UDC 519.626, 519.6,004.942:519.876.2
DOI: 10.32626/2308-5916.2019-20.16-25

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# METHODS OF COMPLEX DYNAMIC SYSTEMS' MODELS' EQUIVALENT CONVERSION

Proposed and considered the formal description of equivalent conversions which can be applied to obtain the various models of different kinds of complex dynamic systems (including electrical systems, power installations, etc.), as well as for transition from one representation to another. The set of basic operations which realize elementary conversions of models is described. The methods and algorithms for conversion of differential equations into integral or integro-differential are considered.

**Key words:** *Dynamic systems, model conversion, integral equations.*

**Introduction.** For investigation of dynamic systems their modeling on the base of differential equations is used in most cases. Appropriate models and methods of their solution are well known and widely applied in practice. At the same time, it is not always evident what kind of model is better to use for a particular system. Selection of the adequate and at the same time enough simple model is actually an art from many points of view. Besides differential equations, there are many other means for description of dynamic systems. Those are, primarily, integral and integro-differential equations. For many problems integral equations are preferable than differential ones. Thus, it is obviously important to create and develop mathematical methods and computer tools which would allow to convert one model description to another.

Formalization of equivalent conversions' description. Let us consider the formalized description of equivalent conversions which can be applied to obtain the various models of the researched dynamic system and for transition from one representation to another.

Let we have an operator model defined by the equation $\Phi_1(u) = \Phi_2(u)$. The basic operations realizing elementary conversions of the model are:

- the additive conversion

$$\Phi_1(u) = \Phi_2(u) \Rightarrow \Phi_1(u)+\Phi_3(u) = \Phi_2(u)+\Phi_3(u);$$

- the multiplicative conversion

$$\Phi_1(u) = \Phi_2(u) \Rightarrow \Phi_1(u)\ \Phi_3(u) = \Phi_2(u)\ \Phi_3(u);$$

 

- the additive splitting

$$\Phi_1(u) = \Phi_2(u) \Rightarrow \Phi_{11}(u) + \Phi_{12}(u) = \Phi_2(u);$$

- the multiplicative splitting

$$\Phi_1(u) = \Phi_2(u) \Rightarrow \Phi_{11}(u)\ \Phi_{12}(u) = \Phi_2(u);$$

- the partial additive inversion

$$\Phi_1(u) = \Phi_2(u) \Rightarrow u = \Phi^{-1}{}_{11}(u)\ (\Phi_2(u) - \Phi_{12}(u));$$

- the partial multiplicative inversion

$$\Phi_1(u) = \Phi_2(u) \Rightarrow \Phi_{12}(u) = \Phi^{-1}{}_{11}(u)\ (\Phi_2(u)).$$

Combining these basic operations, we can obtain more complex conversions of mathematical models.

Let's consider some realizations of algorithms of equivalent converting differential equations to integral or integro-differential ones [1, 2, 4]. In general case, it should be noted that precise reverse transition from integral to differential form of mathematical model is not always possible. The integral form of mathematical models representation is more universal than the differential one. It allows to describe much more physical objects, both with lumped and distributed parameters.

Method of analytical inversion with operator splitting. Let a model of the object is given in a form of the ordinary differential equation (ODE)

$$D[y] \equiv y^n(t) + \sum_{i=1}^{n} a_i y^{(n-i)}(t) = f(t),\ y^{(i)}(0) = C_i,\ i = \overline{0, n-1}, \tag{1}$$

or, in the functional form,

$$D[y] = f. \tag{2}$$

To obtain a series of equivalent integral dynamic models [3, 5, 6], i. e. relations containing integral operators, rather general method based on different versions of splitting the initial differential operator can be applied. Indeed, splitting the operator $D$ into two operators, i. e. putting $D = D_1 + D_2$, we obtain the following differential equation

$$D_1[y] = \psi, \tag{3}$$

where $\psi(t) = f(t) - D_2[y]$. Choosing such form of decomposition which admits analytical solution (3) is available, can allow us to get the equation

$$y = D_1^{-1}[\psi], \tag{4}$$

The operator $D_1^{-1}$ which is inverse to $D_1$ is in integral operator, therefore (4) is the integral or integro-differential equation.

The considered method of equivalent conversion can be applied both to linear and nonlinear problems. If for example the nonlinear differential equation $D_n[y] = f$ with a given nonlinear operator $D_n$, then for its decomposition it is reasonable to separate its linear part, i. e. to use the representation $D_n = D_1 + D_{2n}$, where $D_1$ is a linear operator. Then the initial equation is reduced to the form (4), which is generally the nonlinear integro-differential equation.

