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Square difference labeling of some union and disjoint union graphs

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Abstract. The paper deals with methods of constructing square difference labeling of caterpillars and graphs derived from two operations: path union of cycles and disjoint union of stars. The existence of the square difference labeling of disjoint union of any SD graph with path is proved.

Introduction

Research studies conducted by Ajitha et. al. in the field of number theory inspired them to create two new types of labelings: square sum labeling and square difference labeling. For the first time, square difference labeling was introduced to scientific world in 2012 [1]. Its authors proved the existence of this labeling for such classes of graphs as paths, stars, cycles, complete graphs, bipartite graphs, friendship graphs, triangular snakes. Tharmaraj and Sarasija identified new classes of graphs that have square difference labeling, such as: fan F_n, graph g_n, middle graphs of path and cycle, total graphs of path and cycle [2,3]. In addition, they proved that the disjoint union of path with such graphs as star, path, cycle, and other constructions of graphs have the square difference labeling. Shiama [4] focused on the particular case of the caterpillar and demonstrated the existence of the square difference labeling for the graph $P_n \odot K_1$. These

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labelings, as well as majority of others, are well presented in the review published by Gallian [5].

In this paper, we prove the existence of the square difference labeling for new types of graphs, which were obtained using disjoint union of stars and path union of cycles. In addition, we show that the caterpillar, and the disjoint union of any SD graph and path are square difference graphs.

1. Definitions

We consider finite undirected graphs without loops and multiple edges. Let $G{=}(V,E)$ be a graph with vertex set $V{=}V(G)$ and edge set $E{=}E(G)$. We assume that $|V(G)| = p$, $|E(G)| = q$.

Definition 1 ([1]). A graph $G = (V(G), E(G))$ with $|V(G)| = p$ is said to be a square difference graph if there exist a bijection $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ such that the induced an injection function $f^* : E(G) \to N$ given by $f^*(uv) = |[f(u)]^2 - [f(v)]^2|$ for every edge $uv \in G$ and the function f is called square difference labeling of the graph G.

Graph, allowing square difference labeling, is called square difference graph or SD graph.

For n copies of the graph $G{=}(V,E)$, we introduce the definition of the path union.

Definition 2 ([6]). Let the graphs $G_1, G_2, \ldots, G_n (n \geqslant 2)$ are n copies of certain graph G. Then the graph obtained via connection of G_i vertex with the corresponding G_{i+1} vertex, for $i = 1, 2, \ldots, n-1$, with edge is called the path graphs union $G_1, G_2, \ldots, G_n$.

Definition 3 ([6]). Disjoint union of graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_m = (V_m, E_m)$ is a graph $G = G_1 \cup G_2 \cup \cdots \cup G_m$ with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_m$ and edge set $E = E_1 \cup E_2 \cup \cdots \cup E_m$, where $V_1 \cap V_2 \cap \cdots \cap V_m = \oslash$.

Definition 4 ([7]). Caterpillar is a tree that after removing its end-vertices (vertices with degree 1), turns into a path. The path is called a tree trunk and end-vertices are leaves of the tree, which grow out of the trunk vertices.

Theorem 1 ([8]). *The square difference of consecutive numbers equals the sum of these numbers.*

Theorem 2 ([8]). *If $n = x^2 - y^2$, then $n \equiv 0, 1, 3 \pmod 4$.*

Let f be square difference labeling for graph G, generating edge labeling f^*, then for each edge uv of the set $E(G)$, the following condition is true: $f^*(u,v) \equiv 0, 1, 3 \pmod 4$. Furthermore, if $f(u) = 0$, then $f^*(u,v) \equiv 0, 1 \pmod 4$.

2. Square difference labeling of caterpillars

Theorem 3. *Any caterpillar is a square difference graph.*

Proof. Let G be the caterpillar with $|V(G)| = kn$. We define $V(P) = \{v_1, v_2, \ldots, v_k\}$ as a set of vertices of the trunk of the graph G and $\{u_{i,1}, u_{i,2}, \ldots, u_{i,n_i}\}$ as a set of leaves, for $i = 1, 2, \ldots, k$.

