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RESEARCH ARTICLE

# On certain homological invariant and its relation with Poincaré duality pairs*

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Abstract. Let $G$ be a group, $\mathcal{S} = \{S_i, i \in I\}$ a non empty family of (not necessarily distinct) subgroups of infinite index in $G$ and $M$ a $\mathbb{Z}_2G$-module. In [4] the authors defined a homological invariant $E_*(G, \mathcal{S}, M)$, which is "dual" to the cohomological invariant $E(G, \mathcal{S}, M)$, defined in [1]. In this paper we present a more general treatment of the invariant $E_*(G, \mathcal{S}, M)$ obtaining results and properties, under a homological point of view, which are dual to those obtained by Andrade and Fanti with the invariant $E(G, \mathcal{S}, M)$. We analyze, through the invariant $E_*(G, S, M)$, properties about groups that satisfy certain finiteness conditions such as Poincaré duality for groups and pairs.

## Introduction

Based in the theory of cohomology of groups, Andrade and Fanti in [1], defined a cohomological invariant denoted by $E(G, \mathcal{S}, M)$, where $G$ is a group and $\mathcal{S} = \{S_i, i \in I\}$ is a family of subgroups of $G$ with $[G : S_i] = \infty$, $\forall i \in I$. Through this invariant they proved results in duality for groups and pairs and splitting of groups (see Andrade and

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Fanti in [1], [3] and Andrade et al in [2]). In [4], Andrade and Gazon defined a homological invariant, denoted by $E_*(G, \mathcal{S}, M)$, which is "dual" to the cohomological invariant defined in [1], and they obtained results when $M = \mathbb{Z}_2$. In this work we study this invariant under a more general point of view, obtaining results about groups that satisfy certain finiteness conditions, such as duality conditions for groups and group pairs .

The results presented in this paper provide an alternative way of obtaining applications and properties in the duality theory of groups and pairs of groups, working with greater emphasis in the homology of groups, instead of cohomology.

We assume that the reader is familiar with the theory of absolute and relative (co)homology of groups. For details see [6] and [7]. We recall here some definitions and results which will be useful in this paper.

Let $G$ be a group, $T$ a subgroup of $G$ and $M$ a $\mathbb{Z}_2T$-module. Consider the $\mathbb{Z}_2G$-modules $\operatorname{Ind}_T^G M = \mathbb{Z}_2G \otimes_{\mathbb{Z}_2T} M$ and $\operatorname{Coind}_T^G M = \operatorname{Hom}_{\mathbb{Z}_2T}(\mathbb{Z}_2G, M)$ ([7], III.5). We have the following result.

**Proposition 1** ([7],III.5)**.** *Let $G$ be a group, $T$ a subgroup of $G$ and $M$ a $\mathbb{Z}_2G$-module. Consider the additive group $\operatorname{Hom}(\mathbb{Z}_2(G/T), M)$ with the diagonal $G$-action given by $(g.f)(\overline{\alpha}) = g.f(g^{-1}\overline{\alpha})$, for all $g \in G$ and $\overline{\alpha} \in \mathbb{Z}_2(G/T)$. Then we have the $\mathbb{Z}_2G$-isomorphism*

$$\operatorname{Coind}_T^G M \simeq \operatorname{Hom}(\mathbb{Z}_2(G/T), M). \quad \square$$

**Definition 1.** Let $G$ be a group and $M$ a $\mathbb{Z}_2G$-module. The group of coinvariants of $M$, denoted by $M_G$, is defined by $M_G = M/A$, where $A = \langle g \cdot m - m;\ g \in G \text{ and } m \in M \rangle$ is an additive subgroup of $M$.

**Remark 1.** (1) Let $G$ be a group and $M$ a $\mathbb{Z}_2G$-module. We have $H_0(G; M) = M_G$ ([7], III.1). Hence, if $M$ is a $\mathbb{Z}_2G$-module with trivial $G$-action then $H_0(G; M) = M$.

(2) If $G$ is a finitely generated group, $S$ is a subgroup of $G$ with $[G : S] = \infty$ and $M$ is a $\mathbb{Z}_2S$-module, then $(\operatorname{Coind}_S^G M)_G = 0$ ([7], III.5).

