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On the number of topologies on a finite set

M. Yasir Kizmaz

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Abstract. We denote the number of distinct topologies which can be defined on a set X with n elements by $T(n)$. Similarly, $T_0(n)$ denotes the number of distinct T_0 topologies on the set X. In the present paper, we prove that for any prime p, $T(p^k) \equiv k+1 \ (mod\ p)$, and that for each natural number n there exists a unique k such that $T(p+n) \equiv k \ (mod\ p)$. We calculate k for $n = 0, 1, 2, 3, 4$. We give an alternative proof for a result of Z. I. Borevich to the effect that $T_0(p+n) \equiv T_0(n+1) \ (mod\ p)$.

1. Introduction

Given a finite set X with n elements, let $\mathfrak{T}(X)$ and $\mathfrak{T}_0(X)$ be the family of all topologies on X and the family of all T_0 topologies on X, respectively. We denote the cardinality of $\mathfrak{T}(X)$ by $T(n)$ and the cardinality of $\mathfrak{T}_0(X)$ by $T_0(n)$. There is no simple formula giving $T(n)$ and $T_0(n)$.

Calculation of these sequences by hand becomes very hard for $n \geqslant 4$. The online encyclopedia of N. J. A. Sloane [1] gives the values of $T(n)$ and $T_0(n)$ for $n \leqslant 18$. For a more detailed discussion of results in the literature, we refer to the article by Borevich [4].

In the present paper, we prove that for any prime p, $T(p^k) \equiv k + 1 \ (mod\ p)$, and that for each natural number n there exists a unique k such that $T(p+n) \equiv k \ (mod\ p)$. We calculate k for $n = 0, 1, 2, 3, 4$. We give an alternative proof for a result of Z. I. Borevich [5] to the effect that $T_0(p+n) \equiv T_0(n+1) \ (mod\ p)$. Our results depend on basic properties of group action, which is the first time used in this problem as far as the author is aware.

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2. Main results

Let G be a group acting on the finite set X. Then the action of G on X can be extended to the action of G on $\mathfrak{T}(X)$ by setting $g\tau = \{gU \mid U \in \tau\}$ where $\tau \in \mathfrak{T}(X)$ and $gU = \{gu \mid u \in U\}$. Now set $\mathrm{Fix}(\mathfrak{T}(X)) = \{\tau \in \mathfrak{T}(X) \mid g\tau = \tau \text{ for all } g \in G\}$. Notice that if G is a p-group, $|\mathfrak{T}(X)| \equiv |\,\mathrm{Fix}(\mathfrak{T}(X))|$ $(mod\ p)$ as every non-fixed element of $\mathfrak{T}(X)$ has orbit of length a positive power of p. For a given topology $\tau \in \mathfrak{T}(X)$ and $x \in X$, we denote the intersection of all open sets of τ including x by O_x. Note that $O_x \in \tau$ as we are in the finite case.

Definition 1. A base $\mathfrak{B}$ of a topology τ is called a minimal base if any base of the topology contains $\mathfrak{B}$.

Example. Let $X = \{a, b, c\}$ be a set and let τ be a topology on X with $\tau = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Now if $\mathfrak{B}$ is an arbitrary base for τ then we have $U = \bigcup_{x\in U} B_x$ where $U \in \tau$ and $B_x \in \mathfrak{B}$ such that $x \in B_x \subseteq U$. Thus, we observe that $\mathfrak{B}' = \{\{a\}, \{a, b\}, \{a, c\}\} \subseteq \mathfrak{B}$. Moreover, $\mathfrak{B}'$ is also a base for τ which implies that $\mathfrak{B}'$ is a minimal base for τ.

Proposition 1. *Let $\tau \in \mathfrak{T}(X)$ and M_τ be the set of all distinct O_x for $x \in X$. A base $\mathfrak{B}$ of τ is a minimal base if and only if $\mathfrak{B} = M_\tau$.*

By the proposition, we extend the definition: a base $\mathfrak{B}$ on X is minimal if $\mathfrak{B} = M_\tau$ where τ is the topology generated by $\mathfrak{B}$.

