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CI-property for the group $(\mathbb{Z}_p)^2 \times \mathbb{Z}_q \times \mathbb{Z}_r$

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Abstract. In this paper we prove that the group $(\mathbb{Z}_p)^2 \times \mathbb{Z}_q \times \mathbb{Z}_r$ is CI-group, where p, q, r are primes such that q and r divide $p-1$, and r divides $q-1$.

1. Introduction

A Cayley graph over a finite group H defined by a connection set $S \subseteq H$ has H as a set of nodes and arc set $\mathrm{Cay}(H, S) := \{(x, y) : y.x^{-1} \in S\}$. Two Cayley graphs $\mathrm{Cay}(H, S)$ and $\mathrm{Cay}(K, T)$ are Cayley isomorphic if there exists a group isomorphism $f : H \to K$ which is a graph isomorphism too.

A subset $S \subseteq H$ is called a CI-subset if for each $T \subseteq H$, the graphs $\mathrm{Cay}(H, S)$, $\mathrm{Cay}(H, T)$ are isomorphic if and only if the sets T and S are conjugate by an element of $\mathrm{Aut}(H)$. A group H is called a CI-group if each subset of H is a CI-subset.

L. Babai and P. Frankl began to investigate arbitrary CI-groups, see [1,2]. They found several necessary conditions for a group to be a CI-group and asked for a complete classification of CI-groups. During last few years this problem was intensively studied by L. Nowitz, C.H. Li, M. Conder, S. Praeger, M.Y. Xu, J.X. Meng and P.Palfy [4, 12]

In order to finish the classification of CI-groups one has to answer two basic questions: Which groups are CI-groups and when a coprime product of two CI-groups is a CI-group. The first question was answered affirmatively for many groups: D_{2p} (Babai, [2]), $E(\mathbb{Z}_p, 4)$ (Li-Palfy, [3]),

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$\mathbb{Z}_p^e : e \leqslant 3$ (Dobson, [4]), $\mathbb{Z}_p^4$ (Hirasaka-Muzychuk, [9]), $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ (I. Kovacs and M. Muzychuk, [11]), $\mathbb{Z}_p^5$ (Yan Feng and I. Kovacs, [8]), $\mathbb{Z}_n$ and $\mathbb{Z}_{2n}$ and $\mathbb{Z}_{4n}$ where n is square-free odd (Muzychuk, [14]). The proofs of CI-property for the groups $H = \mathbb{Z}_p^n$ for $n \in \{4, 5\}$ and $H = \mathbb{Z}_p^2 \times \mathbb{Z}_q$ are based on the method of S-rings. In fact, in these proofs it was checked that every Schurian S-ring over H is a CI-S-ring. Due to the result of Hirasaka and Muzychuk, this is sufficient for the proof that H is a CI-group. In this paper we prove the following.

Theorem 1. *The group $(\mathbb{Z}_p)^2 \times \mathbb{Z}_q \times \mathbb{Z}_r$ is CI-group, where p, q, r are primes such that q and r divide $p - 1$, and r divides $q - 1$.*

The text of the paper is organized in the following way. Section 2 contains a background of S-rings. In Section 3 we prove a necessary condition for an S-ring over an abelian group H such that Sylow subgroups of H are elementary abelian to be a CI-S-ring. In Section 4 we prove Theorem 1.

2. Preliminaries

Let X be a nonempty finite set, $\mathrm{Sym}(X)$ denote to the group of all permutations of X. And H is an arbitrary finite multiplicatively written group with identity 1. If $S \subseteq H$ and $f \in \mathrm{Sym}(H)$, then

$$\mathrm{Cay}(H, S)^f := \{(x^f, y^f) : (x, y) \in \mathrm{Cay}(H, S)\}.$$

The automorphism group $\mathrm{Aut}(\mathrm{Cay}(H, S))$ of a Cayley graph $\mathrm{Cay}(H, S)$ consists of all permutations $f \in \mathrm{Sym}(H)$ which satisfy $\mathrm{Cay}(H, S)^f = \mathrm{Cay}(H, S)$. And this group always contains the regular group H_R which consists of the right shifts $h_R : h \in H$ the action of which on H is determined as follows: $x^{h_R} = xh : x \in H$.