**Proposition 2** (Proposition 1.1, [6])**.** *Let $G$ be a group, $\mathcal{S} = \{S_i, i \in I\}$ a family of subgroups of $G$ and $M$ a $\mathbb{Z}_2G$-module. Denote $\bigoplus_{i \in I} H_*(S_i; M)$ by $H_*(\mathcal{S}; M)$. We have the following long exact sequence:*

$$\cdots \to H_1(\mathcal{S}; M) \xrightarrow{\operatorname{cor}_{\mathcal{S}}^G} H_1(G; M) \xrightarrow{J} H_1(G, \mathcal{S}; M) \xrightarrow{\delta}$$
$$\to H_0(\mathcal{S}; M) \to H_0(G; M) \to 0,$$

*which is natural in the module $M$ and the group pair $(G, \mathcal{S})$.* $\square$

**Remark 2.** Let $(G,\mathcal{S})$ a group pair with $\mathcal{S}=\{S_i, i\in I\}$ and $M$ a $\mathbb{Z}_2G$-module. For each $i\in I$, the corestriction map

$$\mathrm{cor}_{S_i}^G : H_1(S_i;M)\longrightarrow H_1(G;M)$$

is induced in homology by the inclusion $S_i \hookrightarrow G$. The map

$$\mathrm{cor}_{\mathcal{S}}^G : \bigoplus_{i\in I} H_1(S_i;M)\longrightarrow H_1(G;M),$$

which appears in the long exact sequence of the Proposition 2, is defined by:

$$\mathrm{cor}_{\mathcal{S}}^G((\alpha_i)_{i\in I})=\sum_{i\in I}\mathrm{cor}_{S_i}^G(\alpha_i),$$

where $(\alpha_i)_{i\in I}\in\bigoplus_{i\in I}H_1(S_i;M)$, with $\alpha_i=0$ for almost all $i$, that is, except possible finitely many $i$.

Now, we present the definition of $E_*(G,\mathcal{S},M)$.

**Definition 2.** Let $(G,\mathcal{S})$ be a group pair, $\mathcal{S}=\{S_i, i\in I\}$ a family of subgroups of $G$ with $[G:S_i]=\infty$, for all $i\in I$, and $M$ a $\mathbb{Z}_2G$-module. We define:

$$E_*(G,\mathcal{S},M)=1+\dim\mathrm{coker}(\mathrm{cor}_{\mathcal{S}}^G),$$

with $\mathrm{coker}(\mathrm{cor}_{\mathcal{S}}^G)=H_1(G;M)/\,\mathrm{Im}(\mathrm{cor}_{\mathcal{S}}^G)$.

Consider now the category $\mathcal{C}$ whose objects are pairs $((G,\mathcal{S});M)$ where $G$ is a group, $\mathcal{S}=\{S_i,\ i\in I\}$ is a family of subgroups of $G$ and $M$ is a $\mathbb{Z}_2G$-module, and whose morphisms are maps:

$$\psi:((G,\mathcal{S});M)\longrightarrow((L,\mathcal{R}=\{R_j, j\in J\});N)$$

consisting of

(a) a homomorphism of groups $\alpha: G\longrightarrow L$;
(b) a map $\pi: I\longrightarrow J$ such that $\alpha(S_i)\subset R_{\pi(i)}$;
(c) a map $\phi: M\longrightarrow N$ such that $\phi(g\cdot m)=\alpha(g)\cdot\phi(m)$, i.e., $\phi$ is a $\mathbb{Z}_2G$-homomorphism via $\alpha: G\longrightarrow L$.

A morphism $\psi$ is an isomorphism in $\mathcal{C}$ when $\alpha$ is an isomorphism of groups, $\pi$ is a bijection and $\phi$ is a $\mathbb{Z}_2G$-isomorphism.

**Theorem 1.** *(Theorem 1, [4]) If in the category $\mathcal{C}$, $((G,\mathcal{S});M)$ and $((L,\mathcal{R});N)$ are isomorphic then*

$$E_*(G,\mathcal{S},M)=E_*(L,\mathcal{R},N),$$

*Hence $E_*(G,\mathcal{S},M)$ is an invariant in $\mathcal{C}$.* □

In the following we present a characterization of the invariant $E_*(G, \mathcal{S}, M)$ in terms of "partial Euler characteristic".

**Proposition 3** (Proposition 1, [4])**.** *Let $(G, \mathcal{S})$ be a group pair, with $[G : S] = \infty$, $\forall S \in \mathcal{S}$, and $M$ a $\mathbb{Z}_2G$-module. If the homology groups $H_0(G; M)$, $H_0(\mathcal{S}; M) = \bigoplus_{S \in \mathcal{S}} H_0(S; M)$ and $H_1(G, \mathcal{S}; M)$ have finite dimension as $\mathbb{Z}_2$-vector spaces then:*

$$E_*(G, \mathcal{S}, M) = 1 + \dim H_0(G; M) - \dim H_0(\mathcal{S}; M) + \dim H_1(G, \mathcal{S}; M).$$