Lemma 1. $\tau \in \mathrm{Fix}(\mathfrak{T}(X))$ *if and only if M_τ is G-invariant.*

Proof. Let $\tau \in \mathrm{Fix}(\mathfrak{T}(X))$ and $O_x \in M_\tau$. We need to show that $gO_x \in M_\tau$. Let $gx = y$ for $g \in G$. As $g\tau = \tau$, $gO_x \in \tau$. Thus, gO_x is an open set containing the element $gx = y$. Hence, $O_y \subseteq gO_x$. Since $g^{-1}y = x$, we can show that $O_x \subseteq g^{-1}O_y$ which forces $gO_x = O_y$. So, $gO_x \in M_\tau$ $\forall g \in G$. Now assume that M_τ is G-invariant. For each $U \in \tau$, we have $U = \bigcup_{x\in U} O_x$. Then $gU \in \tau$ as $gO_x \in M_\tau$ for all $g \in G$, and hence $\tau \in \mathrm{Fix}(\mathfrak{T}(X))$. □

Theorem 1. *Let p be a prime and let k be a natural number. Then $T(p^k) \equiv k + 1$ $(mod\ p)$.*

Proof. Without loss of generality, let X be the cyclic group of order p^k, that is, $X = C_{p^k}$. Clearly X acts on X by left multiplication. By extending this action, X acts on $\mathfrak{T}(X)$. Notice that $|\mathfrak{T}(X)| \equiv |\operatorname{Fix}(\mathfrak{T}(X))|$ $(mod\ p)$ as X is a p-group. It is left to show that $\operatorname{Fix}(\mathfrak{T}(X))$ has $k+1$ elements.

Let $\tau \in \operatorname{Fix}(\mathfrak{T}(X))$ and let $O_x, O_y \in M_\tau$ for $x, y \in X$. Then $(yx^{-1})O_x$ is an open set including y. Hence, $O_y \subseteq (yx^{-1})O_x$ which means $|O_y| \leqslant |O_x|$. The other inclusion can be done similarly so $|O_x| = |O_y|$ for all $x, y \in X$. Now if $m \in O_x \cap O_y$, then $O_m \subseteq O_x \cap O_y$. As their orders are equal, we must have $O_x = O_y$ or $O_x \cap O_y = \varnothing$. Thus, X is a disjoint union of elements of M_τ, that is, $X = \biguplus_{O_x \in M_\tau} O_x$. It follows that $|X| = |M_\tau||O_x|$ for any $x \in X$.

Note that X also acts on M_τ by Lemma 1. Let e be the identity element of X and $\operatorname{Stab}(O_e)$ be the stabilizer of O_e in the action of X on M_τ. Then we obtain that $|X : \operatorname{Stab}(O_e)| = |M_\tau|$ as the action is transitive. It follows that $|\operatorname{Stab}(O_e)| = |O_e|$ as $|X| = |M_\tau||O_e|$ also holds. Since our action is induced from left multiplication in the group X, the product $\operatorname{Stab}(O_e)O_e$ is exactly set multiplication in the group X. Then we obtain that $O_e = \operatorname{Stab}(O_e)O_e \supseteq \operatorname{Stab}(O_e)e = \operatorname{Stab}(O_e)$. But the equality $|O_e| = |\operatorname{Stab}(O_e)|$ forces that $O_e = \operatorname{Stab}(O_e)$.

Hence, the set M_τ is the set of left cosets of a subgroup of X. Since the chosen topology τ from $\operatorname{Fix}(\mathfrak{T}(X))$ uniquely determines M_τ and M_τ uniquely determines a subgroup O_e of X, we have an injection from $\operatorname{Fix}(\mathfrak{T}(X))$ to the set of all subgroups of X. Conversely, for a subgroup H of X, the set of left cosets of H forms an X-invariant minimal base for a topology. The topology τ generated by this base is an element of $\operatorname{Fix}(\mathfrak{T}(X))$ by Lemma 1. Hence, the cardinality of $\operatorname{Fix}(\mathfrak{T}(X))$ is equal to the number of the subgroups of X, which is $k+1$. □

By applying the same method in the above proof, we can also show that $T_0(p^k) \equiv 1$ $(mod\ p)$. Actually, Z. I. Borevich proved a more general result about $T_0(n)$. Now we establish an alternative proof for the theorem of Z. I. Borevich.