2.1. S-rings. Let H be a finite group. We denote by $\mathbb{Z}H$ the group ring of H. For a subset $T \subseteq H$, let $\underline{T}$ denote the group ring element $\sum_{x \in H} a_x \underline{x}$ with: $a_x = 1$ if $x \in T$, and $a_x = 0$ otherwise. Such elements called simple quantities. A subring $\mathcal{A}$ of $\mathbb{Z}H$ is called a Schur ring (or S-ring) over H if the following axioms are satisfied:

1) There exists a basis of $\mathcal{A}$ consisting of simple quantities $\underline{T_0}, \underline{T_1}, \ldots, \underline{T_r}$.
2) $T_0 = \{1\}$, $H = \cup_{i=0}^{r} T_i$ and $T_i \cap T_j = \phi$ for $1 \leqslant i \neq j \leqslant r$.
3) For every $i \in \{1, 2, \ldots, r\}$ there exists $j \in \{1, 2, \ldots, r\}$ such that $\underline{T_i^{-1}} = \underline{T_j}$.

The basis $\underline{T_0}, \underline{T_1}, \ldots, \underline{T_r}$ are unique and are called the standard basis of $\mathcal{A}$. In this case, we adopt the notation $\mathcal{A} = \langle T_0, T_1, \ldots, T_r \rangle$ to convey two essential facts: $\mathcal{A}$ is an S-ring and its standard basis are $\underline{T_0}, \underline{T_1}, \ldots, \underline{T_r}$. Then the sets T_i, $0 \leqslant i \leqslant r$, are called the basic sets of $\mathcal{A}$ and we indicate this by writing $\mathrm{Bsets}(\mathcal{A}) = \{T_0, T_1, \ldots, T_r\}$.

An S-ring $\mathcal{A}' \subseteq \mathbb{Z}H$ is called S-subring of $\mathcal{A}$, if every element $z \in \mathcal{A}'$ is equal to sum of elements from $\mathcal{A}$.

Let $\mathcal{A} = \langle T_0, T_1, \ldots, T_r \rangle$ be an S-ring over a group H. Following O. Tamaschke [21], a subgroup $F \leqslant H$ for which $\underline{F} \in \mathcal{A}$ is called an $\mathcal{A}$-subgroup. There are two trivial $\mathcal{A}$-subgroups: $\{1\}$ and H. If $\mathrm{Bsets}(\mathcal{A})$ $= \{\{1\}, H \backslash \{1\}\}$ then $\mathcal{A}$ is trivial S-ring over H.

For F is an $\mathcal{A}$-subgroup, define $\mathcal{A}_F := \mathcal{A} \cap \mathbf{Z}F$. It is easy to check that $\mathcal{A}_F$ is an S-ring over the group F and that $\mathrm{Bsets}(\mathcal{A}_F) = \{T \in \mathrm{Bsets}(\mathcal{A}) : T \subset F\}$. Such S-rings $\mathcal{A}_F$ are called induced S-subrings of $\mathcal{A}$. If F is an $\mathcal{A}$-subgroup which is normal in H then the natural homomorphism $\pi : H \to H/F$ can be canonically extended to a homomorphism $\mathbf{Z}H \to \mathbf{Z}H/F$ which we shall also denote by π. We introduce the following notation: $T/F := \pi(T) = \{\pi(t) : t \in T\}$ for $T \subset H$, $\mathcal{A}/F := \pi(\mathcal{A}) = \{\pi(x) : x \in \mathcal{A}\}$. We call $\mathcal{A}/F$ a quotient S-ring (over the factor group H/F), and from [21] $\mathcal{A}/F$ is an S-ring over H/F with basic sets are given by: $\mathrm{Bsets}(\mathcal{A}/F) = \{T/F : t \in \mathrm{Bsets}(\mathcal{A})\}$.

The thin radical of an S-ring $\mathcal{A}$ is defined by the set $O_\theta(\mathcal{A}) = \{h \in H : \{h\} \in \mathrm{Bsets}(\mathcal{A})\}$. It is easy to see that $O_\theta(\mathcal{A})$ is an $\mathcal{A}$-subgroup.

Now, let G be an arbitrary group such that $H_R \leqslant G \leqslant \mathrm{Sym}(H)$ and let $T_0 = \{1\}, T_1, \ldots T_d$ be the set of all G_1-orbits, (where G_1 is stabilizer of the element 1). The vector space spanned by $T_0 = \{1\}, T_1, \ldots T_d$ is called the transitivity module of G and is denoted by $\mathcal{V}(H, G_1)$. By ([22]), the transitivity module $\mathcal{V}(H, G_1)$ is an S-ring over H. But the converse is not true, i.e., not every S-ring is the transitivity module of an appropriate group. An S-ring over H will be called Schurian if it is the transitivity module of some $G \leqslant \mathrm{Sym}(H)$ with $H_R \leqslant G$.

