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STRESS—STRAIN AND LIMITING STATE OF SPOKED FLYWHEELS FORMED FROM COMPOSITE MATERIALS.

REPORT 1. COMPUTATIONAL RELATIONSHIPS

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Computation of the stress-strain state of a flywheel built from composite materials in the form of an energy-storing element — a rim connected with spokes using a chorded star-shaped polygon (Fig. 1) — is performed in a rather complete statement with allowance for the interaction between spokes at the points of their intersection. In addition to this, the effect of this interaction on the adequacy of the solution, which has been found to be rather cumbersome, has not been evaluated.

The purpose of the present study is to obtain a rather simple solution for this problem which is convenient for practical engineering computations, and to evaluate its discrepancy with the accurate [1] solution, and also its correspondence with experimental data on the limiting bearing capacity for two flywheel models of various composite materials.

We computed the stress—strain state of a flywheel with consideration given to the joint deformation of the rim and spokes. In this stage of the investigation, it was proposed that stress and strains are caused by short-term loading, and the behavior of the material is elastic, and conforms to Hooke's law. The shape that the rim assumes and other parameters are determined by the plane axisymmetric character of the loading, and also by the axisymmetric anisotropy, or quasi-isotropy of the material used. It was therefore assumed that the flywheel has a plane median surface to which the load is applied. In this case, the stressed state was considered plane. The loading consisted of a mass force caused by rotation, and uniformly distributed radial boundary forces.

1. The equation of equilibrium for an elementary volume bounded by two radial sections and two concentric circles placed at a distance r from the disk's axis of rotation assumes the form (Fig. 2)

$$\frac{d}{dr}(r\sigma_r) - \sigma_\varphi + P_r = 0, \tag{1}$$

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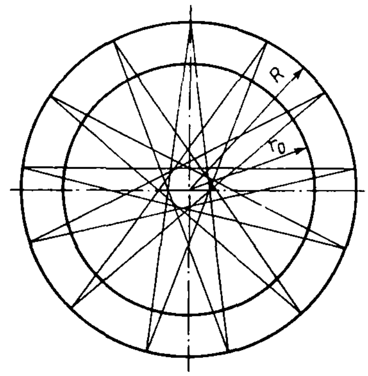

Fig. 1. Diagram showing spoked flywheel.

where $P = \rho\omega^2 r$; σ_r and σ_φ are the radial and circumferential stresses, ρ is the density of the material, and ω is the angular rotation frequency of the object.

A cerain relation exists between the deformations and displacements for the plane case with consideration given to axial symmetry:

$$\varepsilon_r = \frac{du}{dr}; \qquad \varepsilon_\varphi = \frac{u}{r}. \tag{2}$$

Making use of Hooke's law for an anisotropic material in the case of a plane stressed state

$$\varepsilon_r = \frac{\sigma_r}{E_r} - \nu_{\varphi r}\frac{\sigma_\varphi}{E_\varphi}; \qquad \varepsilon_\varphi = \frac{\sigma_\varphi}{E_\varphi} - \nu_{r\varphi}\frac{\sigma_r}{E_r}, \tag{3}$$

and also the condition of interaction for the elastic constants

$$E_r \nu_{\varphi r} = E_\varphi \nu_{r\varphi}, \tag{4}$$

we obtain the following differential equation for radial displacement:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - k^2\frac{u}{r^2} + qr = 0, \tag{5}$$

where

$$k^2 = E_\varphi / E_r; \qquad q = \frac{1 - \nu_{\varphi r}\nu_{r\varphi}}{E_r}\rho\omega^2.$$

Equation (5) has the following solution:

$$u = C_1 r^k + C_2 r^{-k} - \frac{q}{9 - k^2} r^3, \tag{6}$$

where the constants of integration C_1 and C_2 are determined from equilibrium conditions near the edges of the disk.

