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RESEARCH ARTICLE

A way of computing the Hilbert series

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Abstract. Let $S = K[x_1, x_2, \ldots, x_n]$ be a standard graded K-algebra for any field K. Without using any heavy tools of commutative algebra we compute the Hilbert series of graded S-module S/I, where I is a monomial ideal.

Let $S = K[x_1, x_2, \ldots, x_n]$ be a standard graded K-algebra for a field K. We present a way of computing the Hilbert series of a graded S-module S/I, when I is a monomial ideal of S. There are some known ways of computing the Hilbert series (for example [3]). Unlike any other method we compute Hilbert series of S/I by skipping heavy tools of commutative algebra.

If $I \subset S$ is a monomial ideal which is minimally generated by monomials $\{u_1, u_2, \ldots, u_s\}$, S_d/I_d is the dth graded component of S/I and $H(S/I, d) = \dim_K S_d/I_d$ is the dimension of S_d/I_d as a K linear space, called the *Hilbert function* of S/I. The series

$$H_{S/I}(t) = \sum_{d=0}^{\infty} H(S/I, d)t^d$$

is called the *Hilbert Series* of S/I. To know more about Hilbert series of general graded modules and related results see [1], [2] or [4].

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For any monomials $u = \prod_{i=0}^{n} x_i^{a_i}$ and $v = \prod_{i=0}^{n} x_i^{b_i}$ in S, we define the *intersecting multiplication* as

$$u * v = \prod_{i=0}^{n} x_i^{\max\{a_i, b_i\}}. \tag{1}$$

Considering this sort of multiplication allowing us to associate a monomial ideal $I = (u_1, u_2, \dots, u_s) \subset S$ to a unique polynomial called *intersecting polynomial* in the following way

$$P_{S/I} = (1 - u_1) * (1 - u_2) * \dots * (1 - u_s).$$

Simplifying we get

$$P_{S/I} = 1 - \sum_{i=1}^{s} (u_i) + \sum_{1 \leqslant i_1 < i_2}^{s} (u_{i_1} * u_{i_2}) - \dots + (-1)^s (u_{i_1} * u_{i_2} * \dots * u_{i_s}). \tag{2}$$

In the polynomial $P_{S/I}$ we have 2^s number of monomial terms with half positive and half negative coefficient. For simplicity, we denote monomials with coefficient $+1$ by $v_1, v_2, \dots, v_{2^{s-1}}$ and monomials with coefficient -1 by $w_1, w_2, \dots, w_{2^{s-1}}$. Using these notations we can write (2) as

$$P_{S/I} = \sum_{i=1}^{2^{s-1}} (v_i - w_i). \tag{3}$$

Note that if an ideal $J = (I, u)$, for monomial $u \notin I$, then $P_{S/J} = P_{S/I} - u * P_{S/I}$.

Once we obtained the intersecting polynomial $P_{S/I}$, we can write Hilbert series of S/I as described in the following theorem.

Theorem 1. *If $P_{S/I} = \sum_{i=1}^{2^{s-1}} (v_i - w_i)$ is the intersecting polynomial of S/I for monomial ideal $I \subset S$, then*

$$H_{S/I}(t) = \frac{\sum\limits_{i=1}^{2^{s-1}} \left(t^{\deg(v_i)} - t^{\deg(w_i)}\right)}{(1-t)^n}.$$

Proof. If $I = (u_1, u_2, \dots, u_s) \subset S$ is a monomial ideal, then by inclusion exclusion principal we can see that the dimension of dth graded component of I is

$$H(I, d) = \sum_{i=1}^{s} |u_i|_d - \sum_{1 \leqslant i_1 < i_2}^{s} |u_{i_1} * u_{i_2}|_d - \dots + (-1)^{s+1} |u_{i_1} * u_{i_2} * \dots * u_{i_s}|_d,$$

where $|u|_d = \binom{n-1+d-\deg(u)}{n-1}$ is the dimension of dth graded component of an ideal generated by a monomial $u \in S$.

Now $H(S/I, d) = H(S, d) - H(I, d)$, for all d and $H(S, d) = |1|_d$, hence

$$\begin{aligned} H(S/I,d) = |1|_d - &\sum_{i=1}^{s} |u_i|_d \\ &+ \sum_{1 \leqslant i_1 < i_2}^{s} |u_{i_1} * u_{i_2}|_d - \ldots + (-1)^s |u_{i_1} * u_{i_2} * \ldots * u_{i_s}|_d. \end{aligned} \tag{4}$$

If we replace each monomial term in the intersecting polynomial defined in (2) by the corresponding dth graded component of ideal generated by that monomial, then we obtain $H(S/I, d)$ as in (4). Hence we can write Hilbert function $H(S/I, d)$ in terms of (3) as

$$\begin{aligned} H(S/I,d) &= \sum_{i=1}^{2^{s-1}} (|v_i|_d - |w_i|_d) \\ &= \sum_{i=1}^{2^{s-1}} \left(\binom{n-1+d-\deg(v_i)}{n-1} - \binom{n-1+d-\deg(w_i)}{n-1} \right) \end{aligned}$$

and the corresponding Hilbert series is

$$H_{S/I}(t) = \sum_{d=0}^{\infty} \left(\sum_{i=1}^{2^{s-1}} (|v_i|_d - |w_i|_d) \right) = \frac{\sum_{i=1}^{2^{s-1}} \left(t^{\deg(v_i)} - t^{\deg(w_i)} \right)}{(1-t)^n}.$$

□

We give an example to illustrate our method.

Example. If $I = (x_1^2x_3, x_1x_2x_3^2, x_2^2x_3^3) \subset S = K[x_1, x_2, x_3]$, then the corresponding intersecting polynomial is

$$\begin{aligned} P_{S/I} &= (1 - x_1^2x_3) * (1 - x_1x_2x_3^2) * (1 - x_2^2x_3^3) \\ &= 1 - x_1^2x_3 - x_1x_2x_3^2 - x_2^2x_3^3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_3^3 + x_1x_2^2x_3^3 - x_1^2x_2^2x_3^3. \end{aligned}$$

The monomials with coefficient $+1$ and -1 in $P_{S/I}$ are

$$v_1 = 1, \qquad v_2 = x_1^2x_2x_3^2, \qquad v_3 = x_1^2x_2^2x_3^3, \qquad v_4 = x_1x_2^2x_3^3$$

and

$$w_1 = x_1^2 x_3, \qquad w_2 = x_1 x_2 x_3^2, \qquad w_3 = x_2^2 x_3^3, \qquad w_4 = x_1^2 x_2^2 x_3^3,$$

respectively.

Now by using Theorem 1 we can write Hilbert series of S/I, that is;

$$H_{S/I}(t) = \frac{t^0 - t^3 + t^5 - t^4 + t^7 - t^5 + t^6 - t^7}{(1-t)^3} = \frac{1 + t + t^2 - t^4 - t^5}{(1-t)^2}.$$

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