

A Ramsey algebraic study of matrices*

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ABSTRACT. The notion of a topological Ramsey space was introduced by Carlson some 30 years ago. Studying the topological Ramsey space of variable words, Carlson was able to derive many classical combinatorial results in a unifying manner. For the class of spaces generated by algebras, Carlson had suggested that one should attempt a purely combinatorial approach to the study. This approach was later formulated and named Ramsey algebra. In this paper, we continue to look at heterogeneous Ramsey algebras, mainly characterizing various Ramsey algebras involving matrices.

1. Introduction

The notion of a Ramsey algebra came as an offshoot of Carlson’s pioneering work on (topological¹) Ramsey spaces [2] when he suggested that the class of Ramsey spaces induced by algebras can be singled out and be studied combinatorially. Of particular importance was the space of multivariable words. By making the right choices of alphabets, Carlson

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¹What is known as a topological Ramsey space in the modern literature is known simply as a Ramsey space in Carlson’s original work. The adjective “topological” is added through Todorovic’s extension [13] of Carlson’s work on the subject.

derived a wide array of classic combinatorial results as corollaries to a result of his concerning the topological Ramsey space of multivariable words, results which were otherwise derived on independent grounds. Among those classical results were the Hales-Jewett theorem, Ellentuck's theorem, and Hindman's theorem in particular.

The initiating studies on Ramsey algebras can be found in the papers [5], [8], and [9]. In [9], the second author addresses a question of Carlson concerning the existence of idempotent ultrafilters for Ramsey algebras. A precise connection between the notion of a (topological) Ramsey space and the notion of a Ramsey algebra can be found in Section 4 of [11]. As the name implies, Ramsey algebras are algebras possessing a certain homogeneity property as is the case with any Ramsey-type result. We will give a precise definition for what is meant by a (heterogeneous) Ramsey algebra in the next section.

An algebra consists of a nonempty domain and a collection of operations on the domain. In this paper, we will look at the Ramsey algebraic aspects of various reducts of the matrix algebra consisting of matrix addition and multiplication, the field operations, and the determinant operation. We will study generalizations to wider classes of algebras and derive the properties concerning matrices as corollaries. Proofs that are omitted in this paper can mostly be found in a parallel paper [10] uploaded to arxiv.org; that parallel submission does not differ in content and outline from this, except that it contains most proofs we have chosen to omit here.

2. Preliminaries

The set of positive integers will be denoted by \mathbb{N} and ω the set of nonnegative integers. The domain of an operation f will be denoted by $\text{Dom}(f)$ and the image set $\text{Im}(f)$. The identity function on any set A is denoted by id_A . A function f is said to be an *operation* on $\{A_\xi\}_{\xi \in I}$ if $\text{Dom}(f) = \prod_{\xi \in J} A_\xi$ for some finite $J \subseteq I$ and the codomain of f is A_ξ for some $\xi \in I$. Let $\{A_\xi\}_{\xi \in I}$ be a family of nonempty sets with I the indexing set and let \mathcal{F} be a family of operations on $\{A_\xi\}_{\xi \in I}$. The structure $(\{A_\xi\}_{\xi \in I}, \mathcal{F})$ is called a *heterogeneous algebra* or *algebra* in short. If I is a singleton, the algebra is referred to as a *homogeneous algebra* for emphasis. The family $\{A_\xi\}_{\xi \in I}$ is called the *domain* of the algebra and each member of the family a *phylum*. Every $\vec{e} \in {}^\omega I$ is called a sort and, if $\vec{b} \in {}^\omega \left(\bigcup_{\xi \in I} A_i \right)$, then \vec{b} is said to be \vec{e} -sorted if $\vec{b}(i) \in A_{\vec{e}(i)}$ for each

$i \in \omega$. If the lists of phyla or operations are not long, we will write them out explicitly. For instance, if $I = \{0, 1\}$ and $\mathcal{F} = \{\circ, +, \times\}$, the algebra would be written $(A_1, A_0, \circ, +, \times)$.

Following Carlson's convention, we require that the phyla in a given algebra be pairwise disjoint. This allows the sort of any sequence in the context of a given algebra be unique.

We will call an operation f *heterogeneous* if $f : A_{\xi_1} \times \cdots \times A_{\xi_N} \rightarrow A_{\xi_{N+1}}$ and there exist $i, j \in \{1, \dots, N+1\}$ such that $A_{\xi_i} \neq A_{\xi_j}$.

For notational convenience, we will sometimes write an n -tuple $\vec{x} = (x_1, \dots, x_n)$ in the notation of a sequence $\vec{x} = \langle x_1, \dots, x_n \rangle$ so that, for instance, if f is an operation whose domain is the Cartesian product of n sets, then $f(\vec{x})$ will mean the same thing as $f(\bar{x})$. The concatenation operation of sequences will be denoted by $*$. Now, let \mathcal{F} be a family of functions on $\{A_\xi\}_{\xi \in I}$. Define $\mathcal{F}_0 = \mathcal{F} \cup \{\text{id}_{A_\xi} : \xi \in I\}$ and, suppose that \mathcal{F}_k has been defined, let

$$\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{f : \varphi(f)\},$$

where $\varphi(f)$ is the statement that there exist some N -ary operation $g \in \mathcal{F}$ and some $h_1, \dots, h_N \in \mathcal{F}_k$ such that $f(\vec{x}) = g(h_1(\vec{x}_1), \dots, h_N(\vec{x}_N))$ and $\vec{x}_1 * \cdots * \vec{x}_N = \vec{x} = \langle x_1, \dots, x_n \rangle$, where n is the arity of f . Then, we have the following definition:

Definition 1 (Orderly Term). *Denote the set $\bigcup_{k \in \omega} \mathcal{F}_k$ by $\text{OT}(\mathcal{F})$. Each member of $\text{OT}(\mathcal{F})$ is called an orderly term over \mathcal{F} .*

Example 1. *Consider the addition $+$ and multiplication \times operations on matrices. The composition $f(x_1, x_2, x_3, x_4) = \times(+ (x_1, x_2), + (x_3, x_4)) = (x_1 + x_2)(x_3 + x_4)$ is an orderly composition over $\{+, \times\}$. Another example is $g(x_1, x_2, x_3, x_4) = \times(+ (\times (x_1, x_2), x_3), x_4) = (x_1 x_2 + x_3)x_4$. However, the compositions $h(x_1, x_2, x_3) = + (x_2, \times (x_1, x_3)) = x_2 + x_1 x_3$ and $k(x_1, x_2, x_3) = \times (\times (x_3, + (x_2, x_1)), x_4) = (x_3(x_2 + x_1))x_4$ are not.*

Definition 2 (Reduction $\leq_{\mathcal{F}}$). *Let $(\{A_\xi\}_{\xi \in I}, \mathcal{F})$ be an algebra and let $\vec{a}, \vec{b} \in \omega \left(\bigcup_{\xi \in I} A_\xi \right)$. Then \vec{a} is said to be a reduction of \vec{b} , written $\vec{a} \leq_{\mathcal{F}} \vec{b}$, if for each $j \in \omega$, there exist a subsequence \vec{b}_j of \vec{b} and an $f_j \in \text{OT}(\mathcal{F})$ such that*

- 1) $\vec{a}(j) = f_j(\vec{b}_j)$ and
- 2) $\vec{b}_0 * \vec{b}_1 * \cdots$ forms a subsequence of \vec{b} .

Note that if \vec{a} is a subsequence of \vec{b} , then $\vec{a} \leq_{\mathcal{F}} \vec{b}$. We will make free use of this fact throughout. Also note that $\leq_{\mathcal{F}}$ is a transitive relation and, if $\mathcal{G} \subseteq \mathcal{F}$ are families of operations and $\vec{a} \leq_{\mathcal{G}} \vec{b}$, then $\vec{a} \leq_{\mathcal{F}} \vec{b}$.

Definition 3. Let $(\{A_\xi\}_{\xi \in I}, \mathcal{F})$ be an algebra and $\vec{e} \in {}^\omega I$. For each \vec{e} -sorted sequence \vec{b} , define

$$\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{b}) = \left\{ \vec{a}(0) : \vec{a} \leq_{\mathcal{F}} \vec{b} \text{ and } \vec{a} \text{ is } \vec{e}\text{-sorted} \right\}. \quad (1)$$

We are now ready for the notion of a heterogeneous Ramsey algebra.

Definition 4 (\vec{e} -Ramsey Algebra). Suppose that $(\{A_\xi\}_{\xi \in I}, \mathcal{F})$ is an algebra and $\vec{e} \in {}^\omega I$. Then $(\{A_\xi\}_{\xi \in I}, \mathcal{F})$ is said to be an \vec{e} -Ramsey algebra if, for each \vec{e} -sorted sequence \vec{b} and each $X \subseteq A_{\vec{e}(0)}$, there exists an \vec{e} -sorted reduction \vec{a} of \vec{b} such that $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a})$ is either contained in or disjoint from X . Such a sequence \vec{a} is said to be homogeneous for X (with respect to \mathcal{F}).

We now look at some examples.

Theorem 1 (Hindman). $(\mathbb{N}, +)$ is a Ramsey algebra. More generally, every semigroup is a Ramsey algebra.

The next theorem can be found in [5] as Theorem 5.5.

Theorem 2. The following are not Ramsey algebras:

- 1) Infinite rings without zero divisors.
- 2) Infinite rings of characteristic zero with identity.

Vector spaces are examples of heterogeneous algebras. A vector space is an algebra with two phyla A_1, A_0 , the former of which we set to be the set of vectors and the latter the underlying scalar field. Classification of Ramsey vector spaces can be found as Theorem 6.1 in [11].

Let I be the indexing set of some algebra. Define

$$\Omega = \{ \vec{e} \in {}^\omega I : \text{if } \vec{e}(i) = \xi \text{ for some } i, \text{ then } \vec{e}(i) = \xi \text{ for infinitely many } i \}.$$

Further, for each $\xi \in I$, the set of all $\vec{e} \in \Omega$ such that $\vec{e}(0) = \xi$ will be denoted by Ω_ξ . For sorts $\vec{e} \in \Omega$, the sets defined by Eq. 1 has the following characterization:

$$c \in \text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{b}) \iff c = f(\tau) \quad (2)$$

for some $f \in \text{OT}(\mathcal{F})$ with codomain $A_{\vec{e}(0)}$ and some finite subsequence τ of \vec{b} . In particular, in the case of homogeneous algebras, the sets given by Eq. 1 can be characterize as

$$\text{FR}_{\mathcal{F}}(\vec{b}) = \{ f(\tau) : f \in \text{OT}(\mathcal{F}), \tau \text{ a subsequence of } \vec{b} \}. \quad (3)$$