Let us analyze the case when $i = 1$. We obtain a caterpillar, which is the star K_{1,n_1}. We define the vertex labeling f of star K_{1,n_1} as follows:

$$f(v_1) = 0,$$
$$f(u_{1,j}) = j, \quad \text{for } j = 1, 2, \ldots, n_1.$$

Function f defines bijection from the set of vertices into set $\{0, 1, \ldots, n_1\}$ and induces a different edge labels, according to Definition 1. Edge labels form a set $\{1, 4, 9, \ldots, n_1^2\}$. So labeling f is a square difference labeling K_{1,n_1}.

Further, assume that $i = l$ and $l > 1$. For the vertices of the trunk $v_1, v_2, \ldots, v_l$ we assign labels $0, 1, 2, \ldots, l-1$; and $u_{1,1}, \ldots, u_{1,n_1}, u_{2,1}, \ldots, u_{2,n_2}, u_{i,1}, \ldots, u_{i,n_i}$ form a set $\{l, l+1, \ldots, l+n_1-1, l+n_1, \ldots, l+m-1\}$, for m being the number of leaves.

Trunk edge labels and end-edge labels will form two sets of numbers $\{1, 3, 5, \ldots, 2l-3\}$, and $\{l^2, (l+1)^2, \ldots, (l+m-1)^2 - (l-1)^2\}$. As a result, we assign labels to the vertices, using all the positive integers from 0 to $kn - 1$.

Edge labeling f^*, generated by labeling f, presents an injection of $E(G)$ onto a certain subset of positive integers. According to Definition 1, labeling f of the graph G is a square difference labeling. □

3. Square difference labeling of path and cycle union

Theorem 4. *An arbitrary path union of n copies of the cycle C_3 is a square difference graph for any positive integer n.*

Proof. Let graph G be formed by the path union of n copies of the cycle C_3 as shown in Figure 1.

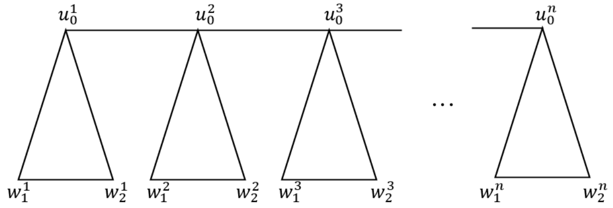

FIGURE 1. A square difference labeling the path union $3C_3$.

Let $u_0^1, u_0^2, \ldots, u_0^n$ be isomorphic images of vertex u of cycle C_3 selected randomly. $V(G)=\{u_0^i, w_1^i, w_2^i\}$ — a set of vertices of graph G, for $i = 1, 2, \ldots, n$. We define the vertex labeling f of graph G with graph order $|V(G)| = 3n$ as follows:

$$f(u_0^i) = i - 1,$$
$$f(w_1^i) = n + 2i - 2,$$
$$f(w_2^i) = n + 2i - 1,$$

for $i = 1, 2, \ldots, n$. Therefore, function f is bijection of vertex set of graph G onto set of numbers $\{0, 1, 2, \ldots, 3n - 1\}$. Then edge labeling f^* generated by the function f, as defined by the Definition 1, is:

$$f^*(u_0^i, u_0^{i+1}) = |[f(u_0^i)]^2 - [f(u_0^{i+1})]^2|, \tag{1}$$

for $i = 1, 2, \ldots, n - 1$;

$$f^*(w_1^i, w_2^i) = |[f(w_1^i)]^2 - [f(w_2^i)]^2|, \tag{2}$$
$$f^*(w_1^i, u_0^i) = |[f(w_1^i)]^2 - [f(u_0^i)]^2|, \tag{3}$$
$$f^*(w_2^i, u_0^i) = |[f(w_2^i)]^2 - [f(u_0^i)]^2|, \tag{4}$$

for $i = 1, 2, \ldots, n$.