We introduce now some notations which will be used in this paper. For the sake of simplicity, we denote the $\mathbb{Z}_2G$-module $\operatorname{Coind}_S^G \mathbb{Z}_2 = \operatorname{Hom}_{\mathbb{Z}_2 S}(\mathbb{Z}_2 G, \mathbb{Z}_2) \simeq \operatorname{Hom}(\mathbb{Z}_2(G/S), \mathbb{Z}_2)$ by $\overline{\mathbb{Z}_2(G/S)}$ and, for a family $\mathcal{S} = \{S_i,\ i \in I\}$ of subgroups of $G$, we denote $\bigoplus_{i \in I} \operatorname{Coind}_{S_i}^G \mathbb{Z}_2$ by $\overline{\mathbb{Z}_2(G/\mathcal{S})}$. If $\mathcal{S} = \{S\}$, we denote $E_*(G, \mathcal{S}, M)$ by $E_*(G, S, M)$. The invariant $E_*(G, \mathcal{S}, M)$, when $M$ is the particular module $\overline{\mathbb{Z}_2(G/\mathcal{S})}$, will be denoted by $E_*(G, \mathcal{S})$. In the particular case in which $\mathcal{S} = \{S\}$, we denote $E_*(G, S, \overline{\mathbb{Z}_2(G/S)})$ by $E_*(G, S)$.

## 1. Some properties of the invariant $E_*(G, \mathcal{S}, M)$

In this section we present some general properties of the invariant $E_*(G, \mathcal{S}, M)$. We also present some properties for the particular invariant $E_*(G, S)$. We begin with some remarks.

**Remark 3.** Let $(G, \mathcal{S})$ be a group pair with $\mathcal{S} = \{S_i,\ i \in I\}$ and $[G : S_i] = \infty$, for all $i \in I$, and let $M$ be a $\mathbb{Z}_2G$-module. It is easy to see that:

(1) If $\mathcal{S}' = \{S_{i_k},\ i_k \in I' \subseteq I\}$ is a subfamily of $\mathcal{S}$, then $E_*(G, \mathcal{S}, M) \leqslant E_*(G, \mathcal{S}', M)$.

(2) If $\mathcal{S}' = \{H_i,\ i \in I\}$ is a family of subgroups of $G$ such that $H_i \leqslant S_i$, for all $i \in I$, then $E_*(G, \mathcal{S}, M) \leqslant E_*(G, \mathcal{S}', M)$.

In particular, if $H, S$ and $T$ are subgroups of $G$ satisfying $H \leqslant S \leqslant T \leqslant G$ with $[G : T] = \infty$ then

$$E_*(G, T, \overline{\mathbb{Z}_2(G/S)}) \leqslant E_*(G, S) \leqslant E_*(G, H, \overline{\mathbb{Z}_2(G/S)}).$$

**Remark 4.** Let $G$ be a group, $T$ a subgroup of $G$ and $M$ a $\mathbb{Z}_2T$-module. We have the following maps:

(a) a canonical $\mathbb{Z}_2G$-monomorphism $\varphi : \operatorname{Ind}_T^G M \longrightarrow \operatorname{Coind}_T^G M$ defined by

$$\varphi(g_0 \otimes m)(g) = \begin{cases} g g_0 m & \text{if } g g_0 \in T \\ 0 & \text{otherwise} \end{cases},$$

which is an isomorphism if $[G:T] < \infty$ ([7], III.5.9);

(b) a canonical $\mathbb{Z}_2T$-monomorphism $i : M \longrightarrow \operatorname{Ind}_T^G M$, defined by $i(m) = 1 \otimes m$, for all $m \in M$, ([7], p.67) and can be easily seen that the isomorphism of Shapiro's lemma $H_*(T; M) \xrightarrow{\simeq} H_*(G; \operatorname{Ind}_T^G M)$ is induced by $(\alpha, i)$, where $\alpha : T \hookrightarrow G$ is the inclusion map;

(c) a canonical $\mathbb{Z}_2G$-isomorphism $\psi : \operatorname{Coind}_T^G \operatorname{Coind}_S^T M \longrightarrow \operatorname{Coind}_S^G M$, where $S \leqslant T \leqslant G$ ([7], p.64 (3.6));

(d) a $\mathbb{Z}_2T$-homomorphism $\chi : \operatorname{Coind}_S^T M \longrightarrow \operatorname{Coind}_S^G M$, where $S \leqslant T \leqslant G$, given by the composition

$$\operatorname{Coind}_S^T M \xrightarrow{i} \operatorname{Ind}_T^G \operatorname{Coind}_S^T M \xrightarrow{\varphi} \operatorname{Coind}_T^G \operatorname{Coind}_S^T M \xrightarrow{\psi} \operatorname{Coind}_S^G M,$$

i.e., $\chi = \psi \circ \varphi \circ i$.