An S-ring $\mathcal{A}$ over H is said to be cyclotomic if $\mathrm{Bsets}(\mathcal{A})$ =$\mathrm{Orbit}(K, H)$ for $K \leqslant \mathrm{Aut}(H)$. In this case we write $\mathcal{A} = \mathrm{Cyc}(K, H)$ and it is eassy to see, $\mathcal{A} = \mathcal{V}(K \cdot H_R, H)$. So every cyclotomic S-ring is Schurian.

Definition 1. Let $\mathcal{A}$ be an S-ring over a group H and N be an $\mathcal{A}$-subgroup such that $N \trianglelefteq H$. Then $\mathcal{A}$ is a wreath product, notation: $\mathcal{A} = \mathcal{A}_N \wr \mathcal{A}_{H/N}$, if for every $T \in \mathrm{Bsets}(\mathcal{A})$ then $T \subset N$, or T is a union of N-cosets.

Definition 2. Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$-subgroups such that $H = EF$ and $E \cap F = \{e\}$. Then $\mathcal{A}$ is a tensor product, notation:

$\mathcal{A} = \mathcal{A}_E \otimes \mathcal{A}_F$ if for every $T \in \text{Bsets}(\mathcal{A})$, if $T \nsubseteq E \cup F$ then $T = RS$ where $R \in \text{Bsets}(\mathcal{A}) \cap E$ and $S \in \text{Bsets}(\mathcal{A}) \cap F$.

Consequently, if $\mathcal{A}$ is an S-ring over the direct product $H = E \times F$ such that both E and F are $\mathcal{A}$-subgroups and $\mathcal{A}_E = \mathbb{Z}E$ or $\mathcal{A}_F = \mathbb{Z}F$, then $A = \mathcal{A}_E \otimes \mathcal{A}_F$.

We say that an S-ring $\mathcal{A}$ over a group H is a p-S-ring if H is a p-group, and all basic sets $T \in Bset(\mathcal{A})$ have p-power size. Following [9], If $\mathcal{A}$ be a p-S-ring over elementary abelian group $\mathbb{Z}_p^n$ then we have: If $n = 1$ then $\mathcal{A} = \mathbb{Z}H$; And if $n = 2$ then $\mathcal{A} = \mathbb{Z}C_p \wr \mathbb{Z}C_p$ or $\mathcal{A} = \mathbb{Z}C_p^2$. And every p-S-ring over $\mathbb{Z}_p^n$ where $n = 1, 2, 3$ is cyclotomic.

Lemma 1. *[11] Let H be an abelian group, $G \leqslant \text{Sym}(H)$ such that $H_R \leqslant G$, And let $\mathcal{A} = \mathcal{V}(H, G_1)$. If $E \leqslant H$ is an $\mathcal{A}$-subgroup such that H/E is a p-subgroup, Then $\mathcal{A}/E$ is a p-S-ring.*

2.2. Isomorphisms of S-rings. Denote by $\text{Iso}(\mathcal{A})$ the set of all isomorphisms from $\mathcal{A}$ to S-rings over H, that is:

$$\text{Iso}(\mathcal{A}) = \left\{ \begin{array}{r} f \in \text{Sym}(H) : f \text{ is an isomorphism from } \mathcal{A} \\ \text{onto an S-ring over } H \end{array} \right\}.$$

And let $\text{Iso}_1(\mathcal{A}) = \{f \in \text{Iso}(\mathcal{A}) : 1^f = 1\}$. Note that, $\text{Iso}(\mathcal{A}) \subseteq \text{Sym}(H)$, but it is not necessarily a subgroup. It follows from the definition that for any $f \in \text{Aut}(\mathcal{A})$ and $g \in \text{Aut}(H)$, their product fg is an isomorphism from $\mathcal{A}$ to an S-ring over H. Therefore, $\text{Aut}(\mathcal{A}).\text{Aut}(H) \subseteq \text{Iso}(\mathcal{A})$. Now, we say that $\mathcal{A}$ is a CI-S-ring, if $\text{Iso}(\mathcal{A}) = \text{Aut}(\mathcal{A})\,\text{Aut}(H)$. This definition was given by Hirasaka and Muzychuk in [9] where the following theorem is proved.

