, y^{x_1}, x_2, \ldots, x_n\}$ is a pp-base of G.

3. Since $[P_1, Q] \neq 1$, there exists $a \in P_1$ such that $Q \neq Q^a$. Moreover $P_1 = [P_1, Q]$. Otherwise $C_{P_1}(Q)$ is a normal subgroup of G, contradicting the minimality of P_1. Hence there exist $c_1, \ldots, c_n \in P_1$ such that $x_1^{a_1} = c_1x_1, \ldots, x_n^a = c_nx_n$. So we obtain $(x_i^a)^{-1}(x_i^a)^y = x_i^{-1}x_i^y \neq 1$. It follows that $\{x_1^a, \ldots, x_n^a, y\}$ is a pp-base of G.

4. In this case $\{g, x_1, x_2 \ldots, x_n, y\}$ and $\{gx_1y, x_2 \ldots, x_n, y\}$ are pp-bases of $P \times H$.

Hence G is not a $\mathcal{B}_{pp}$-group in all the cases. So the proof is complete. □

Remark 3.5. Let G be a solvable group with a minimal normal p-subgroup P, where p is a prime and let H be a complement to P in G. Assume that $H/\Phi(H)$ is a scalar extension. It follows, by Theorem 2.1, that H is a $\mathcal{B}_{pp}$-group and we may assume that d is the size of every pp-bases of H, for some positive integer d. Then from proofs of Lemmas 3.2, 3.4 we immediately deduce that there exist pp-bases B_1 and B_2 of G such that $|B_1| = n+1$ and $|B_2| < n+1$.

Proof of Theorem 1.2. Let G be a Frattini-free solvable group with property $\mathcal{B}_{pp}$. We use induction on $|G|$. Let $P = O_p(G)$ be a maximal normal p-group of G, for some prime p. Hence $\Phi(P) \leqslant \Phi(G) = 1$ and P is an elementary abelian p-group. By [3, Theorem 10.6], there exists a subgroup H of G such that $G = P \rtimes H$. From Theorem 2.3, H is a $\mathcal{B}_{pp}$-group. So by the induction assumption $H = H_1 \times \ldots \times H_k$, where $H_i/\Phi(H)_i$ is an elementary abelian q-group or a scalar extension for $i = 1, \ldots, k$. By [3, Theorem 10.6], $P = P_1 \times \ldots \times P_n$, where P_i is a minimal normal subgroup of G, for $i \in \{1, \ldots, n\}$.

Let $a_i \in P_i$ be a non-trivial element for $i \in \{1, \ldots, n\}$ and $\{h_1, \ldots, h_r\}$ be a pp-base of H. Then $B = \{a_1, \ldots, a_n, h_1, \ldots, h_r\}$ is a pp-base of G. Assume that P_i and P_j are not isomorphic as $\mathbb{F}_p[H]$-module for some $i \neq j$. Then $B' = (B \setminus \{a_i, a_j\}) \cup \{a_i a_j\}$ is a pp-base of G. Since $|B'| = |B| - 1$, G is not a $\mathcal{B}_{pp}$-group, a contradiction. So all the P_i are isomorphic to each another as $\mathbb{F}_p[H]$-modules. In particular, this implies that $C_H(P_i) = 1$ for each P_i.

Again, by Theorem 2.3, we may suppose that P is a minimal normal subgroup of G. Thus $G = P \rtimes (H_1 \times \ldots \times H_k)$ and $P = \langle a \rangle^H$, for every $1 \neq a \in P$. Assume that B_i is a pp-base of H_i for $i = 1, \ldots, k$. Then $\{a\} \cup B_1 \cup \ldots \cup B_k$ is a pp-base of $P \rtimes H$, as $(|H_i|, |H_j|) = 1$, for $i \neq j$.