Expressions for the radial and circumferential stresses can be obtained from Eqs. (3) and (6). Introducing the new constants c_1 and c_2 and making use of condition of interaction (4), we can write:

$$\begin{aligned} \sigma_r &= c_1 r^{k-1} + c_2 r^{-k-1} - \frac{3 + \nu_{\varphi r}}{9 - k^2}\rho\omega^2 r^2; \\ \sigma_\varphi &= c_1 k \cdot r^{k-1} - c_2 k r^{-k-1} - \frac{k^2(3\nu_{r\varphi} + 1)}{9 - k^2}\rho\omega^2 r^2. \end{aligned} \tag{7}$$

The relation between c_1 and c_2 and the constants of integration used in (6) is expressed in the following manner:

$$c_1 = \frac{E_r (k + \nu_{\varphi r}) C_1}{1 - \nu_{r\varphi}\nu_{\varphi r}}; \qquad c_2 = \frac{E_r (\nu_{\varphi r} - k) C_2}{1 - \nu_{r\varphi}\nu_{\varphi r}}. \tag{8}$$

The problem for a disc with an opening free of perimeter loads can be solved in the first stage of the computation of the stress–strain state of a spoked flywheel. Considering the conditions $\sigma_r(R) = \sigma_r(r_0) = 0$, we obtain

$$c_1 = \frac{3+\nu_{\varphi r}}{9-k^2}\rho\omega^2\left(\frac{R^{k+3}-r_0^{k+3}}{R^{2k}-r_0^{2k}}\right);$$

$$c_2 = \frac{3+\nu_{\varphi r}}{9-k^2}\rho\omega^2\left(\frac{r_0^{k+3}R^{2k}-R^{k+3}r_0^{2k}}{R^{2k}-r_0^{2k}}\right) \tag{9}$$

for the case in question.

Let us compute the values of σ_r and σ_φ with consideration given to the relationships derived for the constants c_1 and c_2. For this purpose, let us introduce the dimensionless quantities

$$\hat{r} = r/r_0; \quad \hat{R} = R/r_0; \quad \hat{r}_0 = 1.$$

Then, we can finally write σ_r and σ_φ in the following manner:

$$\sigma_r = \frac{3+\nu_{\varphi r}}{9-k^2}\rho\omega^2 r^2\left[\frac{\hat{R}^{k+3}-1}{\hat{R}^{2k}-1}\hat{r}^{k-3} + \frac{\hat{R}^{k-3}-1}{\hat{R}^{2k}-1}\left(\frac{\hat{R}}{\hat{r}}\right)^{k+3} - 1\right]; \tag{10}$$

$$\sigma_\varphi = \frac{3+\nu_{\varphi r}}{9-k^2}\rho\omega^2 r^2\left[\frac{\hat{R}^{k+3}-1}{\hat{R}^{2k}-1}\cdot\hat{r}^{k-3}\cdot k - \frac{\hat{R}^{k-3}-1}{\hat{R}^{2k}-1}\left(\frac{\hat{R}}{\hat{r}}\right)^{k+3}\cdot k - \frac{k^2+3\nu_{\varphi r}}{3+\nu_{\varphi r}}\right]. \tag{11}$$

In the case of radial displacements, we can derive an expression for u from (6):

$$u = \frac{3+\nu_{\varphi r}}{E_\varphi(9-k^2)}\rho\omega^2 r^3\left[\frac{\hat{R}^{k+3}-1}{\hat{R}^{2k}-1}\hat{r}^{k-3}(k-\nu_{\varphi r}) - \frac{\hat{R}^{k-3}-1}{\hat{R}^{2k}-1}\left(\frac{\hat{R}}{\hat{r}}\right)^{k+3}(k+\nu_{\varphi r}) - \frac{k^2-\nu_{\varphi r}^2}{3+\nu_{\varphi r}}\right]. \tag{12}$$

If the outer perimeter of the disk is not free of loads, i.e., the conditions $\sigma_r(R) = \sigma_R \neq 0$, $\sigma_r(r_0) = 0$ are satisfied, we have from (7)

$$c_1 = \frac{\frac{\sigma_R}{R^2}\hat{R}^{k+3} + \frac{3+\nu_{\varphi r}}{9-k^2}\rho\omega^2(\hat{R}^{k+3}-1)}{(\hat{R}^{2k}-1)r_0^{k-3}}; \tag{13}$$

$$c_2 = \frac{\frac{\sigma_R}{R^2}\hat{R}^{3-k} + \frac{3+\nu_{\varphi r}}{9-k^2}\rho\omega^2(\hat{R}^{3-k}-1)}{(\hat{R}^{-2k}-1)r_0^{-(k+3)}}. \tag{14}$$