Theorem 2. *Let H be an abellian group, then H is CI-group if and only if every Schurian S-rings over H is CI-S-ring.*

Therefore, instead of Theorem 1, we prove the following theorem:

Theorem 3. *Let $H = (\mathbb{Z}_p)^2 \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p, q, r are primes such that q and r divide $p - 1$, and r divides $q - 1$. Let $\mathcal{A} = \mathcal{V}(H, G_1)$ where $G \leqslant \text{Sym}(H)$ and $H_R \leqslant G$ then $\mathcal{A}$ is CI-S-ring.*

Let G, $G' \leqslant \text{Sym}(H)$, then G and G' are called 2-equivalent if $\text{Orbit}(G, H \times H) = \text{Orbit}(G', H \times H)$, and we write $G \underset{2}{\approx} G'$ in this case. If $\mathcal{A} = \mathcal{V}(H, G_1)$ for some $G \leqslant \text{Sym}(H)$ with $H_R \leqslant G$ then $\text{Aut}(\mathcal{A})$ is the largest group which is 2-equivalent to G. If $\mathcal{A}$ is an S-ring over H then we put $\mathrm{Aut}_H(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}) \cap \mathrm{Aut}(H)$. If K_1, $K_2 \leqslant \mathrm{Aut}(H)$ then K_1 and K_2 are called Cayley equivalent if $\mathrm{Orbit}(K_1,H) = \mathrm{Orbit}(K_2,H)$, and then we write $K_1 \underset{\mathrm{Cay}}{\approx} K_2$. If $\mathcal{A} = \mathrm{Cyc}(K,H)$ for some $K \leqslant \mathrm{Aut}(H)$ with $H_R \leqslant K$, then $\mathrm{Aut}_H(\mathcal{A})$ is the largest group which is Cayley equivalent to K. So a cyclcotomic S-ring $\mathcal{A}$ over H is called Cayley minimal if

$$\{K \leqslant \mathrm{Aut}(H) : K \underset{\mathrm{Cay}}{\approx} \mathrm{Aut}_H(\mathcal{A})\} = \{\mathrm{Aut}_H(\mathcal{A})\}.$$

It easy to see that the trivial S-ring $\mathbb{Z}H$ is Cayley minimal, and every cyclotomic S-ring over $\mathbb{Z}_n$ is Cayley minimal. If $q|p-1$ then the group $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_q)$ can be embedded into $\mathrm{Aut}(\mathbb{Z}_p)$. Therefore, if $q|p-1$ then every S-ring over $\mathbb{Z}_p \times \mathbb{Z}_q$ is cyclotomic and Cayley minimal.

3. Generalized wreath product

Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$-subgroups such that $E \leqslant F$ and E is normal in H. Then $\mathcal{A}$ is a generalized wreath product (or wedge product), $\mathcal{A} = \mathcal{A}_F \wr_{F/E} \mathcal{A}_{H/E}$ if for every $T \in \mathrm{Bsets}(\mathcal{A})$ such that $T \nsubseteq F$, T is a union of E-cosets. And $\mathcal{A}$ is non-trivial generalized wreath product if $E \neq 1$ and $F \neq H$, in this case, $S = F/E$ is called $\mathcal{A}$-section.

Let H and H' be finite groups. For a bijection $f : H \to H'$ and a set $X \subseteq H$, the induced bijection from X onto X^f is denoted by f^X. For a set $\triangle \subseteq \mathrm{Sym}(H)$ and a section S of H we set $\triangle^S = \{f^S : f \in \triangle, S^f = S\}$. If S is an $\mathcal{A}$-section then, $\mathrm{Aut}_H(\mathcal{A})^S \leqslant \mathrm{Aut}_S(\mathcal{A}_S)$. In 2013 Evdokimov and Ponomarenko proved the following theorem [15].