Theorem 2 (Z. I. Borevich)**.** *Let p be a prime. If $k \equiv l$ $(mod\ p-1)$, then $T_0(k) \equiv T_0(l)$ $(mod\ p)$.*

Proof. It is equivalent to show that $T_0(n+p) \equiv T_0(n+1)$ $(mod\ p)$ for an integer $n \geqslant 0$. Let $C = C_p$ be the cyclic group of order p and N be a set with n elements. Without loss of generality, let X be the disjoint union of C and N so that the cardinality of X is $p+n$. We define the action of C on X in the following way: for $c \in C$ and for $x \in X$, $c * x = cx$ if $x \in C$ and $c * x = x$ if $x \in N$. Then the action of C on X can be extended to the action of C on $\mathfrak{T}_0(X)$. As $|\mathfrak{T}_0(X)| \equiv |\operatorname{Fix}(\mathfrak{T}_0(X))|$ $(mod\ p)$, it is left to show that $|\operatorname{Fix}(\mathfrak{T}_0(X))| = T_0(n+1)$.

Let $\tau \in \operatorname{Fix}(\mathfrak{T}_0(X))$. Pick $x, y \in C$ and $a \in N$ where $x \neq y$. We know that $yx^{-1}O_x = O_y$. Then we can observe that $O_x \cap N = O_y \cap N$ as N is fixed by C. Similarly, $O_x \cap C$ and $O_y \cap C$ are disjoint or equal. But we can not have $O_x = O_y$ as it is a T_0 topology. Hence, $(O_x \cap C) \cap (O_y \cap C) = \varnothing$. Then we obtain that $O_x \cap C = \{x\}$, and hence $O_x = \{x\} \cup S$ for a subset S of N. On the other hand, $x(C \cap O_a) = C \cap O_a$, which forces that $C \cap O_a$ is either C or $\varnothing$, and hence O_a is either $C \cup T$ or T for a subset T of N. Set $Y = \{x\} \cup N$ then for each $\tau \in \operatorname{Fix}(\mathfrak{T}_0(X))$, we have subspace topology $\tau_Y = \{U \cap Y \mid U \in \tau\}$ on Y. Note that τ_Y is a T_0 topology as it is the subspace topology on Y induced from τ.

Now our aim is to show that the mapping $\tau \mapsto \tau_Y$ is a bijection from $\operatorname{Fix}(\mathfrak{T}_0(X))$ to $T_0(Y)$. Let $\pi, \tau \in \operatorname{Fix}(\mathfrak{T}_0(X))$ such that $\tau_Y = \pi_Y$. Then we have $O_t \cap Y = O'_t \cap Y$ for all $t \in Y$ where $O_t \in M_\tau$ and $O'_t \in M_\pi$. Note that $O_x \subseteq Y$ by the properties deduced in previous paragraph, and hence we obtain $O_x = O'_x$. We also have the equality $O_a \cap Y = O'_a \cap Y$ for $a \in N$, which forces that $O_a \cap C = O'_a \cap C$ is either C or $\varnothing$ by previous paragraph. Since $O_a \cap N = O'_a \cap N$, we get $O_a = O'_a$ for $a \in N$. Then it follows that $O_t = O'_t$ for all $t \in Y$. Since $O_z = zx^{-1}O_x$ for $z \in C$, we obtain that $O_t = O'_t$ for all $t \in X$, which implies the equality $\tau = \pi$. Thus, our map is one to one.

Now, let $\pi \in \mathfrak{T}_0(Y)$ and let M_π be the minimal base of π. Set

$$O_t = \begin{cases} tx^{-1}O'_x & if\ t \in C \\ O'_t & if\ t \in N\ and\ x \notin O'_t \\ C \cup O'_t & if\ t \in N\ and\ x \in O'_t \end{cases}$$