Let's consider this method in more details on the example of the equation (1), which can be rewritten as

$$y^{(n)}(t)+\sum_{i=1}^{m} a_i y^{(n-i)}(t)=f(t)-\sum_{i=m+1}^{n} a_i y^{(n-i)}(t). \qquad (5)$$

After the substitution of variables

$$u(t)=y^{(n-m)}(t),\ u'(t)=y^{(n-m+1)}(t),...,u^{(m)}(t)=y^{n}(t), \qquad (6)$$

we obtain the *m*-th order equation

$$u^{(m)}(t)+\sum_{i=1}^{m} a_i u^{(i-1)}(t)=\psi(t), \qquad (7)$$

where

$$\psi(t)=f(t)-\sum_{i=m+1}^{n} a_i y^{(n-i)}(t). \qquad (8)$$

Converting the equation (7) to an equivalent the 1st order ODEs system and building its solution, e. g. using fundamental solutions of this system, we obtain the equation with exponential kernel:

$$u(t)=e^{At}u_0+\int_0^t e^{At}\Phi(a,u,\tau), \qquad (9)$$

where

$$u(t) = [u'(t), u''(t), \dots, u^{(m)}(t)],\ u_0(t)=[u'(0), u''(0), \dots, u^{(m)}(0)],$$

$$\Phi(a, u, t)=[0, 0, \dots, \psi(t)],$$

and we get the following *m*-th order matrix *A*:

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -a_m & -a_{m-1} & -a_{m-2} & -a_1 \end{pmatrix}.$$

The unknown variables in the equations (1) and (9) are connected by the relation

$$y(t)=\geq \underbrace{\int_0^t \dots \int_0^t}_{n-m} u(s)\,ds=\int_0^t \frac{(t-s)^{n-m-1}}{(n-m-1)!}u(s)\,ds. \qquad (10)$$

The transition from one form of a model to another is carried by modifying value of *m* from 1 to *n*.

*The method of sequential integration.* If we put $m = n$ in the regarded method, decomposition of the operator *D* is reduced to solution of the initial equation with respect to the higher derivative. In this case the solution of the equation (3) is carried out by *n* sequential integrations. As the result we get the following integral equation:

$$y(t)+\int_0^t K(t-s)\,y(s)\,ds = F(t), \tag{11}$$

where

$$K(t-s)=\sum_{i=1}^{n} q_i \frac{(t-s)^{i-1}}{(i-1)!},$$

$$F(t)=\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)\,ds+\sum_{i=1}^{n-1} C_i \frac{t^i}{i!}+C_0\sum_{i=1}^{n-1} q_i \frac{t^i}{i!}+\ldots+ \tag{12}$$
$$+C_1\sum_{i=1}^{n-2} q_i \frac{t^{i+1}}{(i+1)!}+\ldots+C_{n-2}\frac{t^{n-1}}{(n-1)!}$$

*The method of higher derivative.* This method is usually considered in the literature. It represents a special case of the splitting method based on the substitution

$$u(t)=y^{(n)}(t),\ \int_0^t u(s)\,ds+c_1=y^{(n-1)}(t),\ldots$$

It allows to obtain the equivalent integral equation with respect to the higher derivative of the initial equation (1):

$$y^{(n)}(t)+\int_0^t \sum_{k=1}^{n} a_k \frac{(t-s)^{k-1}}{(k-1)!} y^{(n)}(s)\,ds=\varphi(t),$$
$$\varphi(t)=f(t)-C_{n-1}a_1-(C_{n-1}t+C_{n-2})\,a_2-\ldots- \tag{14}$$
$$-\left(C_{n-1}\frac{t^{n-1}}{(n-1)!}+\ldots+C_1t+C_0\right)a_n.$$

The analytical methods of equivalent transition from the ordinary differential equations to integral ones can be effectively implemented using the packages oriented to analytical conversions (Mathematica, Maple etc.).

The structure of the algorithm which allows to carry out the most general method of analytical inversion with operator splitting is shown in the Fig. 1.

This method at $m = n$ is reduced to the sequential integration method, and at $m = 0$ — to the higher derivative method. In general case it allows us to obtain an integro-differential equation, and in the two last special cases we obtain pure integral equations.

It is useful to develop intelligent program environments for selecting appropriate models oriented to simulation and modeling of dynamic systems. Some approaches to solving this problem were discussed in [3].