We will mainly be concerned with Ramsey algebraic properties involving sorts of the class Ω since results concerning sorts of this class is more uniform. The following theorem, which appears as Theorem 5.3 of [11], gives a precise formulation of this uniformity:

Theorem 3. *Let $\mathcal{A} = \left(\bigcup_{\xi \in I} A_\xi, \mathcal{F}\right)$ be an algebra, $J \subseteq I$, and define $\vec{e} \in \Omega_\eta^J$ if and only if $\vec{e} \in \Omega$, $\vec{e}(0) = \eta$, and $\{\vec{e}(i) : i \in \omega\} = J$. Then \mathcal{A} is an \vec{e} -Ramsey algebra for some $\vec{e} \in \omega$ if and only if \mathcal{A} is an \vec{e} -Ramsey algebra for all $\vec{e} \in \omega$.*

Before ending this section, we mention a fact concerning subalgebras. If $\mathcal{A} = (\{A_\xi\}_{\xi \in I}, \mathcal{F})$ is an algebra, then a *subalgebra* $\mathcal{A}' = (\{A'_\xi\}_{\xi \in I}, \mathcal{F}')$ of \mathcal{A} is an algebra such that $A'_\xi \subseteq A_\xi$ for each $\xi \in I$ and, for each $f' \in \mathcal{F}'$, there exists an $f \in \mathcal{F}$ with $f : A_{\xi_1} \times \cdots \times A_{\xi_n} \rightarrow A_{\xi_{n+1}}$ such that $f' = f \upharpoonright (A'_{\xi_1} \times \cdots \times A'_{\xi_n})$ (restriction property).

Proposition 1. *For any sort \vec{e} , every subalgebra of an \vec{e} -Ramsey algebra is an \vec{e} -Ramsey algebra.*

Proof. Suppose \vec{e} is a sort, $\mathcal{A} = (\{A_\xi\}_{\xi \in I}, \mathcal{F})$ is an \vec{e} -Ramsey algebra, and $\mathcal{A}' = (\{A'_\xi\}_{\xi \in I}, \mathcal{F}')$ is a subalgebra of \mathcal{A} . Let $X \subseteq A'_{\vec{e}(0)}$ and let an \vec{e} -sorted sequence \vec{b} of $\bigcup_{\xi \in I} A'_\xi$ be given.

Since $A'_\xi \subseteq A_\xi$, \vec{b} is also a sequence of $\bigcup_{\xi \in I} A_\xi$. Thus, pick an \vec{e} -sorted $\vec{a} \leq_{\mathcal{F}} \vec{b}$ homogeneous for X , i.e. $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \subseteq X$ or $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \subseteq A_{\vec{e}(0)} \setminus X$. Since the terms of \vec{b} consist of elements of $\bigcup_{\xi \in I} A'_\xi$, the terms of \vec{a} are also members of $\bigcup_{\xi \in I} A'_\xi$ by the restriction and closure properties of operations within a subalgebra, whereby $\vec{a} \leq_{\mathcal{F}'} \vec{b}$. In addition, $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) = \text{FR}_{\mathcal{F}'}^{\vec{e}}(\vec{a}) \subseteq A'_{\vec{e}(0)}$. Consequently, we have $\text{FR}_{\mathcal{F}'}^{\vec{e}}(\vec{a}) \subseteq X$ or $\text{FR}_{\mathcal{F}'}^{\vec{e}}(\vec{a}) \subseteq A'_{\vec{e}(0)} \setminus X$. \square

3. Ramsey-type theorems for various matrix algebras

Throughout the paper, the set of $n \times n$ square matrices over a field \mathbb{F} will be denoted by $\mathcal{M}_n(\mathbb{F})$. Addition and multiplication of matrices will be denoted by $+$ and \times , respectively. Addition and multiplication of field elements will come with a subscript \mathbb{F} . The field \mathbb{F} will be assumed to be of characteristic 0 throughout; in such a case, the rational numbers are embedded within \mathbb{F} , hence we will speak freely of the isomorphic copies of the integers, the nonnegative integers, or the rationals in the \mathbb{F} in question simply as *the* integers and so on. Throughout, we fix the indexing in such

a way that the set \mathbb{F} of scalars receives the index 0 while the set \mathbb{V} of vectors receives the index 1.

Assumption 1. *The field \mathbb{F} is assumed to be infinite with characteristic 0 throughout this section.*

In this section, we begin the study of the Ramsey-algebraic properties of various matrix algebras. We call the algebra $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +, \times, +_{\mathbb{F}}, \times_{\mathbb{F}}, | * |)$ the *full* matrix algebra and any reduct of it is known as a matrix algebra. We will be studying these algebras by looking at slightly more general algebras.