Let us introduce the operator

$$A(k,\hat{r}) = \frac{\left[\frac{\sigma_R}{R^2}\hat{R}^{k+3} + \frac{3+\nu_{\varphi r}}{9-k^2}\rho\omega^2(\hat{R}^{k+3}-1)\right]\hat{r}^k}{(\hat{R}^{2k}-1)(k+\nu_{\varphi r})}. \tag{15}$$

Considering this designation, we obtain the relationship for displacements from expressions (6), (13), and (14):

$$u = \frac{r_0^3(1-\nu_{\varphi r}\nu_{r\varphi})}{E_r}\left[A(k,\hat{r}) + A(-k,\hat{r}) - \hat{r}^3\frac{\rho\omega^2}{9-k^2}\right]. \tag{16}$$

The stress distribution in the disc is described by the equations

$$\sigma_r = \left\{[A(k,\hat{r})(k+\nu_{\varphi r}) + A(-k,r)(\nu_{\varphi r}-k)]\hat{r}^{-1} - \frac{3+\nu_{\varphi r}}{9-k^2}\rho\omega^2\hat{r}^2\right\}r_0^2; \tag{17}$$

$$\sigma_\varphi = \left\{[A(k,\hat{r})(k+\nu_{\varphi r})k - A(-k,\hat{r})(\nu_{\varphi r}-k)k]\hat{r}^{-1} - \frac{k^2(3\nu_{r\varphi}+1)}{9-k^2}\rho\omega^2\hat{r}^2\right\}r_0^2. \tag{18}$$

2. Let us derive the equilibrium and deformation equations for the spokes. Let us isolate in the spoke an elementary volume dv bounded by the radial planes of the flywheel with angle of opening $d\varphi$. Let us direct the ordinate of the XOY coordinate system along the axis of the spoke toward the rim (Fig. 2b).

The centrifugal force acting on the isolated volume of the spoke is determined by the relationship

$$dC = \rho\omega^2 r dv, \tag{19}$$

where dv = Sdy and S is the cross-sectional area of the spoke.

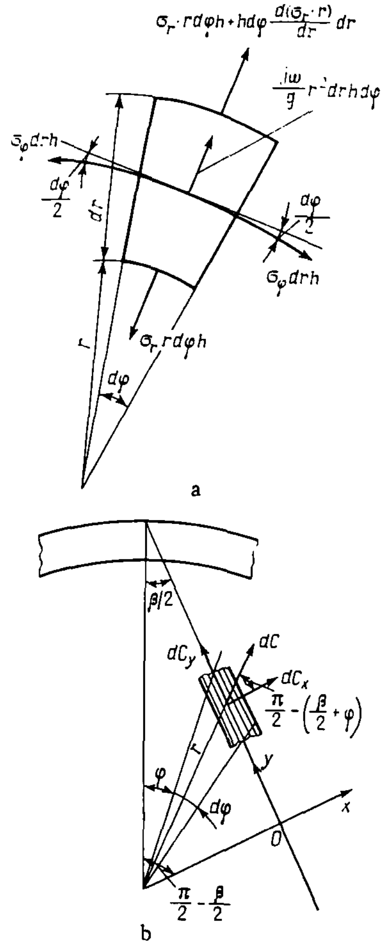

Fig. 2. Forces acting on small volume of disk (a) and spoke (b) existing in centrifugal-force field.