for each $O'_t \in M_\pi$. We need to show that $\mathfrak{B} = \{O_t \mid t \in X\}$ is a minimal base and the topology τ generated by $\mathfrak{B}$ is a T_0 topology. To show that it is a minimal base, we need to show that if $a \in O_t$ then $O_a \subseteq O_t$ for any $a, t \in X$. In fact, we will observe that $O_a \subset O_t$ which shows that τ is a T_0 topology. It can be done case by case but here we present only nontrivial cases. Fix $a \in N$ and let $a \in O_x$. Note that $a \in O'_x$ as $O'_x = O_x$, and hence we obtain $O'_a \subseteq O'_x$. If $x \in O'_a$ then we have $O'_a = O'_x$ which is not possible as π is a T_0 topology. Thus, $x \notin O'_a$ which implies $O_a = O'_a$ by our setting. Then we obtain that $O_a \subset O_x$. Now assume that $x \in O_a$. Then we get that $x \in O'_a$ otherwise $x \notin O_a$ by our setting. Thus, we obtain $O'_x \subset O'_a$ in a similar way. It follows that $O_x \subset O_a$ as $O_x = O'_x$. As a result $\tau \in \operatorname{Fix}(\mathfrak{T}_0(X))$ as $\mathfrak{B}$ is a C-invariant minimal base. Moreover, the equality $\tau_Y = \pi$ holds, which concludes that the mapping $\tau \mapsto \tau_Y$ from $\mathrm{Fix}(\mathfrak{T}_0(X))$ to $\mathfrak{T}_0(Y)$ is a bijection, which completes the proof. $\square$

Corollary 1. *$T_0(p^k) \equiv 1$ (mod p) where k is a natural number and p is a prime number.*

The proof of next theorem is similar to the proof of Theorem 2 in the sense that both use the same technique. For clarity, we repeat some arguments.

Theorem 3. *For each natural number n, there exists a unique integer k such that $T(p+n) \equiv k$ (mod p) for all primes p.*

Proof. If $n = 0$, $T(p) \equiv 2$ (*mod* p) by Theorem 1. Now we can assume that $n > 0$. Let $C = C_p$ be the cyclic group of order p and N be a set with n elements. We define the action of C on X as in the proof the previous theorem. Then the action of C on X can be extended to the action of C on $\mathfrak{T}(X)$. As $|\mathfrak{T}(X)| \equiv |\mathrm{Fix}(\mathfrak{T}(X))|$ (*mod* p), it is left to show that $|\mathrm{Fix}(\mathfrak{T}(X))|$ does not depend on the choice of prime p.

Due to Lemma 1, $|\mathrm{Fix}(\mathfrak{T}(X))|$ is equal to the number of C-invariant minimal bases on X. Let $x, y \in C$. Notice that O_x completely determines O_y as $(yx^{-1})O_x = O_y$. Then $|O_x \cap C| = |O_y \cap C|$ as $yx^{-1}(O_x \cap C) = O_y \cap C$. Now it is easy to see that $O_x \cap C$ and $O_y \cap C$ are either disjoint or equal. As cardinality of C is prime, $O_x \cap C$ is either $\{x\}$ or C. If $a \in N$ then we must have $gO_a = O_a$ for all $g \in C$ as $ga = a$. Hence, $O_a \cap C$ is either $\varnothing$ or C. Thus, the elements of C have no contribution to the number of possible minimal bases. Now we know that such k exists. If k' is also such an integer then $k \equiv k'$ (*mod* p) for all primes p. Hence $k - k'$ is divisible by all primes which forces $k - k' = 0$. $\square$

By the previous theorem, we see that k is uniquely determined by n, which leads the following definition.

Definition 2. We define the integer sequence $k(n)$ for $n \geqslant 0$ to be the unique number such that $k(n) \equiv T(p+n)$ (*mod* p) for all primes p.

The sequence $k(n)$ has appeared in the online encyclopedia of N. J. A. Sloane [2] with reference number (A265042) after this article appeared in arxiv. The proof of the theorem also gives an algorithm to calculate $k(n)$ for a given n.

Corollary 2. *$T(p+1) \equiv 7$ (mod p) for all primes p, that is, $k(1) = 7$.*

Proof. By following the proof the Theorem 3, it can be easily counted that there are exactly seven C-invariant minimal bases. Then the result follows. $\square$

However, it is difficult to count C-invariant minimal bases for larger n to determine $k(n)$. We develop a new method to calculate $k(n)$ for larger n. But the method requires knowing values of $T(s)$ for some s.

Theorem 4. *The sequence $k(n)$ satisfies the following inequality*

$$T(n+1)+T_0(n+1) \leqslant k(n) < 2T(n+1)$$

for $n \geqslant 1$.

Proof. We follow the proof of Theorem 3. We have $k(n) = |\operatorname{Fix}(\mathfrak{T}(X))|$ where $|X| = n+p$ and $|C| = p$. Thus, we need to count C-invariant minimal bases. According to the proof, we have two main cases.