Assumption 2. *The algebras $\mathcal{A} = (A_0, A_1, \mathcal{F})$ studied in this section are of the form $\mathcal{F} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{H}$, where \mathcal{G}_0 consists of operations on A_0 , \mathcal{G}_1 consists of operations on A_1 , and \mathcal{H} consists only of unary operations from A_1 into A_0 and is assumed to be nonempty. We will also denote the algebra (A_0, \mathcal{G}_0) by \mathcal{A}_0 and the algebra (A_1, \mathcal{G}_1) by \mathcal{A}_1 .*

Situations when \mathcal{H} is empty can be found in Theorem 5.2 (1) of [11]. It states that, for each $i = 0, 2$ and each $\vec{e} \in \Omega_i$, the algebra $(A_0, A_1, \mathcal{G}_0, \mathcal{G}_1)$ is an \vec{e} -Ramsey algebra if and only if \mathcal{A}_i is a Ramsey algebra. This leads to the following corollary.

Corollary 1. *Let $n \in \mathbb{N}$, $\vec{e} \in \Omega$, and let \mathcal{A}' be a reduct of the full matrix algebra not containing the determinant operation. Then \mathcal{A}' is an \vec{e} -Ramsey algebra if and only if $\mathcal{G}_{\vec{e}(0)}$ consists of at most one of the two ring operations pertaining to $A_{\vec{e}(0)}$.*

Proof. The proof is immediate; we only need to look at the appropriate \mathcal{A}_0 or \mathcal{A}_1 to decide if \mathcal{A}' is a Ramsey algebra. Of a priori importance is the fact that any ring of matrices $(\mathcal{M}_n(\mathbb{F}), +, \times)$ is not a Ramsey algebra. This hinges upon the fact that \mathbb{F} can be embedded into $\mathcal{M}_n(\mathbb{F})$ as diagonal matrices all of whose diagonal elements are equal to some field element r . Diagonal matrices of this form thus form a subalgebra of $(\mathcal{M}_n(\mathbb{F}), +, \times)$ and, since the subalgebra $(\mathbb{F}, +_{\mathbb{F}}, \times_{\mathbb{F}})$ is not a Ramsey algebra, $(\mathcal{M}_n(\mathbb{F}), +, \times)$ is not a Ramsey algebra either. \square

We have, therefore, identified the Ramsey algebraic properties of all reducts of the full matrix algebra for which the determinant operation is absent.

As per Assumption 1, the algebras of concern are such that any heterogeneous operations are unary from A_1 into A_0 . This condition allows us to derive the following theorem.

Theorem 4. *For each $\vec{e} \in \Omega_1$, \mathcal{A} is an \vec{e} -Ramsey algebra if and only if \mathcal{A}_1 is a Ramsey algebra.*

Proof. (\Rightarrow) Suppose that \mathcal{A} is an \vec{e} -Ramsey algebra and $\vec{\beta}$ is an infinite sequence of A_1 . Pick any \vec{e} -sorted sequence \vec{b} so that $\vec{\beta}$ forms the subsequence of \vec{b} all of whose terms belong in A_1 .

By hypothesis, let $\vec{a} \leq_{\mathcal{F}} \vec{b}$ be \vec{e} -sorted and homogeneous for X . We observe that, if $\vec{\alpha}$ is the subsequence of \vec{a} all of whose terms are members of A_1 , then (1) $\vec{\alpha} \leq_{\mathcal{G}_1} \vec{\beta}$ and (2) $\vec{\alpha}$ is homogeneous for X . This is so because every orderly term over \mathcal{F} with codomain A_1 must have as domain a Cartesian power of A_1 . It then follows that (A_1, \mathcal{G}_1) is a Ramsey algebra.

(\Leftarrow) Suppose (A_1, \mathcal{G}_1) is a Ramsey algebra. Given $X \subseteq A_1$ and an \vec{e} -sorted sequence \vec{b} , let $\vec{\beta}$ be the subsequence of \vec{b} consisting of elements of A_1 . By hypothesis, pick an $\vec{\alpha} \leq_{\mathcal{G}_1} \vec{\beta}$ homogeneous for X . In fact, by carefully going through the definition of reduction, we can pick such an $\vec{\alpha}$ so that, for any \vec{e} -sorted \vec{a} such that $\vec{\alpha}$ is the subsequence of \vec{a} whose terms are members of A_1 , we have $\vec{a} \leq_{\mathcal{F}} \vec{b}$.

Thus, again by the observation we made in the (\Rightarrow) case, we have that $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) = \text{FR}_{\mathcal{G}_1}(\vec{\alpha})$, whence the homogeneity of \vec{a} for X is established. \square

Theorem 4 offers a complete answer as to when a matrix algebra is an \vec{e} -Ramsey algebra in the case when $\vec{e} \in \Omega_1$:

Corollary 2. *Let $n \in \mathbb{N}$. For any $\vec{e} \in \Omega_1$, all reducts of the full matrix algebra $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +, \times, +_{\mathbb{F}}, \times_{\mathbb{F}}, |*|)$ is an \vec{e} -Ramsey algebra except for those reducts that keep both matrix operations.*

Proof. This is because $(\mathcal{M}_n(\mathbb{F}), +)$ and $(\mathcal{M}_n(\mathbb{F}), \times)$ are Ramsey algebras (because they are semigroups) and, since \mathbb{F} is embedded in $(\mathcal{M}_n(\mathbb{F}), +, \times)$, it is not a Ramsey algebra by Theorem 2. \square

Thus, we should now focus on the situations when $\vec{e} \in \Omega \setminus \Omega_1$ as well as when \mathcal{H} is nonempty. Henceforth, we make the following assumption:

Assumption 3. *\mathcal{H} is assumed to be a singleton and its sole member will be denoted by h henceforth.*

