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APPLICATIONS OF PERTURBATIONS ON ACCRETIVE MAPPINGS TO NONLINEAR ELLIPTIC SYSTEMS INVOLVING (p, q)-LAPLACIAN

ЗАСТОСУВАННЯ ЗБУРЕНЬ АКРЕТИВНИХ ВІДОБРАЖЕНЬ ДО НЕЛІНІЙНИХ ЕЛІПТИЧНИХ СИСТЕМ, ЩО МІСТЯТЬ (p, q)-ЛАПЛАСІАН

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Using perturbation results on sums of ranges of nonlinear accretive mappings of Calvert and Gupta, we present some abstract results about the existence of solutions of nonlinear Neumann elliptic systems involving (p, q)-Laplacian. The systems discussed in this paper and the method used extend and complement some of the previous work.

З використанням результатів, отриманих Калвертом та Гуптою, про збурення сум образів нелінійних акретивних відображень, наведено деякі абстрактні результати про існування розв'язків нелінійних еліптичних систем Неймана, що містять (p, q)-лапласіан. Розглянуті системи та використані методи продовжують та доповнюють результати попередніх робіт.

1. Introduction. By using perturbation results on the ranges of m-accretive mappings, Calvert and Gupta [1] and Gupta and Hess [2] provided sufficient conditions so that some nonlinear boundary-value problems involving the Laplacian operator have solutions in $L^p(\Omega)$. Inspired by their ideas, we obtained a sufficient condition in [3] so that the zero boundary-value problem

$$\begin{aligned} -\Delta_p u + g(x, u(x)) &= f(x) \quad \text{a.e. in} \quad \Omega, \\ -\frac{\partial u}{\partial n} &= 0 \quad \text{a.e. on} \quad \Gamma \end{aligned} \tag{1.1}$$

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has solutions in $L^p(\Omega)$, where $2 \leq p < +\infty$. In [4], we studied (1.1) in a more general space $L^p(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$, and the restrictions are weaker than those in [3]. In [5], we showed that the nonlinear boundary-value problem

$$\begin{aligned} &-\Delta_p u + g(x, u(x)) = f(x) \quad \text{a.e. in} \quad \Omega, \\ &-\left\langle \vartheta, |\nabla u|^{p-2}\nabla u \right\rangle \in \beta_x(u(x)) \quad \text{a.e. on} \quad \Gamma \end{aligned} \tag{1.2}$$

has solutions in $L^p(\Omega)$, where $2 \leq p < +\infty$. We further showed that (1.2) has solutions in $L^s(\Omega)$, where $\max\{N, 2\} \leq p \leq s < +\infty$ and $N \geq 1$, see [6]. In [7], we used the ideas of [6] and continued our studies of (1.2) in the Hilbert space $L^2(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ for $N \geq 1$. Recently, we examined the following boundary-value problem:

$$\begin{aligned} &-\Delta_p u + |u|^{p-2}u + g(x, u(x)) = f(x) \quad \text{a.e. in} \quad \Omega, \\ &-\left\langle \vartheta, |\nabla u|^{p-2}\nabla u \right\rangle \in \beta_x(u(x)) \quad \text{a.e. on} \quad \Gamma \end{aligned} \tag{1.3}$$

which in addition contains a perturbation term $|u|^{p-2}u$. In [8], we proved that (1.3) has solutions in $L^s(\Omega)$, where $\frac{2N}{N+1} < p \leq 2$ for $N \geq 1$ and $2 \leq s < +\infty$. Further, in [9] we showed that (1.3) has solutions in $L^p(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ for $N \geq 1$. In [10] and [11], we extended our work to the following problem

$$\begin{aligned} &-\operatorname{div}\left[\left(C(x) + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u\right] + |u|^{p-2}u + g(x, u(x)) = f(x) \quad \text{a.e. in} \quad \Omega, \\ &-\left\langle \vartheta, (C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right\rangle \in \beta_x(u(x)) \quad \text{a.e. on} \quad \Gamma. \end{aligned} \tag{1.4}$$

In [10], we established that (1.4) has solutions in $L^p(\Omega)$, where $2 \leq p < +\infty$; and in [11] we proved that (1.4) has solutions in $L^s(\Omega)$, where $\max\{N, 2\} \leq p \leq s < +\infty$. Clearly, if $C(x) \equiv 0$, then equation (1.4) reduces to contain only p-Laplacian operators.