**Theorem 2.** *Let $S$ and $T$ be subgroups of $G$ with $S \leqslant T \leqslant G$ and $M$ a $\mathbb{Z}_2S$-module. If $[T:S] = \infty$ and $\varphi_* : H_1(G; \operatorname{Ind}_T^G(\operatorname{Coind}_S^T M)) \to H_1(G; \operatorname{Coind}_T^G(\operatorname{Coind}_S^T M))$ is an epimorphism, where $\varphi_*$ is the induced map of the embedding $\varphi : \operatorname{Ind}_T^G(\operatorname{Coind}_S^T M) \to \operatorname{Coind}_T^G(\operatorname{Coind}_S^T M)$, then*

$$E_*(T, S, \operatorname{Coind}_S^T M) \leqslant E_*(G, S, \operatorname{Coind}_S^G M).$$

*Proof.* By considering the maps from Remark 4, we have the following commutative diagram

$$\begin{tikzcd}
H_1(S; \operatorname{Coind}_S^T M) \arrow[r, "\operatorname{cor}_S^T"] \arrow[d, "(\mathrm{id}_S, \chi)_*"] & H_1(T; \operatorname{Coind}_S^T M) \arrow[d, "(\alpha, \chi)_*"'] \\
H_1(S; \operatorname{Coind}_S^G M) \arrow[r, "\operatorname{cor}_S^G"'] & H_1(G; \operatorname{Coind}_S^G M)
\end{tikzcd}$$

where the induced map $(\alpha, \chi)_* : H_1(T; \operatorname{Coind}_S^T M) \longrightarrow H_1(G; \operatorname{Coind}_S^G M)$ is given by $(\alpha, \chi)_* = \psi_* \circ \varphi_* \circ (\alpha, i)_*$ with $\varphi_* \equiv (\mathrm{id}, \varphi)_*$ and $\psi_* \equiv (\mathrm{id}, \psi)_*$. Since $(\alpha, i)_*$ and $\psi_*$ are isomorphisms and $\varphi_*$ is an epimorphism by hypothesis, it follows that $(\alpha, \chi)_*$ is an epimorphism. Furthermore, $(\alpha, \chi)_*(\operatorname{Im}(\operatorname{cor}_S^T)) \subset \operatorname{Im}(\operatorname{cor}_S^G)$. In fact, by the commutative diagram, $(\alpha, \chi)_* \circ \operatorname{cor}_S^T = \operatorname{cor}_S^G \circ (\mathrm{id}, \chi)_*$. Hence,

$$\begin{aligned}
\forall y \in \operatorname{Im}(\operatorname{cor}_S^T) &\Rightarrow y = \operatorname{cor}_S^T(x), \text{ for some } x \in H_1(S; \operatorname{Coind}_S^T M) \\
&\Rightarrow (\alpha, \chi)_*(y) = (\alpha, \chi)_*(\operatorname{cor}_S^T(x)) = \operatorname{cor}_S^G \circ (\mathrm{id}, \chi)_*(x) \in \operatorname{Im}(\operatorname{cor}_S^G).
\end{aligned}$$

Then we have a well-defined map

$$\overline{(\alpha,\chi)_*}: \frac{H_1(T;\operatorname{Coind}_S^T M)}{\operatorname{Im}(\operatorname{cor}_S^T)} \longrightarrow \frac{H_1(G;\operatorname{Coind}_S^G M)}{\operatorname{Im}(\operatorname{cor}_S^G)}$$

given by $\overline{(\alpha,\chi)_*}(a+\operatorname{Im}(\operatorname{cor}_S^T)) = (\alpha,\chi)_*(a)+\operatorname{Im}(\operatorname{cor}_S^G)$. Hence, we have $\dim\operatorname{coker}(\operatorname{cor}_S^G) \geqslant \dim\operatorname{coker}(\operatorname{cor}_S^T)$ and therefore $E_*(T,S,\operatorname{Coind}_S^T M) \leqslant E_*(G,S,\operatorname{Coind}_S^G M)$. $\square$

The next result provides a relation between $E_*(T,S)$ and $E_*(G,S)$, with $S \leqslant T \leqslant G$.

**Corollary 1.** *Let $S$, $T$ be subgroups of $G$, satisfying $S \leqslant T \leqslant G$, and $M = \mathbb{Z}_2$ the trivial $\mathbb{Z}_2G$-module. If $[G:S]=\infty$ and $[G:T]<\infty$, then $E_*(T,S) \leqslant E_*(G,S)$.* $\square$

## 2. $E_*(G,\mathcal{S},M)$ and duality

In this section, through the invariant $E_*(G,\mathcal{S},M)$ we prove some results about groups and group pairs satisfying duality conditions.

Before proving the main results, we recall some definitions about duality due to Bieri and Eckmann (for details see [5], [6] and [7]).