If for some $i \in \{1, \ldots, k\}$, H_i is a q-group, then $q \neq p$, by the choice of P. So suppose that $H_1/\Phi(H_1)$ is a scalar extension and $[P, H_1] \neq 1$. Let $P_1 = \langle a \rangle^{H_1} \leqslant P$, for some $a \in P_1$. Then $\{a\} \cup B_1$ is a pp-base of $P_1 \rtimes H_1$. Moreover, by Remark 3.5, there exists a pp-base, say B of $P_1 \rtimes H_1$, such that $|B| < |B_1| + 1$. Observe that $B \cup B_2 \cup \ldots \cup B_k$ is a generating set of pp-elements of G. So there exists a pp-base $C \subseteq B \cup B_2 \cup \ldots \cup B_k$ of G such that $|C| < |\{a\} \cup B_1 \cup \ldots \cup B_k|$, a contradiction. Hence either $[P, H_1] = 1$ or H_1 is a p-group. If $[P, H_1] = 1$ and P and H have not coprime orders, then by Case 4. of Lemma 3.4 and analogous consideration as the above, we obtain that G is not a $\mathcal{B}_{pp}$-group.

It follows that if H_i is not a q-group, then H_i centralises P. It implies that $G = [P \rtimes (H_1 \times \ldots \times H_r)] \times H_{r+1} \times \ldots \times H_k$. Moreover H_i is a q_i- group, where $[H_i, P] \neq 1$ and $(q_i, p) = 1$ for $i = 1, \dots, r$ while $(|H_i|, |p|) = 1$, for $i = r+1, \dots, k$.

Further, by Theorem 2.3, $P \rtimes (H_1 \times \dots \times H_r)$ is a $\mathcal{B}_{pp}$-group. Then, by Lemma 3.3, only one H_i acts non-trivially on P and such H_i is cyclic. It follows that $G = (P \rtimes H_1) \times H_2 \times \dots \times H_k$, where H_1 is a cyclic q-group, and $(|P \rtimes H_1||, H_j|) = 1$ for $j = 2, \dots, n$. Moreover, by Theorems 2.1, 2.3, $P \rtimes H_1$ is a scalar extension or is abelian.

Conversely, let $G = G_1 \times \dots \times G_n$, where G_i is either an elementary abelian p-group or a scalar extension and $(|G_i|, |G_j|) = 1$ for $i \neq j$. Then, by Theorem 2.1, every direct factor of G is a $\mathcal{B}_{pp}$-group. Hence, by Theorem 2.4, G is a $\mathcal{B}_{pp}$-group. So the proof is complete. □

4. Solvable groups with the pp-basis exchange property

In this section we investigate the structure of finite solvable groups which have the pp-basis exchange property. We start from the statement analogous to Theorem 2.4.

Proposition 4.1. *Let G_1 and G_2 be groups with coprime orders. Then G_1 and G_2 have the pp-basis exchange property if and only if $G_1 \times G_2$ has the pp-basis exchange property.*

Proof. Since G_1 and G_2 have coprime orders, an element $g = g_1 g_2 \in G_1 \times G_2$, where $g_1 \in G_1, g_2 \in G_2$, is a pp-element if and only if $g = g_1$ or $g = g_2$. Hence B is a pp-base of $G_1 \times G_2$ if and only if $B = B_1 \cup B_2$, where B_1, B_2 are pp-bases of G_1, G_2, respectively. From here the result follows immediately. □

Proposition 4.2. *Let G be a Frattini-free solvable group. If G has the pp-basis exchange property, then G has property $\mathcal{B}_{pp}$ and all pp-elements of G have prime orders.*

Proof. Assume that B_1, B_2 are two pp-bases of G such that $|B_1| < |B_2|$. We choose B_1 and B_2 with the property that $|B_2 \setminus B_1|$ is minimal. Let $x \in B_2 \setminus B_1$. Since G has the pp-basis exchange property, there exists $y \in B_1 \setminus B_2$ such that $(B_2 \setminus \{x\}) \cup \{y\}$ is a pp-base of G. Moreover $|(B_2 \setminus \{x\}) \cup \{y\} \setminus B_1| < |B_2 \setminus B_1|$. This contradicts the minimality of $|B_1 \setminus B_2|$. So G is a $\mathcal{B}_{pp}$-group.