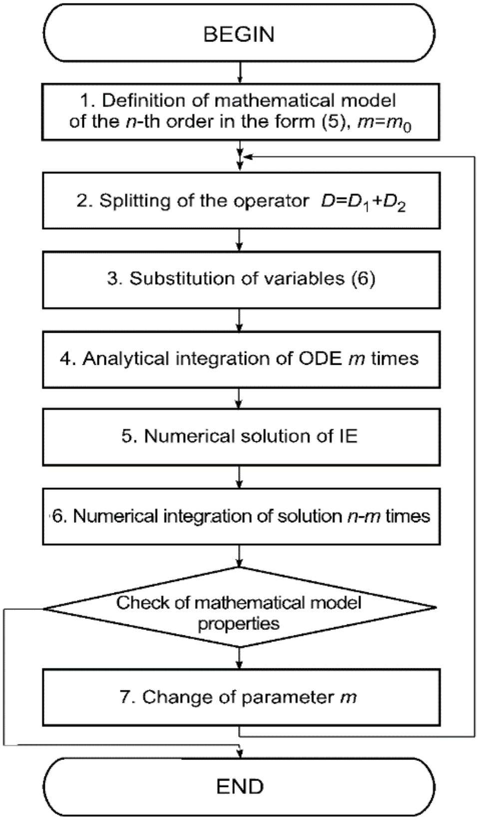


***Fig. 1.*** *Algorithm of analytical inversion with operator splitting*

Comparing the sequential integration method and the higher derivative method, we have to note that only in the first respect to the required function $u(t) = y(t)$ is obtained. In the rest cases (at $m < n$) we obtain the equation with respect to derivative of required function, and obtained solution should be integrated $m - n$ times.

Continuing a comparison of the methods of transition from differential to integral form of mathematical models, we have to note that for the sequential integration method in algorithm in the Fig. 1 the item 6, and in the higher derivative method the item 4 are omitted. Anyway the problem of numerical integration remains. In the sequential integration method we have to integrate the right hand side of the differential equation, and in the higher derivative method — the obtained solution.

For an investigation of dynamic models with approximate initial data (for example, obtained by measurements) the sequential integration method is preferable. In this case approximate initial data are integrated on the first step of algorithm. In this case the influence of errors in initial data (especially if errors look like a white noise) can be considerably reduced.

*Conversion of nonlinear models.* Let's consider a possibility to obtain the equivalent integral equations for nonlinear object in the case when one of derivative is included into the initial differential equation under the sign of continuous nonlinear function, i. e. when the nonlinear differential equation has the form

$$\sum_{i=0}^{n} a_i(t)\, y^{(i)}(t) + F\left(y^{(m)}(t)\right) = f(t), \qquad (15)$$

with the initial conditions $y^{(i)}(t_0) = C_i$, and continuous variable coefficients $a_i(t)$, $i = 0, \ldots, n-1$.

We suppose that $a_n = 1$, $a_m = 0$ and $m \neq n$.

An integral equation with respect to the $m$-th derivative can be obtained integrating the equation (15) $n - m$ times.

Let's consider at first the case of $m = 0$. Integrating the equation (15) $n + 1$ times, we reduce it to the following form that does not contain derivatives:

$$\int_{t_0}^{t}\left[1 + \int_{s}^{t} M_n(\xi, s)\, d\xi\right] y(s)\, ds + \int_{t_0}^{t} \frac{(t-s)^n}{n!} F\left(y(s)\right) ds = \int_{t_0}^{t} \Phi_n(s)\, ds. \qquad (16)$$

Here

$$M_n(\xi, s) = \sum_{i=1}^{n} (-1)^i \sum_{j=1}^{n-1} C_{n-i}^{j} a_{n-i}^{(j)}(s) \frac{(s-\xi)^{i+j-1}}{(i+j-1)!},$$

$$\Phi_n(t) = \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s)\, ds + P_n(t),$$

$$P_n(t) = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (1)^i\, a_k^{(i)} \sum_{j=1}^{k-i} C_{j-1} C_{k-j}^{i} \frac{(t-t_0)^{i+j-1}}{(i+j-1)!},$$

$$a_k^{(i)} = \frac{d^i}{dt^i} a_k(t)\bigg|_{t=s}, \quad C_n^m = \frac{n!}{m!(n-m)!}.$$