The determinant operation is a homomorphism from the set of matrices equipped with matrix multiplication to the multiplicative group of the underlying field. Suppose that two homogeneous algebras \mathcal{A}_0 and \mathcal{A}_1 have the same signature, i.e. if there exists a one-to-one correspondence between \mathcal{G}_0 and \mathcal{G}_1 such that, if $F \in \mathcal{G}_1$ is an n -ary operation, then the

corresponding operation $f \in \mathcal{G}_0$ is also n -ary. Then $h : A_1 \rightarrow A_0$ is said to be a *homomorphism* from \mathcal{A}_1 into \mathcal{A}_0 if for each corresponding n -ary operation $F \in \mathcal{G}_1$ and $f \in \mathcal{G}_0$, and for all (a_1, \dots, a_n) in the domain of f ,

$$h(F(a_1, \dots, a_n)) = f(h(a_1), \dots, h(a_n)). \quad (4)$$

Lemma 1. *Suppose that $\vec{e} \in \Omega_0$ and h is a homomorphism from \mathcal{A}_1 into \mathcal{A}_0 . If $\vec{\alpha} \in {}^\omega A_1$ and the \vec{e} -sorted sequence \vec{a} are related by Eq. 3, then for each N -ary $f \in \text{OT}(\mathcal{F})$ having codomain A_0 and each $n_1 < \dots < n_N$, there exists an N -ary $F \in \text{OT}(\mathcal{G}_1)$ such that $f(\vec{a}(n_1), \dots, \vec{a}(n_N)) = h(F(\vec{\alpha}(n_1), \dots, \vec{\alpha}(n_N)))$. In addition, $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) = \{h(c) : c \in \text{FR}_{\mathcal{G}_1}(\vec{\alpha})\}$.*

Proof. We prove the lemma by induction on the generation of f . For the base case, we consider $f \in \mathcal{G}_0 \cup \{h\} \cup \{\text{id}_{A_0}\}$. If $f = \text{id}_{A_0}$, then

$$f(\vec{a}(i)) = \text{id}_{A_0}(h(\vec{a}(i))) = h(\text{id}_{A_1}(\vec{a}(i)))$$

which clearly shows that $f(\vec{a}(i))$ is in the stipulated form. For $f = h$, the proof is similar. Thus, suppose now that f is an N -ary operation belonging in \mathcal{G}_0 and $n_1 < \dots < n_N$. Let F be the corresponding operation in \mathcal{G}_1 under the homomorphism h . Then

$$\begin{aligned} f(\vec{a}(n_1), \dots, \vec{a}(n_N)) &= f(h(\vec{\alpha}(n_1)), \dots, h(\vec{\alpha}(n_N))) \\ &= h(F(\vec{\alpha}(n_1), \dots, \vec{\alpha}(n_N))), \end{aligned}$$

which again is in the stipulated form. The last base case to consider is when $f = h$, but this is immediate.

Next, for the inductive step, suppose that f is such that $f(\vec{a}(n_1), \dots, \vec{a}(n_N)) = G(G_1(\tau_1), \dots, G_N(\tau_N))$, where $G \in \mathcal{G}_0 \cup \{h\} \cup \{\text{id}_{A_0}\}$ and $\tau_1 * \dots * \tau_N = \langle \vec{a}(n_1), \dots, \vec{a}(n_N) \rangle$. For each finite subsequence $\tau = \langle \vec{a}(m_1), \dots, \vec{a}(m_M) \rangle$ of \vec{a} , let $\tilde{\tau}$ denote the finite subsequence $\langle \vec{\alpha}(m_1), \dots, \vec{\alpha}(m_M) \rangle$ of $\vec{\alpha}$. We omit the case when $f = h$ as the proof is immediate. Thus, by induction hypothesis, let $F_1, \dots, F_N \in \text{OT}(\mathcal{G}_1)$ be such that $G_i(\tau_i) = h(F_i(\tilde{\tau}_i))$ for each $i \in \{1, \dots, N\}$. Then, denoting by $G' \in \mathcal{G}_1$ the operation corresponding to G under the homomorphism h , we now have

$$\begin{aligned} f(\vec{a}(n_1), \dots, \vec{a}(n_N)) &= G(G_1(\tau_1), \dots, G_N(\tau_N)) \\ &= G(h(F_1(\tilde{\tau}_1)), \dots, h(F_N(\tilde{\tau}_N))) \\ &= h(G'(F_1(\tilde{\tau}_1), \dots, F_N(\tilde{\tau}_N))). \end{aligned}$$

Since $G', F_1, \dots, F_N \in \text{OT}(\mathcal{G}_1)$ and $\sigma_1 * \dots * \sigma_N = \langle \vec{\alpha}(n_1), \dots, \vec{\alpha}(n_N) \rangle$, it follows that $f(\vec{a}(n_1), \dots, \vec{a}(n_N))$ can be expressed in the stipulated form. This completes the induction proof of the first conclusion of the lemma.