Theorem 4. *Let $\mathcal{A}$ be an S-ring over an Abelian group H. Suppose $\mathcal{A} = \mathcal{A}_F \wr_{F/E} \mathcal{A}_{H/E}$ for $\mathcal{A}$ subgroups E, F of H. Then $\mathcal{A}$ is Schurian if and only if so are the S-rings $\mathcal{A}_{H/F}$ and $\mathcal{A}_F$ and there exist $\triangle_F$ and $\triangle_{H/E}$ satisfying $F_R \leqslant \triangle_F \leqslant \mathrm{Aut}(\mathcal{A}_F)$ and $(H/E)_R \leqslant \triangle_{H/E} \leqslant \mathrm{Aut}(\mathcal{A}_{H/E})$, $(\triangle_F)^{F/E} = (\triangle_{H/E})^{F/E}$ such that $\triangle_F \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_F)$ and $\triangle_{H/E} \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{H/E})$. In this case $\mathrm{Aut}(\mathcal{A}) \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_F) \wr_{F/E} \mathrm{Aut}(\mathcal{A}_{H/E})$.*

Therefore, we conclude the following.

Theorem 5. *Let H be an abelian group such that Sylow subgroups of H are elementary abelian, and let $\mathcal{A}$ be an S-ring over H, such that $\mathcal{A} = \mathcal{A}_F \wr_{F/E} \mathcal{A}_{H/E}$ for $\mathcal{A}$ subgroups E, F of H. Suppose $\mathcal{A}_F$ and $\mathcal{A}_{H/E}$ are CI-S-rings and*

$$\mathrm{Aut}_F((A_F)^{F/E}) = \mathrm{Aut}_{F/E}((A_{F/E})) = \mathrm{Aut}_{H/E}((A_{H/E})^{F/E})$$

then $\mathcal{A}$ is CI-S-ring.

Proof. Let $\mathcal{A}'$ be an S-ring over H, and $f \in \mathrm{Iso}(\mathcal{A}, \mathcal{A}')$. By lemma (5.5) of [18], $\mathcal{A}'$ is F'/E'-wreath product, where $E' = E^f$ and $F' = F^f$. But H is abelian group, so we can found an $\tau \in \mathrm{Aut}(H) : E^f = E^\tau, F^f = F^\tau$. Therefore, we can take $f.\tau^{-1}$ without f. So $\mathcal{A}'$ is F/E-wreath product. And $f^F \in \mathrm{Iso}(\mathcal{A}_F, \mathcal{A}'_F)$, $f^{H/E} \in \mathrm{Iso}(\mathcal{A}_{H/E}, \mathcal{A}'_{H/E})$. But $\mathcal{A}_F$ and $\mathcal{A}_{H/E}$ are CI-S-rings. So $f^F = \varphi\sigma_1 : \varphi \in \mathrm{Aut}(\mathcal{A}_F), \sigma_1 \in \mathrm{Aut}(F)$, and $f^{H/E} = \psi\sigma_0 : \psi \in \mathrm{Aut}(\mathcal{A}_{H/E}), \sigma_0 \in \mathrm{Aut}(H/E)$. By the condition of the theorem we have $f^{F/E} = \sigma_1^{F/E} = \sigma_0^{F/E}$, and so $\varphi^{F/E} = \psi^{F/E}$.

Because H is abelian group such that sylow subgroups of H are elementary abelian, we can write $H = D \times F$ and $F = V \times E$.

Now, let $D = \langle x_1 \rangle \times \cdots \times \langle x_t \rangle$ and $(x_i E)^\psi = d_i v_i E$, where $d_i \in D$ and $v_i \in V$, $i \in \{1, 2, .., t\}$. We note that the elements $d_i.v_i : i = 0, 1, 2, \ldots, t$ generate the group D', where $D' \cap F = \{e\}$. We can found $\alpha \in \mathrm{Aut}(H)$ such that $\alpha^F = \varphi$, $(x_i)^\alpha = d_i.v_i : i = 1, 2, \ldots, t$. Therefore we have $E^\alpha = E$, $F^\alpha = F$ and $(dL)^\psi = d^\alpha L$.

Now, let $T \in \mathrm{Bsets}(\mathcal{A})$. If $T \subseteq F$ then $T^f = T^\alpha$ by the definition of α. If $T \nsubseteq F$ then $T = d_1 v_1 E \cup d_2 v_2 E \cdots \cup d_t v_t E$. So,

$$\begin{aligned} T^\alpha &= (d_1)^\alpha (v_1)^\alpha E \cup (d_2)^\alpha (v_2)^\alpha E \cup \cdots \cup (d_t)^\alpha (v_t)^\alpha E \\ &= (d_1)^\alpha (v_1)^\varphi E \cup (d_2)^\alpha (v_2)^\varphi E \cup \cdots \cup (d_t)^\alpha (v_t)^\varphi E \\ &= (d_1)^\alpha (v_1 E)^\varphi \cup (d_2)^\alpha (v_2 E)^\varphi \cup \cdots \cup (d_t)^\alpha (v_t E)^\varphi. \end{aligned}$$