In a special case when the higher derivative is included into the equation under the sign of continuous nonlinear function $F$, i. e. when the nonlinear differential equation has the form

$$F\left(y^{(n)}(t)\right) + \sum_{i=0}^{n-1} a_i(t)\, y^{(i)}(t) = f(t), \qquad (17)$$

with the initial conditions $y^{(i)}(t_0) = C_i$ and continuous variable coefficients $a_i(t)$, $i = 0, \ldots, n-1$, the equivalent integral equation can be written as

$$F\left(Z_n(t)\right) - \int\limits_{t_0}^{t} K_n(t,s) Z_n(s)\,ds = \varphi_n(t), \tag{18}$$

where

$$K_n(t,s) = -\sum_{i=1}^{n-1} \frac{\left(t-t_0\right)^{n-i-1}}{(n-i-1)!} a_i(t),$$

$$Z_n(t) = y^{(n)}(t),$$

$$\varphi_n(t) = y(t) - \sum_{i=0}^{n-1}\sum_{j-1}^{n-1} a_i(t) \frac{\left(t-t_0\right)^{n-j-1}}{(n-j-1)!} C_{i-j-1}.$$

The equation (18) is obtained by substituting the derivatives $y^{(i)}(t)$, expressed in terms of $y^{(n)}(t)$, into (17):

$$y^{(i)}(t) = \int\limits_{t_0}^{t} \frac{(t-s)^{n-i-1}}{(n-i-1)!} Z_n(s)\,ds + \sum_{j=1}^{n-i} \frac{\left(t-t_0\right)^{n-i-j}}{(n-i-j)!} C_{i+j-1}.$$

In the case $m \neq 0$ the equation (15) can be represented as

$$\sum_{i=m+1}^{n} a_i(t) y^{(i)}(t) + F\left(y^{(m)}(t)\right) + \sum_{i=0}^{m-1} a_i(t) y^{(i)}(t) = f(t).$$

Taking into account (18) it can be written in the following form:

$$\sum_{i=0}^{n-m-1} a_{m-1+i}(t) Z_m^{(i+1)}(t) + F\left(Z_m(t)\right) - \int\limits_{t_0}^{t} K_m(t,s) Z_m(s)\,ds = \varphi_m(t). \tag{19}$$

Sequentially integrating (19) and considering (16), we obtain the nonlinear integral equation

$$\int\limits_{t_0}^{t} \left[1 - \int\limits_{s}^{t} M_{n-m-1}(\xi,s)\,d\xi\right] Z_m(s)\,ds + \int\limits_{t_0}^{t} \frac{(t-s)^{n-m-1}}{(n-m-1)!} F\left(Z_m(s)\right) ds -$$

$$-\int\limits_{t_0}^{t} I^{n-m-1}\left[K_m(t,s)\right] Z_m(s)\,ds = \psi(t),$$

where

$$\psi(t) = \int\limits_{t_0}^{t} \left[\int\limits_{t_0}^{t} \frac{(t-s)^{n-m-1}}{(n-m-1)!} \varphi_m(s)\,ds + Pn-n-1(t)\right] dt,$$

$I^{n-m-1}[K_m(t, s)]$ means the application of the operator

$$I\left[K_m(t,s)\right] = \int\limits_{s}^{t} K_m(\xi,s)\,d\xi$$

$n - m - 1$ times. The operator $I$ arises when the integration limits are changed:

$$\int_{t_0}^{t} ds \int_{t_0}^{s} K_m(s,\xi) Z_m(\xi) d\xi = \int_{t_0}^{t} Z_m(\xi) d\xi \int_{\xi}^{t} K_m(s,\xi) ds =$$
$$= \int_{t_0}^{t} Z_m(s) ds \int_{s}^{t} K_m(\xi,s) d\xi = \int_{t_0}^{t} Z_m(s) I\left[K_m(t,s)\right] ds.$$

Thus, the integral equation equivalent to (15) has the following form

$$\int_{t_0}^{t} \frac{(t-s)^{n-m-1}}{(n-m-1)!} F\left(Z_m(s)\right) ds - \int_{t_0}^{t} Q(t,s) Z_m(s) ds = \psi(t), \qquad (20)$$

where

$$Q(t,s) = -1 + \int_{s}^{t} M_{n-m-1}(\xi,s) d\xi + I^{n-m-1}\left[K_m(t,s)\right], \left(m = \overline{0,n}\right).$$

It follows from the formulas (16), (18), (20), that the left-hand side of the integral equation, equivalent to the given nonlinear differential equation, in which one of derivatives is included nonlinearly, consists of two components. One of these components is the nonlinear with respect to the function $y^{(m)}(t)$, and the second one is an application of integral operator with the kernel $Q(t-s)$ to $y^{(m)}(t)$.

If a model is represented by a system of differential equations, we have more wide opportunities to transform it to integral or integro-differential form then in the case of a single equation. Every equation in the system can be transformed in different way, equations can be combined etc. We can also reduce the model dimension decreasing the number of governing equations. This can be done more flexible than if we use the differential approach: the higher derivatives demanding special treatment at numerical solution do not appear, and resulting integral equations are solved in usual way.