The other conclusion of the lemma can now be deduced easily. \square

Theorem 5. *Suppose that \mathcal{A}_1 is a Ramsey algebra and h is a homomorphism. Then, \mathcal{A} is an \vec{e} -Ramsey algebra for each $\vec{e} \in \Omega_0$.*

Proof. Suppose that $\vec{e} \in \Omega_0$, \vec{b} is an \vec{e} -sorted sequence, and $X \subseteq A_0$. Let $\vec{\beta}$ be the subsequence of \vec{b} whose terms are members of A_1 . As such, let $\vec{\alpha} \leq_{\mathcal{G}_1} \vec{\beta}$ be homogeneous for $h^{-1}[X]$. Take note that the relation $\vec{\alpha} \leq_{\mathcal{F}} \vec{\beta} \leq_{\mathcal{F}} \vec{b}$ holds by the transitivity of $\leq_{\mathcal{F}}$ and the fact that $\mathcal{G}_1 \subseteq \mathcal{F}$.

Now let the \vec{e} -sorted sequence \vec{a} be defined by

$$\vec{a}(i) = \begin{cases} \vec{\alpha}(i) & \text{if } \vec{e}(i) = 1, \\ h(\vec{\alpha}(i)) & \text{otherwise.} \end{cases}$$

Noting that $\vec{a} \leq_{\mathcal{F}} \vec{\alpha}$, we have $\vec{a} \leq_{\mathcal{F}} \vec{b}$ by the transitivity of $\leq_{\mathcal{F}}$. We may now apply Lemma 1. Namely, each member of $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a})$ is the image of some $c \in \text{FR}_{\mathcal{G}_1}(\vec{\alpha})$ under h since h is a homomorphism. From this, we conclude that $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \subseteq X$ or $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \subseteq A_0 \setminus X$ depending respectively on whether $\text{FR}_{\mathcal{G}_1}(\vec{\alpha}) \subseteq h^{-1}[X]$ or $\text{FR}_{\mathcal{G}_1}(\vec{\alpha}) \subseteq A_1 \setminus h^{-1}[X]$. This is a statement about the homogeneity of \vec{a} for X , hence \mathcal{A} is an \vec{e} -Ramsey algebra. \square

Corollary 3. *$(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, \times_{\mathbb{F}}, \times, | * |)$ is an \vec{e} -Ramsey algebra for each $\vec{e} \in \Omega_0$ and $n \in \omega$.*

Noting that $\text{FR}_{\emptyset}^{\vec{e}}(\vec{a}) \subseteq \text{FR}_{\mathcal{G}}^{\vec{e}}(\vec{a})$ for any family of operation \mathcal{G} , we see that a slight modification of the proof above yields the following theorem. See Theorem 3.2 of [10] for a complete proof.

Theorem 6. *Suppose that $\vec{e} \in \Omega_0$. If $\mathcal{G}_0 = \emptyset$ and \mathcal{A}_1 is a Ramsey algebra, then \mathcal{A} is an \vec{e} -Ramsey algebra.*

Corollary 4. *For each $\vec{e} \in \Omega_0$ and $n \in \omega$, the algebras $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, | * |)$, $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +, | * |)$, and $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, \times, | * |)$ are \vec{e} -Ramsey algebras.*

If the heterogeneous operation h is a one-to-one function, then we may deduce a result on the negative spectrum:

Theorem 7. *Suppose that \mathcal{A}_1 is not a Ramsey algebra as witnessed by $\vec{\beta}$ and X_1 and that h is one-to-one on $\text{FR}_{\mathcal{G}_1}(\vec{\beta})$. Then, \mathcal{A} is not an \vec{e} -Ramsey algebra for all nonconstant $\vec{e} \in \Omega_0$.*

Proof. Let $\vec{e} \in \Omega_0$ be nonconstant. Begin by defining an \vec{e} -sorted sequence \vec{b} as follows:

$$\vec{b}(i) = \begin{cases} \vec{\beta}(i) & \text{if } \vec{e}(i) = 1, \\ h(\vec{\beta}(i)) & \text{otherwise.} \end{cases}$$

Observe that, for every $\vec{a} \leq_{\mathcal{F}} \vec{b}$, the subsequence \vec{a} consisting of terms belonging in A_1 is such that $\vec{a} \leq_{\mathcal{G}_1} \vec{\beta}$ and, therefore, $\text{FR}_{\mathcal{G}_1}(\vec{a}) \not\subseteq X$ and $\text{FR}_{\mathcal{G}_1}(\vec{a}) \not\subseteq A_1 \setminus X$ by hypothesis. Consequently, since \vec{e} is not constant, we see that $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a})$ is such that $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \not\subseteq h(X \cap \text{FR}_{\mathcal{G}_1}(\vec{\beta}))$ and $\text{FR}_{\mathcal{F}}^{\vec{e}} \not\subseteq A_0 \setminus h(X \cap \text{FR}_{\mathcal{G}_1}(\vec{\beta}))$ owing to the fact that h is a one-to-one function. Hence, \mathcal{A} is not an \vec{e} -Ramsey algebra. \square

Note that the theorem above requires that \vec{e} being nonconstant. If \vec{e} is constant, then the conclusion depends solely on whether \mathcal{A}_0 is a Ramsey algebra or not.