As a summary of our previous work, in [12] we studied the following problem:

$$\begin{aligned} &-\operatorname{div}\left[(C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\right] + \varepsilon|u|^{q-2}u + g(x, u(x)) = f(x) \quad \text{a.e. in} \quad \Omega, \\ &-\left\langle \vartheta, (C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right\rangle \in \beta_x(u(x)) \quad \text{a.e. on} \quad \Gamma, \end{aligned} \tag{1.5}$$

where $0 \leq C(x) \in L^p(\Omega)$, ε is a non-negative constant and ϑ denotes the exterior normal derivative of Γ. We showed that (1.5) has solutions in $L^s(\Omega)$ under some conditions, where $\frac{2N}{N+1} < p \leq s < +\infty$, $1 \leq q < +\infty$ if $p \geq N$, and $1 \leq q \leq \frac{Np}{N-p}$ if $p < N$, for $N \geq 1$.

In this paper, we shall extend our study to the case of nonlinear Neumann elliptic systems. Specifically, by using perturbations of accretive mappings, we shall find a sufficient condition for the existence of a solution in $L^p(\Omega) \times L^q(\Omega)$ of the following system:

$$\begin{aligned}
&-\Delta_p u + \varepsilon_1 |u|^{p-2}u + g(x, u(x), v(x)) = f_1(x) \quad \text{a.e. in} \quad \Omega, \\
&-\Delta_q v + \varepsilon_2 |v|^{q-2}v + g(x, v(x), u(x)) = f_2(x) \quad \text{a.e. in} \quad \Omega, \\
&-\left\langle \vartheta, |\nabla u|^{p-2}\nabla u \right\rangle \in \beta_x(u(x)) \quad \text{a.e. on} \quad \Gamma, \\
&-\left\langle \vartheta, |\nabla v|^{q-2}\nabla v \right\rangle \in \beta_x(v(x)) \quad \text{a.e. on} \quad \Gamma.
\end{aligned} \tag{1.6}$$

Here, Ω is a bounded conical domain of the Euclidean space R^N with its boundary $\Gamma \in C^1$ (see [5]), $f_1(x) \in L^p(\Omega)$ and $f_2(x) \in L^q(\Omega)$ are given functions, ε_1 and ε_2 are non-negative constants, and ϑ denotes the exterior normal derivative of Γ. We shall assume that the Green's formula is available.

2. Preliminaries. Let X be a real Banach space with a strictly convex dual space X'. We shall use „$\to$” and „$w-\lim$” to denote strong and weak convergences, respectively. For any subset G of X, we denote by $\operatorname{int} G$ its interior and $\overline{G}$ its closure, respectively. A mapping $T : X \to X'$ is said to be *hemi-continuous* on X if $w - \lim\limits_{t\to 0} T(x+ty) = Tx$ for any $x, y \in X$.

Let J denote the *duality mapping* from X into $2^{X'}$ defined by

$$J(x) = \{f \in X' : (x, f) = \|x\|\|f\|, \ \|f\| = \|x\|\} \quad \forall x \in X,$$

where $(\cdot,\cdot)$ denotes the generalized duality pairing between X and X'. Since X' is strictly convex, J is a single-valued mapping.

A multivalued mapping $A : X \to 2^X$ is said to be *accretive* if $(v_1 - v_2, J(u_1 - u_2)) \geq 0$, for any $u_i \in D(A)$ and $v_i \in Au_i$, $i = 1, 2$. An accretive mapping A is said to be m*-accretive* if $R(I + \lambda A) = X$ for some $\lambda > 0$. We say that $A : X \to 2^X$ is *boundedly-inversely-compact* if, for any pair of bounded subsets G and G' of X, the subset $G \bigcap A^{-1}(G')$ is relatively compact in X.

A multivalued operator $B : X \to 2^{X'}$ is said to be *monotone* if its graph $G(B)$ is a monotone subset of $X \times X'$ in the sense that $(u_1 - u_2, w_1 - w_2) \geq 0$ for any $[u_i, w_i] \in G(B)$, $i = 1, 2$. A monotone operator B is said to be *maximal monotone* if $G(B)$ is maximal among all monotone subsets of $X \times X'$ in the sense of inclusion.