We apply Theorem 5 to formulas (1) and (2), and obtain

$$f^*(u_0^i, u_0^{i+1}) = f(u_0^i) + f(u_0^{i+1}), \tag{5}$$

$$f^*(w_1^i, w_2^i) = f(w_1^i) + f(w_2^i). \tag{6}$$

Edge labels calculated by formulas (5) and (6) form sets of numbers $S_1 = \{1, 3, 5, \ldots, 2n - 3\}$, $S_2 = \{2n + 1, 2n + 5, 2n + 9, \ldots, 6n - 3\}$. Meanwhile, edge labels obtained by formulas (3) and (4) form set $S_3 = \{n^2, (n + 1)^2, \ldots, (2n - 1)(4n - 3), 2n(4n - 3)\}$.

Let us consider minimum number of n^2 in set S_3, and compare it with the maximum number of $6n-3$ from set $S_1 \cup S_2$. If $n \geqslant 6$, then $n^2 > 6n-3$ and elements of set $S = S_1 \cup S_2 \cup S_3 \subset N$ are different, where N is the set of positive integers. They form a set $S = S_1 \cup S_2 \cup S_3 \subset N$. Thus, for $n \geqslant 6$ function f^* is an injection of $E(G)$ into the S set. Labeling f, according to Definition 1, is a square difference labeling, while graph G is a SD graph.

For $n = 1, 2, 3, 4, 5$ square difference labeling of an arbitrary path union of n copies of the cycle C_3 is shown in Figure 2, 3. □

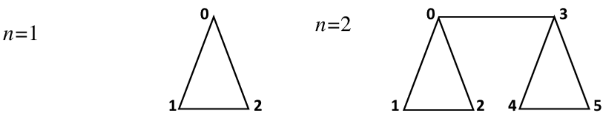

FIGURE 2. Square difference labeling the path union nC_3, for $n = 1, 2$.

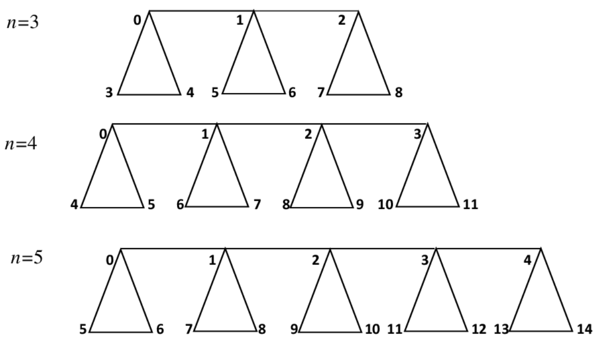

FIGURE 3. Square difference labeling the path union nC_3, for $n = 3, 4, 5$.

4. Square difference labeling of the disjoint union of graphs

Theorem 5. *The disjoint union of graphs $G_1, G_2, \ldots, G_m$ of the family of graphs $G_i = K^i_{1,n_i}$, for $i = 1, 2, \ldots, m$ is a square difference graph for any positive integers m and n_i.*

Proof. The graph G is obtained by disjoint union of stars K_{1,n_i}, for $i = 1, 2, \ldots, m$. Let $V(K_{1,n_i}) = \{u^i_0, u^i_1, u^i_2, \ldots, u^i_{n_i}\}$ be the set of vertices of the graph G, for n_i can be any value starting from 1. We assume that u_0^i is a central vertex of star K_{1,n_i}. We define the vertex labeling f of the graph G with graph order $|V(G)| = (n_1 + n_2 + \cdots + n_m) + m$ as following:

$$f(u_0^i) = i - 1, \tag{7}$$

$$f(u_j^i) = \begin{cases} m - 1 + j & \text{if } i = 1, \\ \Sigma_{l=1}^{i-1} n_l + m - 1 + j & \text{otherwise}, \end{cases} \tag{8}$$

for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n_i$. Function f defines bijection from the set of vertices into a set of numbers $\{0, 1, 2, \ldots, (n_1 + n_2 + \cdots + n_m) + m - 1\}$. At the same time it induces the following edge labels for the first stars K_{1,n_1}:

$$\begin{aligned} f^*(u_1^1, u_0^1) &= |[f(u_1^1)]^2 - [f(u_0^1)]^2| = |m^2 - 0| = m^2, \\ f^*(u_2^1, u_0^1) &= |[f(u_2^1)]^2 - [f(u_0^1)]^2| = |(m+1)^2 - 0| = (m+1)^2, \\ f^*(u_3^1, u_0^1) &= |[f(u_3^1)]^2 - [f(u_0^1)]^2| = |(m+2)^2 - 0| = (m+2)^2, \\ &\cdots \\ f^*(u_{n_1}^1, u_0^1) &= |[f(u_{n_1}^1)]^2 - [f(u_0^1)]^2| = |(m+n_1-1)^2 - 0| = (m+n_1-1)^2. \end{aligned}$$

The calculated edge labels form a set of different numbers $S_1 = \{m^2, (m+1)^2, \ldots, (m + n_1 - 1)^2\}$. Also, the function f generates edge labels for the second star K_{1,n_2}:

$$\begin{aligned} f^*(u_1^2, u_0^2) &= |[f(u_1^2)]^2 - [f(u_0^2)]^2| = |(m + n_1)^2 - 1| \\ &= |1 + 2(m + n_1 - 1) + (m + n_1 - 1)^2 - 1| \\ &= 2(m + n_1 - 1) + (m + n_1 - 1)^2, \\ f^*(u_2^2, u_0^2) &= |[f(u_2^2)]^2 - [f(u_0^2)]^2| = |(m + n_1 + 1)^2 - 1| \\ &= |1 + 2(m + n_1) + (m + n_1)^2 - 1| \\ &= 2(m + n_1) + (m + n_1)^2, \\ &\cdots \\ f^*(u_{n_2}^2, u_0^2) &= |[f(u_{n_2}^2)]^2 - [f(u_0^2)]^2| = |(m + n_1 + n_2 - 1)^2 - 1| \\ &= |1 + 2(m + n_1 + n_2 - 2) + (m + n_1 + n_2 - 2)^2 - 1| \\ &= 2(m + n_1 + n_2 - 2) + (m + n_1 + n_2 - 2)^2. \end{aligned}$$

Thus, we get a set of numbers $S_2 = \{2(m + n_1 - 1) + (m + n_1 - 1)^2, 2(m + n_1) + (m + n_1)^2, \ldots, 2(m + n_1 + n_2 - 2) + (m + n_1 + n_2 - 2)^2\}$.

Similar actions are performed for all the stars in the disjoint union. Thus, the edge labels for the latest star K_{1,n_m}, induced by function f, form a set of numbers $S_m = \{2(n_1 + n_2 + \cdots + n_{m-1} + 1)(m-1) + (n_1 + n_2 + \cdots + n_{m-1} + 1)^2, \ldots, 2(n_1 + n_2 + \cdots + n_m)(m-1) + (n_1 + n_2 + \cdots + n_m)^2\}$.

Consequently, set $S = S_1 \cup S_2 \cup \cdots \cup S_m \subset N$, consisting of different numbers is formed. Therefore, function f^*, is the injection of $E(G)$ into the set of positive integers. Labeling f for the graph G, according to Definition 1, is square difference labeling. $\square$

In the case of graph $mK_{1,n}$, formulas (7) and (8) can be written as following:

$$f(u_0^i) = i - 1, \tag{9}$$

$$f(u_j^i) = n(i-1) + m - 1 + j, \tag{10}$$

for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Example 1. We apply these formulas (9) and (10) to the graph $3K_{1,4}$, and get square difference labeling of the graph $3K_{1,4}$ as shown in Figure 4.