**Definition 3.** A group $G$ is called a duality group of dimension $n$, or simply a $D^n$-group, if there exist a $\mathbb{Z}_2G$-module $C$, called the dualizing module of $G$, and natural isomorphisms

$$H^k(G;M) \simeq H_{n-k}(G;C\otimes M)$$

for all integers $k$ and all $\mathbb{Z}_2G$-modules $M$. In the special case where $C=\mathbb{Z}_2$, we say that $G$ is a Poincaré duality group of dimension $n$, or simply a PD$^n$-group.

**Definition 4.** A duality pair of dimension $n$, or simply a $D^n$-pair, consists of a group pair $(G,\mathcal{S})$, where $\mathcal{S}=\{S_i, i\in I\}$ is a finite family of $D^{n-1}$-subgroups of $G$, a $\mathbb{Z}_2G$-module $C$ and natural isomorphisms

$$H^k(G;M) \simeq H_{n-k}(G,\mathcal{S};C\otimes M),$$
$$H^k(G,\mathcal{S};M) \simeq H_{n-k}(G;C\otimes M),$$

for all $\mathbb{Z}_2G$-modules $M$ and all $k\in\mathbb{Z}$. $C$ is called the dualizing module of the $D^n$-pair $(G,\mathcal{S})$. If $C=\mathbb{Z}_2$ the duality pair $(G,\mathcal{S})$ is called a Poincaré duality pair, or simply a PD$^n$-pair.

**Remark 5** ([5], [6]). (1) If $G$ is a $D^n$-group, then $G$ is finitely generated and its cohomological dimension, $\mathrm{cd}(G)$, is $n$.

(2) If $G$ is a $D^n$-group and $S$ is subgroup of $G$, with $[G:S]<\infty$, then $S$ is a $D^n$-group (with the same dualizing module).

(3) If $(G,\mathcal{S})$ is a $\mathrm{PD}^n$-pair, then $G$ is a $D^n$-group and each $S\in\mathcal{S}$ is a $\mathrm{PD}^{n-1}$-group.

**Lemma 1.** *Let $(G,\mathcal{S})$ be a $\mathrm{PD}^n$-pair, with $\mathcal{S}=\{S_i,\ i=1,\dots,r\}$. Then*

(i) $H_1(G,\mathcal{S};\bigoplus_{i=1}^r \mathrm{Coind}_{S_i}^G M_i)=\bigoplus_{i=1}^r (M_i)_{S_i}$, *where $M_i$ is a $\mathbb{Z}_2S_i$-module, for all $i\in I$. In particular, if $M_i=\mathbb{Z}_2$ is the trivial $\mathbb{Z}_2S_i$-module for $i=1,\dots,r$, then $H_1(G,\mathcal{S};\overline{\mathbb{Z}_2(G/\mathcal{S})})=\bigoplus_{i=1}^r\mathbb{Z}_2$.*

(ii) $H_1(G,\mathcal{S};\mathrm{Coind}_S^G M)=M_S$ *where $S$ is a $\mathrm{PD}^{n-1}$-subgroup of $G$ (which does not necessarily belong to the family $\mathcal{S}$) and $M$ is a $\mathbb{Z}_2S$-module.*

*Proof.* (i) By using Definitions 3 and 4, Remark 1 and Shapiro's Lemma, for $(G,\mathcal{S})$ a $\mathrm{PD}^n$-pair, we have:

$$\begin{aligned} H_1(G,\mathcal{S};\bigoplus_{i=1}^r \mathrm{Coind}_{S_i}^G M_i) &= H^{n-1}(G;\bigoplus_{i=1}^r \mathrm{Coind}_{S_i}^G M_i)\\ &=\bigoplus_{i=1}^r H^{n-1}(G;\mathrm{Coind}_{S_i}^G M_i)=\bigoplus_{i=1}^r H^{n-1}(S_i;M_i)\\ &=\bigoplus_{i=1}^r H_0(S_i;M_i)=\bigoplus_{i=1}^r (M_i)_{S_i}. \end{aligned}$$

(ii) It is similar to (i). □

**Lemma 2.** *If $G$ is a group and $S$ is a subgroup of $G$, then $(\overline{\mathbb{Z}_2(G/S)})_S\neq 0$. More specifically, there exists $\overline{f}\in(\overline{\mathbb{Z}_2(G/S)})_S$ such that $\langle\overline{f}\rangle\simeq\mathbb{Z}_2$.*

*Proof.* Consider $\overline{\mathbb{Z}_2(G/S)}\simeq\mathrm{Hom}(\mathbb{Z}_2(G/S),\mathbb{Z}_2)$ with the diagonal $G$-action (see Proposition 1). Since $\mathbb{Z}_2$ is a trivial $\mathbb{Z}_2G$-module, it follows that $g\cdot f(g^{-1}\overline{\alpha})=f(g^{-1}\overline{\alpha})$, for all $g\in G$ and $\overline{\alpha}\in\mathbb{Z}_2(G/S)$. In particular, for $\overline{\alpha}=\overline{1}=1\cdot S$ and $s\in S$ we have

$$(s\cdot f)(\overline{1})=f(s^{-1}\cdot\overline{1})=f(\overline{s^{-1}})=f(\overline{1}).$$