Definition 2.1 [1]. *The duality mapping $J : X \to X'$ is said to satisfy condition (2.1) if there exists a function $\eta : X \to [0, +\infty)$ such that for $u, v \in X$,*

$$\|Ju - Jv\| \leq \eta(u - v). \tag{2.1}$$

Lemma 2.1 [1]. *Let Ω be a bounded domain in R^N and let $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ denote the duality mapping. Then, J_p satisfies condition (2.1). Moreover, for $2 \leq p < +\infty$, $J_p u = |u|^{p-1}\operatorname{sgn} u \|u\|_p^{2-p}$ $\forall u \in L^p(\Omega)$; for $1 < p \leq 2$, $J_p u = |u|^{p-1}\operatorname{sgn} u$ $\forall u \in L^p(\Omega)$, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.*

Definition 2.2 [1]. *Let $A : X \to 2^X$ be an accretive mapping and $J : X \to X'$ be a duality mapping. We say that A satisfies condition (2.2) if, for any $f \in R(A)$ and $a \in D(A)$, there exists a constant $C(a, f)$ such that, for any $u \in D(A)$, $v \in Au$,*

$$(v - f, J(u - a)) \geq C(a, f). \tag{2.2}$$

Lemma 2.2 [1]. *Let Ω be a bounded domain in R^N and $g : \Omega \times R \to R$ be a function satisfying Carathéodory's conditions (see [13]) such that*

(i) $g(x, \cdot)$ is monotonically increasing on R;

(ii) the mapping $u \in L^p(\Omega) \to g(x, u(x)) \in L^p(\Omega)$, $1 < p < +\infty$, is well defined.

Then, the mapping $B : L^p(\Omega) \to L^p(\Omega)$ defined by $(Bu)(x) = g(x, u(x))$, for any $x \in \Omega$, satisfies condition (2.2).

Theorem 2.1 [1]. *Let X be a real Banach space with a strictly convex dual X'. Let $J : X \to \to X'$ be a duality mapping on X satisfying condition (2.1). Let $A, C_1 : X \to 2^X$ be accretive mappings such that*

(i) either both A and C_1 satisfy condition (2.2), or $D(A) \subset D(C_1)$ and C_1 satisfies condition (2.2);

(ii) $A + C_1$ is m-accretive and boundedly-inversely-compact.

If $C_2 : X \to X$ is a bounded continuous mapping such that, for any $y \in X$, there is a constant $C(y)$ satisfying $(C_2(u + y), Ju) \geq -C(y)$ for any $u \in X$, then

(a) $\overline{[R(A) + R(C_1)]} \subset \overline{R(A + C_1 + C_2)}$;

(b) $\operatorname{int}[R(A) + R(C_1)] \subset \operatorname{int} R(A + C_1 + C_2)$.

We also require the following basic results for the product space, which can be found in [14].

The product space of Banach spaces X_1 and X_2, which is denoted by $X_1 \times X_2$, is a set of all (ordered) pairs (x_1, x_2) of elements where $x_1 \in X_1$ and $x_2 \in X_2$. $X_1 \times X_2$ is a vector space if the linear operation is defined by

$$k_1(x_1, y_1) + k_2(x_2, y_2) = (k_1x_1 + k_2x_2, k_1y_1 + k_2y_2).$$

Furthermore, $X_1 \times X_2$ becomes a normed space if the norm is defined by

$$\|(x_1, x_2)\| = (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}}.$$

Other choices of the norm are possible, we employ the above norm mainly because it ensures that $(X_1 \times X_2)' = X_1' \times X_2'$, where X_1' and X_2' are dual spaces of X_1 and X_2, respectively. $(X_1 \times X_2)' = X_1' \times X_2'$ means the following: (i) each element $(f, g) \in X_1' \times X_2'$ defines an element $F \in (X_1 \times X_2)'$ by $((x_1, x_2), F) = (x_1, f) + (x_2, g)$ and, conversely, each $F \in (X_1 \times X_2)'$ is expressed in this form by a unique $(f, g) \in X_1' \times X_2'$; (ii) the norm of the above $F \in (X_1 \times X_2)'$ is exactly equal to $\|(f, g)\| = (\|f\|^2 + \|g\|^2)^{\frac{1}{2}}$.

It is easily seen that $X_1 \times X_2$ is a Banach space since both X_1 and X_2 are Banach spaces.

3. Main results. In this paper, unless otherwise stated, we shall assume that

$$\frac{2N}{N+1} < p < +\infty \quad \text{and} \quad \frac{2N}{N+1} < q < +\infty,$$

where $N \geq 1$. We shall use $\|\cdot\|_p$ and $\|\cdot\|_q$ to denote the norms of spaces $L^p(\Omega)$ and $L^q(\Omega)$, respectively.