Corollary 5. *Neither $(\mathcal{M}_n, \mathbb{F}, +, \times, | * |)$, $(\mathcal{M}_n, \mathbb{F}, +, \times, +_{\mathbb{F}}, | * |)$, $(\mathcal{M}_n, \mathbb{F}, +, \times, \times_{\mathbb{F}}, | * |)$, nor the full matrix algebra is an \vec{e} -Ramsey algebra for any nonconstant $\vec{e} \in \Omega_0$.*

Proof. (A sketch.) Note that matrix ring has zero divisors, so we cannot apply Theorem 2 directly. However, since our field \mathbb{F} of interest are of characteristic 0, we may take as bad sequence \vec{b} the sequence of diagonal matrices whose diagonal elements are terms of the bad sequence $\langle 2^{2^i} \rangle_{i \in \omega}$ as given by Example 2.4.1 of [6]. Hence $| * |$ will be one-to-one on $\text{FR}_{\mathcal{G}_1}(\vec{\beta})$ and the conclusion follows from the theorem above. \square

We have another result of the negative nature, but first we need a lemma.

Lemma 2. *Suppose that h is a homomorphism, $\vec{\beta} \in {}^\omega \text{Im}(h)$, $\vec{e} \in \Omega_0$, and \vec{b} is \vec{e} -sorted such that $\vec{\beta}(i) = h(\vec{b}(i))$ if $\vec{e}(i) = 1$ and $\vec{b}(i) = \vec{\beta}(i)$ otherwise. Then we have:*

- 1) *If f is an N -ary member of $\text{OT}(\mathcal{F})$ having codomain A_0 , then for each $n_1 < \dots < n_N$, there exists $F' \in \text{OT}(\mathcal{G}_0)$ such that $f(\vec{b}(n_1), \dots, \vec{b}(n_N)) = F'(\vec{\beta}(n_1), \dots, \vec{\beta}(n_N))$.*
- 2) *If $\vec{u} \leq_{\mathcal{F}} \vec{b}$ and $\vec{u} \in {}^\omega A_0$, then $\vec{u} \leq_{\mathcal{G}_0} \vec{\beta}$.*

For this lemma, 2) follows easily from 1), which can be obtained by an application of Lemma 1.

Theorem 8. *Suppose that h is a homomorphism and suppose that \mathcal{A}_0 is not a Ramsey algebra. If there exists a sequence $\vec{\beta} \in {}^\omega \text{Im}(h)$ and a set $X \subseteq A_0$ witnessing the failure of \mathcal{A}_0 being a Ramsey algebra, then \mathcal{A} is not an \vec{e} -Ramsey algebra for all $\vec{e} \in \Omega_0$.*

This theorem appears correspondingly as Theorem 3.5 in [10]. Although a proof can be found in the accompanying paper, the reader who is familiar with the technique above should now be able to derive the result.

Corollary 6. $(\mathcal{M}_n, \mathbb{F}, +_{\mathbb{F}}, \times_{\mathbb{F}}, |*|)$ and $(\mathcal{M}_n, \mathbb{F}, \times, +_{\mathbb{F}}, \times_{\mathbb{F}}, |*|)$ are not \vec{e} -Ramsey algebras for any $\vec{e} \in \Omega_0$.

4. More on matrix algebras

In this section, we tackle the remaining matrix algebras obtainable as reducts of the full matrix algebra $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +, \times, +_{\mathbb{F}}, \times_{\mathbb{F}}, |*|)$. With the exception of 1., Theorem 9, results in this section are of a negative nature. Note again that the case with $\vec{e} \in \Omega_1$ follows from Theorem 4.

We begin with a lemma from which, either in its full power or special cases of it, many results can be derived as we shall see. We will make use of the uniqueness of binary representation of the positive integers (UBR).

Lemma 3. Let $0 < i_1 < \dots < i_L$ and $0 < j_1 < \dots < j_M < k_1 < \dots < k_N$ be integers. Then,

$$\left(2^{2^{i_1}} + \dots + 2^{2^{i_L}}\right)^2 \neq \left(2^{2^{j_1}} + \dots + 2^{2^{j_M}}\right)^2 + \left(2^{2^{k_1}} + \dots + 2^{2^{k_N}}\right)^2. \quad (5)$$

Proof. Let Γ denote the set $\{1, \dots, L\}$. Expanding the left hand side of Inequality 5, we obtain

$$\begin{aligned} \sum_{(p,q) \in \Gamma^2} 2^{2^p + 2^q} &= \sum_{p \in \Gamma} 2^{2(2^p)} + \sum_{p,q \in \Gamma, p \neq q} 2 \cdot 2^{2^p + 2^q} \\ &= \sum_{p \in \Gamma} 2^{2^{i_{p+1}}} + \sum_{p,q \in \Gamma, p \neq q} 2^{2^p + 2^q + 1}. \end{aligned} \quad (6)$$

Call this quantity, a positive integer, N_1 . Similar expansion of the right hand side of Inequality 5 gives us

$$\begin{aligned} &\sum_{r \in \{1, \dots, M\}} 2^{2^{j_{r+1}}} + \sum_{r,s \in \{1, \dots, M\}, r \neq s} 2^{2^{j_r} + 2^{j_s} + 1} \\ &\quad + \sum_{u \in \{1, \dots, N\}} 2^{2^{k_u + 1}} + \sum_{u,v \in \{1, \dots, N\}, u \neq v} 2^{2^{k_u} + 2^{k_v} + 1} \\ &= \sum_{r \in \{1, \dots, M\}} 2^{2^{j_{r+1}}} + \sum_{u \in \{1, \dots, N\}} 2^{2^{k_u + 1}} \\ &\quad + \sum_{r,s \in \{1, \dots, M\}, r \neq s} 2^{2^{j_r} + 2^{j_s} + 1} + \sum_{u,v \in \{1, \dots, N\}, u \neq v} 2^{2^{k_u} + 2^{k_v} + 1}. \end{aligned} \quad (7)$$

Call this integer N_2 .