Let us write the relationships of certain quantities in rectangular coordinates in terms of polar coordinates:

$$y = r\sin\left(\frac{\pi}{2} - \frac{\beta}{2} - \varphi\right) = r\cos\left(\frac{\beta}{2} + \varphi\right). \tag{20}$$

Expressing r in terms of R, we obtain

$$\begin{gathered} y = R\sin\frac{\beta}{2}\cot\left(\frac{\beta}{2} + \varphi\right); \\ dy = -R\sin\frac{\beta}{2}\sin^{-2}\left(\frac{\beta}{2} + \varphi\right)d\varphi. \end{gathered} \tag{21}$$

Thus,

$$dC = -\rho\omega^2 R^2 \frac{\sin\frac{\beta}{2}}{\sin^3\left(\frac{\beta}{2} + \varphi\right)} \cdot S d\varphi, \tag{22}$$

or

$$\begin{gathered} dC_y = -\rho\omega^2 R^2 \frac{\sin\frac{\beta}{2}\cos\left(\frac{\beta}{2} + \varphi\right)}{\sin^3\left(\frac{\beta}{2} + \varphi\right)} \cdot S d\varphi; \\ dC_x = -\rho\omega^2 R^2 \frac{\sin^2\frac{\beta}{2}}{\sin^2\left(\frac{\beta}{2} + \varphi\right)} \cdot S d\varphi \end{gathered} \tag{23}$$

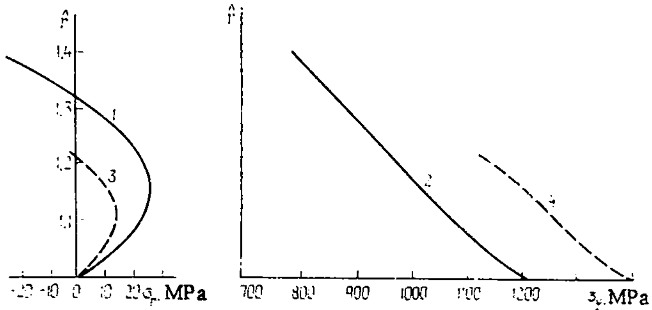

Fig. 3. Distribution of radial (1, 3) and circumferential (2, 4) stresses in rim of flywheel at rotation frequency of 65,600 rpm (1), 70,800 rpm (2) 53,600 rpm (3), and 59,900 rpm (4). (Solid lines denote rim of fiberglass flywheel, and broken lines rim of carbon-fiber-reinforced-plastic flywheel.)

in projections on the coordinate axes.

In this stage of the computations, the effect of the components C_x on the stress—strain state of the spoke can be neglected in view of its smallness. The expression for the stress in the cross section of the spoke due to C_y assumes the following form:

$$\sigma_y = \frac{C_y}{S} = -\rho\omega^2 R^2 \sin^2\frac{\beta}{2}\int_0^{\varphi}\frac{\cos\left(\frac{\beta}{2}+\varphi\right)}{\sin^3\left(\frac{\beta}{2}+\varphi\right)}d\varphi. \tag{24}$$

The integration of this expression yields the relationship

$$\sigma_y = \frac{1}{2}\rho\omega^2 R^2 \sin^2\frac{\beta}{2}\left[\sin^{-2}\left(\frac{\beta}{2}+\varphi\right)-\sin^{-2}\frac{\beta}{2}\right]. \tag{25}$$

The extremum of expression (25) can be determined from the condition $\frac{d\sigma_y}{d\varphi} = 0$; hence,

$$\cos\left(\frac{\beta}{2}+\varphi\right) = 0.$$

The resultant condition is satisfied when $\frac{\beta}{2}+\varphi=\frac{\pi}{2}$ i.e., when $\varphi=\varphi_{max}$. The maximum value of the stresses in the spoke is

$$\sigma_{y\max} = \sigma_y(\varphi_{\max}) = -\frac{1}{2}\rho\omega^2 R^2\cos^2\frac{\beta}{2}. \tag{26}$$

Analysis indicates that $\cos^2\beta/2 > 0.9$ when $R/r_1 > 3$. Consequently, the maximum stresses in the inclined ($\beta \neq 0$) spokes differ from those in the straight ($\beta = 0$) spokes by no more than 10%. Here, this difference diminishes markedly with increasing ratio R/r_0. Hereinafter, the assumption that the spokes are straight is therefore made to simplify computations in studying the stress—strain state of flywheel elements. The equation of equilibrium of a straight spoke (rod) in a centrifugal-force field can be written as

$$\frac{d\sigma_r}{dr} + \rho\omega^2 r = 0. \tag{27}$$

After integrating and considering the condition $\sigma_r(R) = 0$, we obtain

$$\sigma_r = \rho\omega^2 (R^2 - r^2)/2. \tag{28}$$

The spoke's strain ε_r is expressed in the following way:

$$\varepsilon_r = \frac{\rho\omega^2}{2E}(R^2 - r^2). \tag{29}$$

Let us find the displacement, assuming that u = 0 when r = 0:

$$u = \int_0^r \varepsilon_r dr = \frac{\rho\omega^2}{2E}\left(R^2 r - \frac{r^3}{3}\right). \tag{30}$$