Hence,

$$s\cdot f(\overline{1})-f(\overline{1})=0,\qquad\forall f\in\overline{\mathbb{Z}_2(G/S)},\quad\forall s\in S.\tag{$*$}$$

Consider now the augmentation map $\varepsilon:\mathbb{Z}_2(G/S)\to\mathbb{Z}_2$. We will show that $\varepsilon$ provides a non-null element $\overline{\varepsilon}$ in $\overline{\mathbb{Z}_2(G/S)}_S=\mathrm{Hom}(\mathbb{Z}_2(G/S),\mathbb{Z}_2)/A$, where $A = \langle sf - f \mid f \in \overline{\mathbb{Z}_2(G/S)},\ s \in S\rangle$. For this, suppose that $\overline{\varepsilon} = \overline{0}$ in $\overline{\mathbb{Z}_2(G/S)}_S$. Thus, $\varepsilon \in A$ and there exist $s_1, s_2, \ldots, s_k \in S$ and $f_1, f_2, \ldots, f_k \in \overline{\mathbb{Z}_2(G/S)}$ such that

$$\varepsilon = (s_1 f_1 - f_1) + (s_2 f_2 - f_2) + \ldots + (s_k f_k - f_k).$$

Now, for $\overline{1} \in \mathbb{Z}_2(G/S)$, one has

$$1 = \varepsilon(\overline{1}) = (s_1 f_1 - f_1) + (s_2 f_2 - f_2) + \ldots + (s_k f_k - f_k)(\overline{1}) \overset{(*)}{=} 0,$$

which gives us a contradiction. Hence, there exists $\overline{f} = \overline{\varepsilon} \neq 0$ in $\mathbb{Z}_2(G/S))_S$ and $\langle \overline{f}\rangle \simeq \mathbb{Z}_2 \subset \overline{\mathbb{Z}_2(G/\mathcal{S})}_S$. $\square$

The next result provides a necessary condition for a group pair $(G, \mathcal{S})$ to be a Poincaré duality pair (PD$^n$-pair).

**Theorem 3.** *Let $(G, \mathcal{S})$ be a group pair with $\mathcal{S} = \{S_i,\ i = 1, \ldots, r\}$ and $[G : S_i] = \infty$, for all $i$. If $E_*(G, \mathcal{S}) > 1$, then $(G, \mathcal{S})$ is not a* PD$^n$*-pair. In other words, if $(G, \mathcal{S})$ is a* PD$^n$*-pair, then $E_*(G, \mathcal{S}) = 1$.*

*Proof.* Consider part of the exact sequence of the Proposition 2 for $M = \overline{\mathbb{Z}_2(G/\mathcal{S})}$:

$$\bigoplus_{i=1}^{r} H_1(S_i; \overline{\mathbb{Z}_2(G/\mathcal{S})}) \xrightarrow{\operatorname{cor}_{\mathcal{S}}^{G}} H_1(G; \overline{\mathbb{Z}_2(G/\mathcal{S})}) \xrightarrow{J} H_1(G, \mathcal{S}; \overline{\mathbb{Z}_2(G/\mathcal{S})})$$
$$\xrightarrow{\delta} \bigoplus_{i=1}^{r} H_0(S_i; \overline{\mathbb{Z}_2(G/\mathcal{S})}) \xrightarrow{\operatorname{cor}_0{}_{,\mathcal{S}}^{G}} H_0(G; \overline{\mathbb{Z}_2(G/\mathcal{S})}) \to 0.$$

Since $(G, \mathcal{S})$ is a PD$^n$-pair, it follows from Remark 5 that $G$ is a $D^n$-group, $S$ is a PD$^{n-1}$-subgroup and $G$ is finitely generated. And, by using Remark 1, we conclude that $\dim H_0(G; \overline{\mathbb{Z}_2(G/\mathcal{S})}) = 0$. In fact

$$H_0(G; \overline{\mathbb{Z}_2(G/\mathcal{S})}) = H_0(G; \bigoplus_{i=1}^{r} \overline{\mathbb{Z}_2(G/S_i)}) = \bigoplus_{i=1}^{r} H_0(G; \overline{\mathbb{Z}_2(G/S_i)})$$
$$= \bigoplus_{i=1}^{r} \overline{\mathbb{Z}_2(G/S_i)}_G = \bigoplus_{i=1}^{r} (\operatorname{Coind}_{S_i}^{G} \mathbb{Z}_2)_G = 0.$$