Note that the exponents in the sum of Eq. 6 are pairwise distinct, hence Eq. 6 is the binary expansion of N_1 . Similarly, Eq. 7 is the binary expansion of N_2 . Thus, if N_1 is to equal N_2 , they must have the exact same binary expansion. In the case $L^2 \neq M^2 + N^2$, clearly $N_1 \neq N_2$ because the total number of summands in Eq. 6 is L^2 while the total number of summands in Eq. 7 is $M^2 + N^2$. Thus, $N_1 \neq N_2$ in this case.

On the other hand, suppose that $L^2 = M^2 + N^2$. We compare the sums $\sum_{p \in \{1, \dots, L\}} 2^{2^p+1}$ and $\sum_{r \in \{1, \dots, M\}} 2^{2^r+1} + \sum_{u \in \{1, \dots, N\}} 2^{2^u+1}$. By inspecting exponents, we conclude that for N_1 to equal N_2 , it is required by UBR that both sums are a fortiori equal. However, this leads to $L = M + N$, contradicting $L^2 = M^2 + N^2$ since neither M or N is 0. \square

Theorem 9. 1) If $n = 1$, then the algebra $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +, +_{\mathbb{F}}, | * |)$ is an \vec{e} -Ramsey algebra for every $\vec{e} \in \Omega_0$.

2) If $n > 1$, then the algebra $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +, +_{\mathbb{F}}, | * |)$ is not an \vec{e} -Ramsey algebra for every nonconstant $\vec{e} \in \Omega_0$.

Proof. We give the proof of Case (2), which uses the full statement of Lemma 3. Case (1) can be obtained by noting that 1×1 matrices are essentially field elements.

For each $i \in \omega$, let D_i be the diagonal matrix all of whose diagonal elements are 2^{2^i} and, given any $\vec{e} \in \Omega_0$, let

$$\vec{b}(i) = \begin{cases} D_i & \text{if } \vec{e}(i) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Note that, if $\vec{a} \leq_{\mathcal{F}} \vec{b}$ is \vec{e} -sorted, then each matrix term of \vec{a} is of the form $D_{i_1} + \dots + D_{i_M}$ for some $i_1 < \dots < i_M$ and the determinant of such a term is given by $(2^{2^{i_1}} + \dots + 2^{2^{i_M}})^n$, which is a member of $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a})$ as we appeal to Eq. 2.

Therefore, define $X \subseteq \mathbb{F}$ by

$$X = \left\{ \left(2^{2^{i_1}} + \dots + 2^{2^{i_M}} \right)^n \in \omega : i_1 < \dots < i_M \right\}.$$

Thus, note that, for each $\vec{a} \leq_{\mathcal{F}} \vec{b}$ that is \vec{e} -sorted, the intersection $X \cap \text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a})$ is nonempty.

Now, let $k_1 < k_2$ be such that $\vec{a}(k_1) = D_{i_1} + \dots + D_{i_M}$ and $\vec{a}(k_2) = D_{j_1} + \dots + D_{j_N}$. Then $(2^{2^{i_1}} + \dots + 2^{2^{i_M}})^n + (2^{2^{j_1}} + \dots + 2^{2^{j_M}})^n$, which is a member of $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a})$ again by Eq. 2, is not a member of X . This is because the quantity cannot be expressed in the form of $2^{2^{i_1}} + \dots + 2^{2^{i_M}}$

owing to Fermat's Last Theorem for $n > 2$, while for $n = 2$, the result is the content of Lemma 3. \square

We end this section with a list of results whose proofs can be obtained using the techniques we have seen in this section. Each corresponding statement in the manuscript [10] is listed in brackets behind the statement here.

Theorem 10. *The following are not \vec{e} -Ramsey algebras for every non-constant $\vec{e} \in \Omega_0$:*

- 1) $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, \times, +_{\mathbb{F}}, | * |)$ [Theorem 4.1],
- 2) $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +_{\mathbb{F}}, | * |)$ [Theorem 4.2],
- 3) $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +, \times_{\mathbb{F}}, | * |)$ [Theorem 4.3],
- 4) $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, +, +_{\mathbb{F}}, \times_{\mathbb{F}}, | * |)$ [Theorem 4.3],
- 5) $(\mathcal{M}_n(\mathbb{F}), \mathbb{F}, \times_{\mathbb{F}}, | * |)$ [Theorem 4.4].

Conclusion

This paper was aimed at further understanding heterogeneous Ramsey algebras. Specifically, we have looked at heterogeneous algebras consisting of two phyla with some "disjoint" set of operations and some heterogeneous unary operations mapping members of a phylum to another. Special cases are when the heterogeneous unary operations are homomorphisms. Such algebras are motivated from the various matrix algebras that we have studied and, as corollaries, we derived results pertaining to the matrix algebras of concern.

While the paper has shed more light on the behavior of heterogeneous Ramsey algebras, algebras for which heterogeneous operations are present remain elusive. The only heterogeneous algebras admitting such operations that have been studied are vector spaces and the results can be found in [11]. A combined look at these two works should be a good starting point for further investigation.

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