The displacement of the end of the spoke (r = R) is determined by the relationship

$$u(R)=\frac{\rho\omega^2R^3}{3E}. \tag{31}$$

3. The problem of acquiring the stress—strain state of all flywheel elements is statically indeterminant. For its solution, let us make use of the compatibility of rim and spoke deformations. The displacements of the rim and spokes on a radius R should be:

$$u^{sp}(\omega)+u^{sp}(Q)=u^{d}(\omega)-u^{d}(Q), \tag{32}$$

where Q is an unknown force acting along the rim of the disk owing to its interaction with the spokes, $u^{sp}(\omega)$ is the displacement in a spoke due to the effect of centrifugal forces, $u^{sp}(Q)$ is the displacement in a spoke due to the effect of the deformed disk, $u^{d}(\omega)$ is the displacement in the disk due to the effect of centrifugal forces, and $u^{d}(Q)$ is the displacement in the disk due to the deformed spokes.

Let the spokes be fixed on the rim at m points; four spokes radiate from each point. Stresses in the spokes generate a perimeter loading for the disk. The relation between the stress through the rim of the disk $\sigma_R{}^d$ and the stress at the end of a spoke $\sigma^{sp}{}_R$ is expressed by the relationship

$$Q=-\sigma_R^d h\cdot 2\pi R=\sigma_R^{sp}4mS, \tag{33}$$

where h is the thickness of the disk, and $\sigma^d{}_R$ corresponds to σ_R in expressions (13)-(18). Thus,

$$\sigma_R^{sp}=-\frac{\pi Rh}{2mS}\sigma_R. \tag{34}$$

Considering what we have stated, let us write the equation for determination of σ_R. The right side of relationship (32) is expression (16) for r = R, where σ_R is taken with a "minus" sign. Let us express the left side of relationship (32), considering expression (31):

$$u^{sp}(\omega)+u^{sp}(Q)=\frac{\rho\omega^2R^3}{3E^{sp}}-\frac{\pi R^2h\sigma_R}{2mSE^{sp}}. \tag{35}$$

Finally, we have

$$\frac{r_0^3(1-\nu_{\varphi r}\nu_{r\varphi})}{E_r^d}\left[A(k,\hat{R})+A(-k,\hat{R})-\hat{R}^3\frac{\rho\omega^2}{9-k^2}\right]=\frac{1}{E^{sp}}\left(\frac{\rho\omega^2R^3}{3}-\frac{\pi R^2h\sigma_R}{2mS}\right). \tag{36}$$

Having determined σ_R from expression (36), it is possible to calculate the distribution of the stresses σ_r and σ_φ in the disk from relationships (17) and (18).

4. Using this solution, we calculated the stress—strain and limiting state of a fiberglass flywheel, data on the geometric and mechanical parameters of which are presented in [1]. According to Eq. (36), we obtain

$$\sigma_R=-0.564\,\omega^2. \tag{37}$$

We then computed the limiting rotational frequencies corresponding to spoke failure $\omega^{sp}{}_{max}$, rim delamination $\omega^d{}_{rmax}$, and rim failure $\omega^d{}_{\varphi max}$, as well as values of σ_R for these cases. The numerical values of the desired parameters are as follows:

$$\omega_{max}^{sp}=7.64\cdot10^4 \text{ rpm} \qquad \sigma_R=-36.1 \text{ MPa}$$

$$\omega_{rmax}^{d}=6.65\cdot10^4 \text{ rpm} \qquad \sigma_R=-27.4 \text{ MPa}$$

$$\omega_{\varphi max}^{d}=7.13\cdot10^4 \text{ rpm} \qquad \sigma_R=-31.4 \text{ MPa}$$

The resultant data are close to the results of the above-described study [1]. Here, we may also conclude that the bearing capacity of the flywheel is governed by the radial strength of the rim. Comparison with experimental data should be considered more satisfactory for the computation performed in the present study, however, since the minimum limiting rotation frequency ($5.84\cdot10^4$ rpm) is lower in [1] than that attained during the failure of a flywheel on an acceleration bench.