Now, by Theorem 1.2, $G = G_1 \times \dots \times G_k$ where G_i is either an elementary abelian p-group or a scalar extension. If G_i is elementary abelian, then obviously all elements have prime orders. So assume that $G_i = P \rtimes \langle x \rangle$ is a scalar extension and x is a q-element. Moreover assume $x^q \notin C_{G_i}(P)$. Let $a_1, \ldots, a_s \in P$ be a minimal set such that $\langle a_1, \ldots a_s \rangle^{\langle x \rangle} = P$. Hence $B_1 = \{a_1 x, a_2, \ldots, a_s, x^q\}$ and $B_2 = \{a_1, \ldots a_s, x\}$ are pp-bases of G_i. Moreover $\langle (B_1 \setminus \{a_1 x\}) \cup \{y\} \rangle \neq G_i$ for every $y \in B_2$. Hence G_i has not the pp-basis exchange property. So, by Theorem 4.1, G also has not the pp-basis exchange property, a contradiction. Hence $x^q \in C_{G_i}(P)$. Since G is Frattini-free, it follows, by Theorem 2.2, that all pp-elements of G_i have prime orders. □

Lemma 4.3. *Let $G = P \rtimes Q$ be a scalar extension. If all pp-elements of G have prime orders, then G has the pp-basis exchange property.*

Proof. Assume that $|Q| = q$, then all pp-elements of G have prime orders and all pp-basis have n elements . Let $B_1 = \{x_1, \ldots, x_n\}$ and $B_2 = \{y_1, \ldots, y_n\}$ be pp-bases of G. Assume that $x_1 \notin B_2$ and $H = \langle x_2, \ldots, x_n \rangle$. We show that H is a maximal subgroup of G. In this purpose we consider two cases:

1. $x_1 \in P$. Then $\langle x_1 \rangle^Q$ is a minimal normal subgroup of G and $H = P/\langle x_1 \rangle^Q \rtimes Q^a$, where $a \in P$. So H is a maximal subgroup of G.

2. $x_1 \notin P$. Then x_1 is a q-element, where q is a prime and Q is a q-group. Since $H \nsubseteq P$ there exists in H another q-element, say x_2. We may assume that $x_1 = x^{a_1}$ and $x_2 = x^{a_2}$, where $a_1 a_2^{-1} \notin C_P(Q)$. Hence $x^{a_1} x^{-a_2} = c \in P$ and $c \notin H$. Indeed, if $c \in H$ and $x^{a_2} \in H$, then $x^{a_1} \in H$, a contradiction. It follows, by analogous as in (1), that H is a maximal subgroup in G.

By assumption, there exists $y_i \notin H$ for some $i \in \{1, \ldots, n\}$. Since H is a maximal subgroup of G, $\langle H, y_i \rangle = G$. It follows that $\langle (B_1 \setminus \{x_1\}) \cup \{y_1\} \rangle = G$ and $|(B_1 \setminus \{x_1\}) \cup \{y_1\}| = n$. Since G is a $\mathcal{B}_{pp}$-group, $(B_1 \setminus \{x_1\}) \cup \{y_1\}$ is a pp-base of G. The proof is complete. □

Proof of Theorem 1.3. It follows immediately from Proposition 4.1, Proposition 4.2 and Lemma 4.3. □

Using Theorem 1.2 we obtain

Corollary 4.4. *Let G be a Frattini-free solvable group. Then G has the pp-basis exchange property if and only if it is one of the following groups:*

1) *an elementary abelian p-group;*
2) *a scalar extension $P \rtimes Q$, where P is an elementary abelian p-group, Q has order q for distinct primes $p \neq q$;*
3) *a direct product of groups given in (1) and (2) with pairwise coprime orders.*

Remark 4.5. By [5], we know that every simple group is generated by an involution and an element of prime order. So a simple group has a 2-element pp-base. On the other hand, by the Classification of Finite Simple Group, we know that every simple group is generated by at least three involutions, so every simple group has a pp-base which has at least 3 elements. It implies that all simple groups do not have property $\mathcal{B}_{pp}$. By the first part of the proof of Proposition 4.2, we may deduced that if a simple group has not property $\mathcal{B}_{pp}$, then it has not the pp-basis exchange property.

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