To illustrate these opportunities, we consider the following example. The simple quarter car model of automotive suspension (Fig. 2) is described by following system of ODEs:

$$\begin{aligned} m_b \ddot{x}_b &= -C_{s1}\left(x_b - x_w - x_{s2}\right) - m_b g, \\ m_w \ddot{x}_w &= C_{s1}\left(x_b - x_w - x_{s2}\right) - C_t\left(x_w - x_r\right) - \mu_t\left(\dot{x}_w - \dot{x}_r\right) - m_w g, \end{aligned} \qquad (21)$$

where $m_b$ and $m_w$ are masses of the body and the wheel, $x_b$, $x_w$ are their displacements, $x_{s2}$ is a displacement in the second section used as the laminated spring model, $C_i$, $\mu_i$ are stiffnesses and viscosities in the models of the spring and the tire, $F_{fr}$ is the friction force in the spring model.

Instead of the last equation in the system (21), on the stage when the second element of spring model is deformed we can use a differential equation with respect to the spring tension $F_s = C_{s1}(x_b - x_w - x_{s2})$:

$$\dot{F}_s + \frac{1}{n} F_s = C_{s1} \left( \dot{\overline{x}} + \frac{C_{s2} \overline{x} + F_f \, sgn(\dot{x}_{s2})}{\mu_s} \right), \tag{22}$$

where $n = \dfrac{\mu_s}{C_{s1} + C_{s2}}$ is a characteristic relaxation time [1], and $\overline{x} = x_b - x_w$ is the spring deformation.

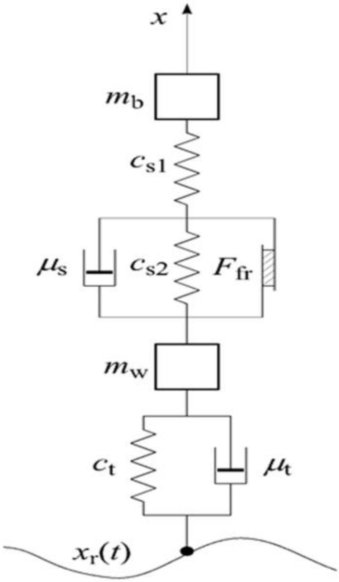


***Fig. 2.*** *Automotive suspension*

Solving this equation analytically, we obtain the following relation which includes integral operators with the relaxation kernel $e^{-\frac{t-s}{n}}$ :

$$F_s(t) = (F_{s0} - C_{s1} \overline{x}_0) e^{-\frac{t-t_0}{n}} +$$
$$+ C_{s1} \left[ \overline{x}(t) - \frac{C_{s1}}{\mu_s} \int_{t_0}^{t} e^{-\frac{t-s}{n}} \overline{x}(s) ds + \int_{t_0}^{t} \frac{F_{fr}}{\mu_s} sgn(\dot{x}_{s2}(s)) e^{-\frac{t-s}{n}} \overline{x}(s) ds \right]. \tag{23}$$

The substitution of (23) into the dynamic equations (the first two ones in the system (21) leads to a system of integro-differential equations. Integrating them twice, we obtain two Volterra integral equations of the 2nd kind [4].

**Conclusions.** Using the proposed approach, we can obtain integral equations for each of the variables $x_b$, $x_w$ and $x_{s2}$ or $F_s$ by the sequential integration method. Also, we can obtain the simple system of integral equations (but with larger number of equations) by transition from (21) to the 1st order ODEs system and integrating it. One integral equation with respect to a single governing function (e. g. the car's body displacement or acceleration) can be obtained instead of two dynamic equations.

Thus, the obtained equivalent conversions' formalized description along with the proposed conversion algorithms provide the possibility to create various models of different kind of complex dynamic systems and make convenient and effective conversion of the models from one representation to another, i.e. from differential equations to integral or integro-differential.

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## МЕТОДИ ЕКВІВАЛЕНТНОГО ПЕРЕТВОРЕННЯ МОДЕЛЕЙ СКЛАДНИХ ДИНАМІЧНИХ СИСТЕМ


Запропоновано та розглянуто формальний опис еквівалентних перетворень, які можна застосувати для отримання різних моделей складних динамічних систем (включаючи електричні системи, енергоустановки тощо), а також для переходу від одного представлення до іншого. Описано набір основних операцій, які реалізують перетворення моделей. Розглянуто методи та алгоритми перетворення диференціальних рівнянь в інтегральні та інтегро-диференціальні.



**Ключові слова:** *динамічні системи, перетворення моделі, інтегральні рівняння.*

Отримано: 21.08.2019