Now, it follows from Lemma 1, that $H_1(G, \mathcal{S}; \overline{\mathbb{Z}_2(G/\mathcal{S})}) = \bigoplus_{i=1}^{r} \mathbb{Z}_2$. Thus, $\dim H_1(G, \mathcal{S}; \overline{\mathbb{Z}_2(G/\mathcal{S})}) = r$. For calculating $\dim \bigoplus_{i=1}^{r} H_0(S_i; \overline{\mathbb{Z}_2(G/\mathcal{S})})$

observe that

$$\begin{aligned}\bigoplus_{i=1}^{r} H_0(S_i;\overline{\mathbb{Z}_2(G/\mathcal{S})}) &= \bigoplus_{j=1}^{r} H_0(S_j;\bigoplus_{i=1}^{r}\overline{\mathbb{Z}_2(G/S_i)}) \\ &= \bigoplus_{j=1}^{r}\Bigg[H_0(S_j;\overline{\mathbb{Z}_2(G/S_j)}) \oplus \bigoplus_{i\neq j, i=1}^{r} H_0(S_j;\overline{\mathbb{Z}_2(G/S_i)})\Bigg] \\ &= \bigoplus_{j=1}^{r}\Bigg[(\overline{\mathbb{Z}_2(G/S_j)})_{S_j} \oplus \bigoplus_{i\neq j, i=1}^{r} (\overline{\mathbb{Z}_2(G/S_i)})_{S_j}\Bigg].\end{aligned}$$

By using Lemma 2, we have that $\mathbb{Z}_2$ is isomorphic to a subset of $(\overline{\mathbb{Z}_2(G/S_j)})_{S_j}$, for all $j = 1, \ldots, r$. Thereby,

$$\bigoplus_{j=1}^{r}\mathbb{Z}_2 \hookrightarrow \bigoplus_{j=1}^{r}(\overline{\mathbb{Z}_2(G/S_j)})_{S_j} \hookrightarrow \bigoplus_{j=1}^{r}\Bigg[(\overline{\mathbb{Z}_2(G/S_j)})_{S_j} \oplus \bigoplus_{i\neq j, i=1}^{r} (\overline{\mathbb{Z}_2(G/S_i)})_{S_j}\Bigg],$$

and thus,

$$\dim \bigoplus_{i=1}^{r} H_0(S_i;\overline{\mathbb{Z}_2(G/\mathcal{S})}) \geqslant r. \tag{$*$}$$

On the other hand, since $H_0(G;\overline{\mathbb{Z}_2(G/\mathcal{S})}) = 0$, the map $\delta$ of the exact sequence (3.1) is surjective. It follows that,

$$\dim \bigoplus_{i=1}^{r} H_0(S_i;\overline{\mathbb{Z}_2(G/\mathcal{S})}) \leqslant \dim\ H_1(G,\mathcal{S};\overline{\mathbb{Z}_2(G/\mathcal{S})}) = r. \tag{$**$}$$

Hence, from $(*)$ and $(**)$, we have $\dim \bigoplus_{i=1}^{r} H_0(S_i;\overline{\mathbb{Z}_2(G/\mathcal{S})}) = r$. Therefore, by using Proposition 3, $E_*(G,\mathcal{S}) = 1 + 0 - r + r = 1$. □

**Example 1.** Let $X$ be a torus minus an open disc and $Y$ the boundary of $X$. We have $G = \pi_1(X) \simeq \mathbb{Z} * \mathbb{Z}$ and $S = \pi_1(Y) \simeq \mathbb{Z}$. It follows from [6], Theorem 6.1, that the pair $(G,S)$ is a PD$^2$-pair and thus, by Theorem 3, $E_*(G,S) = 1$.. More generally, if $X$ is a closed surface $F$ minus $k$ open discs ($k \geqslant 2$ if $F = S^2$) and $Y = \partial X = \bigcup_{i=1}^{k} Y_i$, where $Y_i$ are the boundary of the $k$-discs, consider $\mathcal{S} = \{S_i = \pi_1(Y_i),\ i = 1, \ldots, k\}$ and $G = \pi_1(X)$. Then, $(G,\mathcal{S})$ is a PD$^2$-pair and, by Theorem 3, $E_*(G,\mathcal{S}) = 1$.

We will see now simple computations of the particular invariant $E_*(G,S)$ when $G$ and $S$ satisfy some finiteness conditions.

**Proposition 4.** *Let $G$ be a* PD$^n$*-group and $S$ a subgroup of $G$.*

(i) *If* $S$ *is a* $\mathrm{PD}^{n-1}$*-group then* $E_*(G,S) \leqslant 2$.
(ii) *If* $\operatorname{cd} S \leqslant n-2$ *then* $E_*(G,S) = 1$.