Let $\varphi : \Gamma \times R \to R$ be a given function such that, for each $x \in \Gamma$, $\varphi_x = \varphi(x, \cdot) : R \to R$ is a proper, convex and lower-semi-continuous function with $\varphi_x(0) = 0$. Let β_x be the subdifferential of φ_x, i.e., $\beta_x \equiv \partial\varphi_x$. Suppose that $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Suppose that $g : \Omega \times R \times R \to R$ is a given function satisfying Carathéodory's conditions such that for any $1 < p < +\infty$, the mapping $u(x) \in L^p(\Omega) \to g(x, u(x), v(x)) \in L^p(\Omega)$ is well-defined for all fixed $v(x) \in L^q(\Omega)$ and for any $1 < q < +\infty$, the mapping $v(x) \in L^q(\Omega) \to$ $\to g(x, v(x), u(x)) \in L^q(\Omega)$ is well-defined for all fixed $u(x) \in L^p(\Omega)$. We shall also assume that there exists a function $0 \leq T(x) \in L^{\min(p,q)}(\Omega)$ such that $g(x, s, t)t \geq 0$, for $|t| \geq T(x)$, $x \in \Omega$, and for fixed number $s \in R$; and $g(x, s, t)s \geq 0$, for $|s| \geq T(x)$, $x \in \Omega$, and for fixed number $t \in R$.

Our main idea to tackle (1.6) is to apply Theorem 2.1, similar arguments are also used in [3–12].

To begin, let Y denote the product of two spaces $L^p(\Omega)$ and $L^q(\Omega)$, i.e., $Y = L^p(\Omega) \times$ $\times L^q(\Omega) = \{(u, v) : u \in L^p(\Omega), v \in L^q(\Omega)\}$. The space Y will be endowed with the norm

$$\|(u, v)\| = \sqrt{\|u\|_p^2 + \|v\|_q^2}, \quad \text{for} \quad (u, v) \in L^p(\Omega) \times L^q(\Omega).$$

The dual space of Y will be denoted by Y'.

We also use $\langle\cdot,\cdot\rangle$ and $|\cdot|$ to denote the Euclidean inner-product and Euclidean norm in R^N.

Lemma 3.1 [9, 12]. *Define the mapping $B_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ by*

$$(w, B_p u) = \int_\Omega \langle|\nabla u|^{p-2}\nabla u, \nabla w\rangle\, dx + \varepsilon \int_\Omega |u(x)|^{p-2}u(x)\, w(x)\, dx$$

for any $u, w \in W^{1,p}(\Omega)$. Then, B_p is everywhere defined, monotone, hemi-continuous and coercive.

Similarly, the mapping $B_q : W^{1,q}(\Omega) \to (W^{1,q}(\Omega))'$ defined by

$$(w, B_q v) = \int_\Omega \langle|\nabla v|^{q-2}\nabla v, \nabla w\rangle\, dx + \varepsilon \int_\Omega |v(x)|^{q-2}v(x)\, w(x)\, dx$$

for any $v, w \in W^{1,q}(\Omega)$, is also everywhere defined, monotone, hemi-continuous and coercive.

Lemma 3.2 [9]. *The mapping $\Phi_p : W^{1,p}(\Omega) \to R$ defined by $\Phi_p(u) = \int_\Gamma \varphi_x(u|_\Gamma(x))d\Gamma(x)$ for any $u \in W^{1,p}(\Omega)$, is proper, convex and lower-semi-continuous on $W^{1,p}(\Omega)$.*

Similarly, the mapping $\Phi_q : W^{1,q}(\Omega) \to R$ defined by $\Phi_q(v) = \int_\Gamma \varphi_x(v|_\Gamma(x))d\Gamma(x)$, for any $v \in W^{1,q}(\Omega)$, is also a proper, convex and lower-semi-continuous on $W^{1,q}(\Omega)$.

Lemma 3.3 [9, 12]. *Define the mapping $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ as follows:*

$$D(A_p) = \{u \in L^p(\Omega) \mid \text{ there exists an } f \in L^p(\Omega) \text{ such that } f \in B_p u + \partial\Phi_p(u)\}.$$

For $u \in D(A_p)$, we set $A_p u = \{f \in L^p(\Omega) \mid f \in B_p u + \partial\Phi_p(u)\}$. Then A_p is m-accretive.