Case 1: $O_x \cap C = C$ for $x \in C$. Then $O_y \cap C = C$ for all $y \in C$ and $O_x = O_y$ for all $x, y \in C$. If $a \in N$ then $O_a \cap C$ is either C or $\varnothing$. Hence we can see the whole C as one element, that is, pass to the quotient $\overline{X} = X/\sim$ where $x \sim y$ if $x, y \in C$ then there is a one to one correspondence between C-invariant topologies and quotient topologies. It follows that we have exactly $T(n+1)$ possible sub-cases.

Case 2: $O_x \cap C = \{x\}$ for $x \in C$. Let $x, y \in C$ then when we define O_x, O_y is completely determined as $O_y = yx^{-1}O_x$. Moreover, we have $O_x \cap N = O_y \cap N$ as C acts trivially on N. We also have $O_a \cap C$ is either C or $\varnothing$. Now set $Y = N \cup \{x\}$ then the mapping $\tau \to \tau_Y$ is one to one from $\operatorname{Fix}(\mathfrak{T}(X))$ to $\mathfrak{T}(Y)$. Hence we can have at most $T(n+1)$ possible sub-cases. But we can not have exactly $T(n+1)$ possible sub-cases as the given map is not onto. To see this, set $O'_x = \{x, a\}$ for $a \in N$ and $O'_a = \{x, a\}$ for a topology $\pi \in \mathfrak{T}(Y)$. Let $\tau \in \operatorname{Fix}(\mathfrak{T}(X))$ such that $\tau_Y = \pi$. Note that $O'_a = O_a \cap Y$ and O_a must be fixed by the action of C. Thus, we have $C \cup \{a\} \subseteq O_a$. Since $O_x = O'_x = \{x, a\}$, we obtain that $O_a \subseteq \{x, a\}$. Then we obtain that $O_a = \{a\}$ as it must be fixed by C, which is a contradiction. Thus, the map is not onto. Then we get that $k(n) < 2T(n+1)$. Moreover, the mapping is onto from $\operatorname{Fix}(\mathfrak{T}_0(X)) \subseteq \operatorname{Fix}(\mathfrak{T}(X))$ to $\mathfrak{T}_0(Y) \subseteq \mathfrak{T}(Y)$.(See the proof of Theorem 2 for details of how the map $\tau \to \tau_Y$ is one to one. It also shows why this map is onto when the target set is $\mathfrak{T}_0(Y)$.) Thus, we have at least $T_0(n+1)$ sub-cases, and hence $T(n+1)+T_0(n+1) \leqslant k(n) < 2T(n+1)$. $\square$

Corollary 3. $\lim_{n\to\infty} \frac{k(n)}{T(n+1)} = 2.$

Proof. We have $1 + \frac{T_0(n+1)}{T(n+1)} \leqslant \frac{k(n)}{T(n+1)} < 2$ by Theorem 4. In [6], it is proved that $lim_{n\to\infty}\frac{T_0(n)}{T(n)} = 1$, and hence the result follows. □

Theorem 5. *The sequence* $k(n) = 2, 7, 51, 634, 12623$ *for* $n = 0, 1, 2, 3, 4$ *respectively.*

Proof. If $n = 0$, the result follows from Theorem 1. Now assume that $n \geqslant 1$. We only show the calculation of $k(2)$ and rest of them follow in a similar way. By previous theorem, we have $T(3) + T_0(3) \leqslant k(2) < 2T(3)$ so $48 \leqslant k(2) < 58$. Clearly, we have

$$\begin{aligned} T(5) &\equiv k(2)\ (mod\ 3) \\ T(7) &\equiv k(2)\ (mod\ 5) \end{aligned}$$

It follows that $k(2) \equiv 6\ (mod\ 15)$ by solving the above congruence relation. We obtain that $k(2) = 51$ as $48 \leqslant k(2) < 58$. For $n = 3, 4$, we have the same procedure. □

We should note that for $n \geqslant 5$, there is no unique solution satisfying the inequality. For example, $k(5) \in \{357593, 387623, 417653\}$. The closed form of $k(n)$ seems to be another open problem. Hence calculation of $k(n)$ for specific n or some better lower and upper bounds can be seen as new problems arising from this article.

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Contact information

M. Y. Kizmaz Department of Mathematics, Middle East Technical University, Ankara 06531, Turkey
E-Mail(s): yasir@metu.edu.tr

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