*Proof.* Since $G$ is $\mathrm{PD}^n$-group, by the hypothesis of (i) or (ii) and Remark 5, it follows that $[G:S]=\infty$. Then, $E_*(G,S)$ can be defined. Now, by using duality and Shapiro's Lemma, we have:

$$H_1(G;\overline{\mathbb{Z}_2(G/S)}) \simeq H^{n-1}(G;\overline{\mathbb{Z}_2(G/S)}) \simeq H^{n-1}(S;\mathbb{Z}_2).$$

(i) If $S$ is a $\mathrm{PD}^{n-1}$-group, then $H^{n-1}(S;\mathbb{Z}_2) \simeq \mathbb{Z}_2$ and thus,

$$\operatorname{coker}(\operatorname{cor}_S^G) = \frac{H^1(G;\overline{\mathbb{Z}_2(G/S)})}{\operatorname{Im}(\operatorname{cor}_S^G)} = \frac{\mathbb{Z}_2}{\operatorname{Im}(\operatorname{cor}_S^G)}$$

can only be $\{0\}$ or $\mathbb{Z}_2$. Therefore, $E_*(G,S) \leqslant 2$.

(ii) If $\operatorname{cd} S \leqslant n-2$ we have $H^{n-1}(S,\mathbb{Z}_2) = \{0\}$. Hence $\operatorname{coker}(\operatorname{cor}_S^G) = \{0\}$ and we have $E_*(G,S) = 1$. □

**Example 2.** Consider $G = \mathbb{Z}^n$ and $S \simeq \mathbb{Z}^r$ a subgroup of $G$ with $n > r \geqslant 2$. Note that $G$ is a $\mathrm{PD}^n$-group and $S$ is a $\mathrm{PD}^r$-group. If $r = n-1$, then $E_*(G,S) \leqslant 2$ and if $r \leqslant n-2$, then $E_*(G,S) = 1$.

Finally, we prove a necessary condition for $(G,S)$ to be not a Poincaré duality pair.

**Theorem 4.** *If* $(G,S)$ *is a group pair with* $[G:S]=\infty$ *and* $S$ *a normal subgroup of* $G$*, then* $(G,S)$ *is not a Poincaré duality pair.*

*Proof.* If $(G,S)$ is a $\mathrm{PD}^n$-pair, then, by using the technique of the proof of Theorem 3, we can prove that $\dim H_0(S;\overline{\mathbb{Z}_2(G/S)}) = 1$ and thus

$$H_0(S;\overline{\mathbb{Z}_2(G/S)}) = \mathbb{Z}_2. \tag{$*$}$$

Since $S$ is normal in $G$, the $S$-action on $\overline{\mathbb{Z}_2(G/S)}$ is trivial. In fact, $\forall f \in \overline{\mathbb{Z}_2(G/S)}$ and $s \in S$,

$$\begin{aligned}(s\cdot f)(\overline{g}) &= f(s^{-1}\overline{g}) \quad \text{(diagonal action)}\\ &= f(\overline{s^{-1}g})\\ &= f(\overline{s^{-1}}\cdot\overline{g}) \quad (S \text{ normal in } G)\\ &= f(\overline{g}).\end{aligned}$$

Therefore, $sf = f$. Hence,

$$H_0(S;\overline{\mathbb{Z}_2(G/S)}) = \overline{\mathbb{Z}_2(G/S)}. \tag{$**$}$$

Since $[G:S]=\infty$ we have, from $(*)$ and $(**)$, a contradiction. □

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## References

[1] M.G.C. Andrade, E.L.C. Fanti, A relative cohomological invariant for pairs of groups. *Manuscripta Math.*, **83** (1994), 1-18.

[2] M.G.C. Andrade, E.L.C. Fanti and J.A. Daccach, On certain relative cohomological invariant. *International Journal of Pure and Applied Mathematics*, **21** (2005), 335-351.

[3] M.G.C. Andrade, E.L.C. Fanti, A note about splittings of groups and commensurability under a cohomological point of view. *Algebra and Discrete Mathematics*, **9** (2), (2010), 1-10.

[4] M.G.C. Andrade; A. B. Gazon, A dual homological invariant and some properties, *International Journal of Applied Mathematics*, **27** (1), (2014), 13-20.

[5] R. Bieri, *Homological Dimension of Discrete Groups*, Queen Mary College Math. Notes, Londres, (1976).

[6] R. Bieri, B. Eckmann, Relative homology and Poincaré duality for Group Pairs. *Journal of Pure and Applied Algebra*, **13** (1978), 277-319.

[7] K.S. Brown, *Cohomology of Groups*. New York: Springer-Verlag, (1982).

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