Define the mapping $A_q : L^q(\Omega) \to 2^{L^q(\Omega)}$ as follows:

$$D(A_q) = \{v \in L^q(\Omega) \mid \quad \textit{there exists an} \quad g \in L^q(\Omega) \quad \textit{such that} \quad g \in B_q v + \partial\Phi_q(v)\}.$$

For $v \in D(A_q)$, we set $A_q v = \{g \in L^q(\Omega) \mid g \in B_q v + \partial\Phi_q(v)\}$. Then A_q is also m-accretive.

Definition 3.1. *Define the mapping $A_{p,q} : Y \to 2^Y$ as*

$$A_{p,q}(u,v) = (A_p u, A_q v) \quad \textit{for} \quad (u,v) \in Y.$$

Proposition 3.1. *The mapping $A_{p,q} : Y \to 2^Y$ is m-accretive.*

Proof. Let J_p denote the duality mapping from $L^p(\Omega)$ to $L^{p'}(\Omega)$, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, and let J_q denote the duality mapping from $L^q(\Omega)$ to $L^{q'}(\Omega)$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.

Let $J : Y \to Y'$ and $J(u,v) = (J_p u, J_q v)$ for $(u,v) \in Y$. We shall show that J is the duality mapping from Y to Y'. For this, let $(u,v) \in Y$, then

$$((u,v), J(u,v)) = ((u,v),(J_p u, J_q v)) = (u, J_p u) + (v, J_q v) = \|u\|_p^2 + \|v\|_q^2 = \|(u,v)\|^2.$$

Moreover, we have

$$\|J(u,v)\|^2 = \|J_p u\|^2 + \|J_q v\|^2 = \|u\|_p^2 + \|v\|_q^2 = \|(u,v)\|^2.$$

Thus, J is the duality mapping from Y to Y'.

Next, we shall show that $A_{p,q}$ is m-accretive. For this, let $u^* \in L^p(\Omega)$ and $v^* \in L^q(\Omega)$, then it follows from Lemma 3.3 that there exist $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ such that $u^* = u + \lambda A_p u$ and $v^* = v + \lambda A_q v$. Therefore,

$$(u^*, v^*) = (u,v) + \lambda(A_p u, A_q v) = (u,v) + \lambda A_{p,q}(u,v),$$

which implies that $R(I + \lambda A_{p,q}) = Y$.

To see that $A_{p,q}$ is accretive, let $u_i \in L^p(\Omega)$, $v_i \in L^q(\Omega)$, $i = 1, 2$, then from Lemma 3.3, we have

$$(A_{p,q}(u_1,v_1) - A_{p,q}(u_2,v_2), J((u_1,v_1) - (u_2,v_2))) =$$
$$= (A_p u_1 - A_p u_2, J_p(u_1 - u_2)) + (A_q v_1 - A_q v_2, J_q(v_1 - v_2)) \ge 0.$$

This completes the proof.

Proposition 3.2. *The duality mapping $J : Y \to Y'$ defined in Proposition 3.1 satisfies condition (2.1).*

Proof. From Lemma 2.1, we know that both J_p and J_q satisfy condition (2.1). Thus, there exists a function $\eta_1 : L^p(\Omega) \to [0, +\infty)$ such that

$$\|J_p u - J_p v\| \le \eta_1(u - v) \quad \forall u, v \in L^p(\Omega);$$

and there exists $\eta_2 : L^q(\Omega) \to [0, +\infty)$ such that

$$\|J_q w - J_q z\| \le \eta_2(w - z) \quad \forall w, z \in L^q(\Omega).$$

Define $\eta : Y \to [0, +\infty)$ as

$$\eta(a, b) = (\eta_1^2(a) + \eta_2^2(b))^{\frac{1}{2}} \quad \forall (a, b) \in Y.$$

Then for any $(u, w), (v, z) \in Y$, we have

$$\|J(u, w) - J(v, z)\| = \|(J_p u - J_p v, J_q w - J_q z)\| =$$

$$= (\|J_p u - J_p v\|^2 + \|J_q w - J_q z\|^2)^{\frac{1}{2}} \le \eta((u - v, w - z)).$$

This completes the proof.

Lemma 3.4 [15]. *Let X be a Banach space and $J : X \to X'$ be the duality mapping. Then, X is strictly convex if and only if*

$$x^* \in Jx, \quad y^* \in Jy, \quad x \ne y \Longrightarrow (x - y, x^* - y^*) > 0.$$

By using Lemma 3.4, we can easily get the following result.