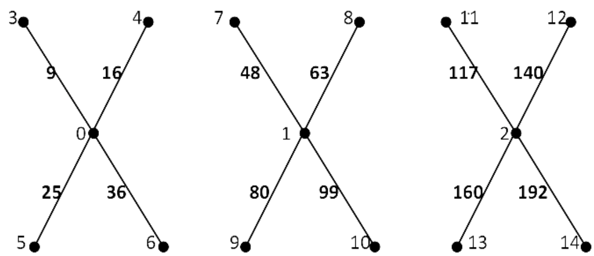

FIGURE 4. Square difference labeling the graph $3K_{1,4}$.

In the following theorem, we summarize some results obtained in the articles [2,3] concerning the square difference labeling of the disjoint union of paths or cycles, or other types of SD graphs with path.

Theorem 6. *The disjoint union of any* SD *graph G with path P_k, is the square difference graph for any k.*

Proof. Let us consider the disjoint union of G and path P_k. Let $V(G) = \{w_1, w_2, \ldots, w_l\}$ be the set of vertices of graph G. We denote $V(P_k) = \{v_1, v_2, \ldots, v_k\}$ the set of vertices of path P_k. Let f be a square difference labeling of G. We define the vertex labeling f_1 of graph $G \cup P_k$, as follows:

$$f_1(v_i) = i - 1 \quad \text{for } i = 1, 2, \ldots, k,$$
$$f_1(w_j) = f(w_j) + k \quad \text{for } j = 1, 2, \ldots, l.$$

Function f_1 defines bijection from $V(G \cup P_k)$ onto the set $\{0, 1, 2, \ldots, k+l-1\}$. Edge labels of path image P_k in $G \cup P_k$, induced by labeling f_1, form a set of numbers $\{1, 3, 5, \ldots, 2k-3\}$.

Let w_n, w_m be any adjacent vertices of G with labels $f(w_n) = n$, $f(w_m) = m$, for $m, n \in \{0, 1, \ldots, l-1\}, n \neq m$ and f^* be edge labeling of G, induced by labeling f. Also, we assume $n > m$. Then

$$f^*(w_n, w_m) = |[f(w_n)]^2 - [f(w_m)]^2 = |n^2 - m^2| = n^2 - m^2.$$

Let us consider edge labeling f_1^* of the graph $G \cup P_k$, induced by labeling f_1 and find an edge label $w_n w_m$:

$$\begin{aligned} f_1^*(w_n, w_m) &= |[f_1^*(w_n)]^2 - [f_1^*(w_m)]^2 \\ &= |(n+k)^2 - (m+k)^2| \\ &= |n^2 + 2nk + k - m^2 - 2mk - k^2| \\ &= n^2 - m^2 + 2k(n-m) \\ &= f^*(w_n, w_m) + 2k(n-m). \end{aligned} \tag{11}$$

Since $2k(n-m) > 2k-3$ for $\forall m, n \in \{0, 1, \ldots, l-1\}$, then $f_1^*(w_n, w_m) > 2k-3$. Consequently, the edge labeling of graph $G \cup P_k$ is different. So f_1^* is an injection. According to Definition 1, f_1 is square difference labeling of graph G. □

Additionally, using analytical description of labeling obtained in Theorems 4 and 5, we developed algorithms for square difference labeling of certain graph examples for path union of cycles and disjoint union of stars. Results verify our theoretical findings.

5. Conclusion

In this paper the class of square difference graphs is expanded. The existence of the square difference labeling of the disjoin union of any SD graph with path is proved. The methods developed for constructing a square difference labeling for caterpillars and graphs derived from two operations: a path union of cycles and disjoint union of stars, may be used in further theoretical studies.

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