The stress distributed in the rim of the flywheel at limiting rotation frequencies as computed from Eqs. (17) and (18) is shown in Fig. 3. Curve 1 (Fig. 3) corresponds to radial stresses when $\omega^d{}_{rmax}=6.65\cdot10^4$ rpm. $\sigma_r=-27.4$ MPa is attained on the outer perimeter of the rim ($\hat{r}=\hat{R}=1.4$), and the largest tensile radial stresses $\sigma_r=27$ MPa on the radius

$\hat{r}$ = 1.15; this corresponds to the limiting strength of the composite in the direction perpendicular to the reinforcement. The distribution of circumferential stresses for a rotation frequency $\omega^d_{\varphi\,max} = 7.13 \cdot 10^4$ rpm is shown in Fig. 3 (curve 2). Circumferential stresses equal to the ultimate strength in the direction of the reinforcement (1200 MPa) are attained on the radius of the internal perimeter $\hat{r} = \hat{r}_0 = 1$.

5. The strength parameters of a compatible model of a flywheel formed from other composite materials are computed in a similar manner.

The material of the flywheel rim is a carbon-fiber-reinforced plastic with 30% of epoxy binder. The physicomechanical characteristics of the rim composite are as follows: $\rho = 1.44 \cdot 10^3$ kg/m³, $E_\varphi = 1.3 \cdot 10^5$ MPa (in the circumferential direction), E_r = 5400 MPa (in the radial direction, Π_φ = 1400 MPa (ultimate strength in the circumferential direction), and Π_r = 14 MPa (ultimate strength in the radial direction). The mass of the rim is 830 g.

The flywheel has 52 spokes, which intersect the outer perimeter at 13 points. Each spoke contains 10 braids of an organoplastic with 30% of epoxy binder. The overall mass of the spokes is 550 g. The physicomechanical characteristics of the spoke composite, which are required for the computation, are as follows: $\rho = 1.31 \cdot 10^3$ kg/m³, $E = 6.5 \cdot 10^4$ MPa, and Π = 1800 MPa. The geometric parameters of the flywheel are as follows: R = 0.165 m, r_0 = 0.135 m, and h = 0.024 m.

Numerical calculation of the stress–strain state of the construction under investigation produced the following results:

$$\omega^{sp}_{max} = 9.07 \cdot 10^4 \text{ rpm} \qquad \sigma_R = -19.8 \text{ MPa}$$

$$\omega^{d}_{r\,max} = 5.82 \cdot 10^4 \text{ rpm} \qquad \sigma_R = -8.2 \text{ MPa}$$

$$\omega^{d}_{\varphi\,max} = 6.06 \cdot 10^4 \text{ rpm} \qquad \sigma_R = -8.9 \text{ MPa}$$

The stress distribution in the carbon-fiber-reinforced-plastic rim of the flywheel under consideration, which corresponds to the limiting rotation frequencies, is shown in Fig. 3 (relationships 3 and 4): radial stresses σ_r for $\omega^d_{r\,max} = 5.82 \cdot 10^4$ rpm and circumferential stresses σ_φ for $\omega^d_{\varphi\,max} = 6.06 \cdot 10^4$ rpm. Maximum radial stresses $\sigma_{r\,max}$ = 14 MPa develop on the radius $\hat{r}$ = 1.1.

Bench tests of the flywheel model that we have presented, which were performed by the Zhitomir Branch of the Kiev Polytechnic Institute, suggest that its ultimate bearing capacity is exhausted at a rotation frequency of 50,000 rpm. This can be considered to be in good agreement with the results of computations, bearing in mind the statistical character of the physicomechanical properties of composite materials.

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