But $\varphi^{F/E} = \psi^{F/E}$. So

$$\begin{aligned} T^\alpha &= (d_1)^\alpha (v_1 E)^\psi \cup (d_2)^\alpha (v_2 E)^\psi \cup \cdots \cup (d_t)^\alpha (v_t E)^\psi \\ &= (d_1 E)^\alpha (v_1 E)^\psi \cup (d_2 E)^\alpha (v_2 E)^\psi \cup \cdots \cup (d_t E)^\alpha (v_t E)^\psi \\ &= (d_1 E)^\psi (v_1 E)^\psi \cup (d_2 E)^\psi (v_2 E)^\psi \cup \cdots \cup (d_t E)^\psi (v_t E)^\psi \\ &= (d_1 v_1 E \cup d_2 v_2 E \cup \cdots \cup d_t v_t E)^\psi = T^\psi. \end{aligned}$$

Consequently, $T^\alpha/E = T^\psi/E = (T/E)^\psi = (T/E)^{f^{H/E}} = (T/E)^f$. Since T is a union of E-cosets, then T is a union of E^α-cosets, and $E^\alpha = E$, we found that $T^\alpha = T^f$. Consequently, $T^{f\alpha^{-1}} = T$, so $f\alpha^{-1} \in \mathrm{Aut}(\mathcal{A})$, and $f \in \mathrm{Aut}(\mathcal{A})\alpha \subseteq \mathrm{Aut}(\mathcal{A}).\,\mathrm{Aut}(H)$. And, So $\mathcal{A}$ is *CI-S*-ring. □

Now, let H be an abelian group such that Sylow subgroups of H are elementary abelian, and suppose $\mathcal{A}$ be an S-ring over H, such that $\mathcal{A} = \mathcal{A}_F \wr_{F/E} \mathcal{A}_{H/E}$, and $\mathcal{A}_F$, $\mathcal{A}_{H/E}$ are CI-S-ring. Then we have

Lemma 2. *If $\mathcal{A}_{F/E} = \mathbb{Z}(F/E)$ then $\mathcal{A}$ is CI-S ring.*

Proof. If $\mathcal{A}_{F/E} = \mathbb{Z}(F/E)$ then $\mathrm{Aut}(\mathcal{A}_{F/E})$ is trivial. But we have $\mathrm{Aut}_F((\mathcal{A}_F)^{F/E}) \leqslant \mathrm{Aut}_{F/E}(\mathcal{A}_{F/E})$. So $\mathrm{Aut}_F((\mathcal{A}_F)^{F/E}) = \mathrm{Aut}_{F/E}(\mathcal{A}_{F/E})$. By theorem 5, $\mathcal{A}$ is CI-S-ring. □

Lemma 3. *If $\mathcal{A}_{F/E}$ is Cayley minimal then $\mathcal{A}$ is CI-S ring.*

Proof. We have $\mathrm{Aut}_F(\mathcal{A}_F)^{F/E} \leqslant \mathrm{Aut}_{F/E}(\mathcal{A}_{F/E})$. So, if $\mathcal{A}_{F/E}$ is Cayley minimal, then $\mathrm{Aut}_F(\mathcal{A}_F)^{F/E} = \mathrm{Aut}_{F/E}(\mathcal{A}_{F/E})$. Therefore, by theorem 5, $\mathcal{A}$ is CI-S-ring. □

Lemma 4. *If $\mathcal{A}_{F/E} = \mathbf{Z}C_p \wr \mathbf{Z}C_p$ then $\mathcal{A}$ is CI-S ring.*

Proof. If $\mathcal{A}_{F/E} = \mathbf{Z}C_p \wr \mathbf{Z}C_p$, then $\mathcal{A}_{F/E}$ is cyclotomic, and $|O_\theta(\mathcal{A}_{F/E})| = p$. By proposition(4.3) of [6],we have $\mathcal{A}_{F/E}$ is Cayley minimal and by lemma 3, $\mathcal{A}$ is CI-S ring. □

4. S-rings over $(C_p)^2 \times C_q \times C_r$

Let $H = M \times P$ where M is an abelian group and $P = C_p$ with prime p coprime to the order of M. And let $\mathcal{A}$ be an S-ring over H, then we have the next lemma:

Lemma 5. *[16] If there exist a maximal $\mathcal{A}$-subgroup N contained in M such that $N \neq M$, then one of the following statements holds:*

1) *$\mathcal{A} = \mathcal{A}_N \wr \mathcal{A}_{H/N}$ where rank$(\mathcal{A}_{H/N})$=2.*
2) *$\mathcal{A}$ is a F/E-wreath product for $\mathcal{A}$-subgroups $F, E \leqslant H$ such that $P \leqslant E < H$ and $F = PN$.*

Therefore, If M is not $\mathcal{A}$-subgroup then the Hypothese of Lemma 5 is satisfied. So one of two statements of that lemma holds. If M is an $\mathcal{A}$-subgroup while P is not, then by lemma 1, $\mathcal{A}/M = \mathbb{Z}C_p$. But the prime p coprime to the order of M. So every $T \in Bsets(\mathcal{A})$ with $T \cap M = \phi$ is equal to M-coset. This gives us that $\mathcal{A} = \mathcal{A}_M \wr \mathcal{A}/M$.

If M and P are $\mathcal{A}$-subgroups, then by lemma 1, $\mathcal{A}/M = \mathbb{Z}C_p$. But $\mathcal{A}_P = \mathbb{Z}C_p$. So in this case $\mathcal{A} = \mathcal{A}_M \otimes \mathbb{Z}C_p$. Consequently, we have the next lemma:

Lemma 6. *Let $H = (C_p)^2 \times C_q \times C_r$ and $\mathcal{A}$ is an S-ring over H. Then one of the following statements holds:*

1) *$\mathcal{A}$ is F/E-wreath product for $\mathcal{A}$-subgroups $F, E \leqslant H$.*
2) *There exist $\mathcal{A}$-subgroups $H_1, H_2 \leqslant H$ such that $\mathcal{A} = \mathcal{A}_{H_1} \otimes \mathcal{A}_{H_2}$.*

5. Proof of theorem 3

Now let $H = (C_p)^2 \times C_q \times C_r$ and $G \leqslant \mathrm{Sym}(H)$ such that $H_R \leqslant G$. Suppose $\mathcal{A} = \mathcal{V}(H, G_1)$. Then we have:

Lemma 7. *$\mathcal{A}$ is CI-S-ring.*

Proof. By lemma 6, suppose at first $\mathcal{A}$ is E/F-wreath product for $\mathcal{A}$-subgroups F,E of H. Then $|E/F| = 1, p, q, r$ or p^2 or $pq\,, pr\,, qr$. If $\mathcal{A}_{F/E}$ is p-S-ring then we have:

Case (1): $|E/F| = p, q, r$. So $\mathcal{A}_{F/E} = \mathbb{Z}(F/E)$. By lemma 2, $\mathcal{A}$ is CI-S-ring.

Case (2): $|E/F| = p^2$. Then either $\mathcal{A}_{F/E} = \mathbb{Z}C_p^2$ and by lemma 2, $\mathcal{A}$ is CI-S-ring, or $\mathcal{A}_{F/E} = \mathbb{Z}C_p \wr \mathbb{Z}C_p$, so $\mathcal{A}_{F/E}$ is Cayley minimal and by lemma 3, $\mathcal{A}$ is CI-S-ring. If $\mathcal{A}_{F/E}$ is not p-S-ring then $\mathcal{A}_{F/E}$ is an S-ring over group of order p, q, r or p^2 or $pq\,, pr\,, qr$. By our conditions on the numbers p, q, r, $\mathcal{A}_{F/E}$ is cyclotomic and so $\mathcal{A}_{F/E}$ is Cayley minimal. By lemma 3, $\mathcal{A}$ is CI-S-ring.

If $|F/E| = 1$, then $\mathcal{A}$ is a wreath product of two proper S-rings over subgroups of H, but from [11] and [14] we can see that every proper subgroup of H is CI-group, thus by theorm 2, every schurian S-ring over a proper subgroup of H is a CI-S-ring, and $\mathcal{A}$ is CI-S-ring. now suppose $\mathcal{A}$ is not E/F-wreath product. If $\mathcal{A}$ is a direct product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1$, $\mathcal{A}_2$ are S-rings over subgroups of H, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are CI-S-rings by theorm 2, and then $\mathcal{A}$ is CI-S-ring. □

By lemma 7 we complete the proof of theorem 3.

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