Proposition 3.3. *The dual space Y' of $Y = L^p(\Omega) \times L^q(\Omega)$ is strictly convex.*

Lemma 3.5 [9, 12]. *Both the mappings $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ and $A_q : L^q(\Omega) \to 2^{L^q(\Omega)}$ have a compact resolvent, for $\frac{2N}{N+1} < p, q \le 2$ and $N \ge 1$.*

Proposition 3.4. *The mapping $A_{p,q} : Y \to 2^Y$ has a compact resolvent, for $\frac{2N}{N+1} < p, q \le 2$ and $N \ge 1$.*

Proof. Since $A_{p,q}$ is m-accretive, it suffices to prove that if $(u, v) + \lambda A_{p,q}(u, v) = (f, h)$, $\lambda > 0$, and $\{(f, h)\}$ is bounded in $L^p(\Omega) \times L^q(\Omega)$, then $\{(u, v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$.

In fact, from $(u, v) + \lambda A_{p,q}(u, v) = (f, h)$, we have $u + \lambda A_p u = f$ and $v + \lambda A_q v = h$. Since $\{(f, h)\}$ is bounded in $L^p(\Omega) \times L^q(\Omega)$, it follows that $\{f\}$ is bounded in $L^p(\Omega)$ and $\{h\}$ is bounded in $L^q(\Omega)$. Lemma 3.5 implies that $\{u\}$ is relatively compact in $L^p(\Omega)$ and $\{v\}$ is relatively compact in $L^q(\Omega)$. Therefore, $\{(u, v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$.

This completes the proof.

Remark 3.1 [9]. If $\beta_x \equiv 0 \ \forall x \in \Gamma$, then $\partial\Phi_p(u) \equiv 0 \ \forall u \in W^{1,p}(\Omega)$ and $\partial\Phi_q(v) \equiv 0 \ \forall v \in W^{1,q}(\Omega)$.

Lemma 3.6 [9, 12]. *If $\beta_x \equiv 0 \ \forall x \in \Gamma$, then we have*

(i) $\{f \in L^p(\Omega) | \int_\Omega f dx = 0\} \subset R(A_p)$ for $\frac{2N}{N+1} < p < +\infty$ and $N \ge 1$, and

(ii) $\{f \in L^q(\Omega) | \int_\Omega f dx = 0\} \subset R(A_q)$ for $\frac{2N}{N+1} < q < +\infty$ and $N \ge 1$.

Proposition 3.5. *If $\beta_x \equiv 0 \; \forall x \in \Gamma$, then we have*

$$\left\{ (f_1, f_2) \in L^p(\Omega) \times L^q(\Omega) \;\middle|\; \int\limits_{\Omega} f_1 \, dx = 0 = \int\limits_{\Omega} f_2 dx \right\} \subset R(A_{p,q}).$$

Proof. From the fact that $(f_1, f_2) \in L^p(\Omega) \times L^q(\Omega)$ with $\int_\Omega f_1 dx = 0 = \int_\Omega f_2 dx$, and Lemma 3.6, we have $f_1 \in R(A_p)$ and $f_2 \in R(A_q)$. Then $(f_1, f_2) \in R(A_{p,q})$ from the definition of $A_{p,q}$.

This completes the proof.

Definition 3.2 [1, 9, 12]. *For $t \in R$ and $x \in \Gamma$, let $\beta_x^0(t) \in \beta_x(t)$ be the element with least absolute value if $\beta_x(t) \neq \varnothing$ and $\beta_x^0(t) = \pm\infty$, where $t > 0$ or < 0, respectively, in case $\beta_x(t) = \varnothing$. Finally, let $\beta_\pm(x) = \lim_{t \to \pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma$. Then, $\beta_\pm(x)$ define measurable functions on Γ.*

Lemma 3.7 [9, 12]. *If $f_1(x) \in L^p(\Omega)$ satisfies*

$$\int\limits_{\Gamma} \beta_-(x) d\Gamma(x) < \int\limits_{\Omega} f_1(x) dx < \int\limits_{\Gamma} \beta_+(x) d\Gamma(x), \tag{3.1}$$

then $f_1 \in \operatorname{int} R(A_p)$, for $\dfrac{2N}{N+1} < p < +\infty$ and $N \geq 1$.

Similarly, if $f_2(x) \in L^q(\Omega)$ satisfies (3.1), then, $f_2(x) \in \operatorname{int} R(A_q)$, for $\dfrac{2N}{N+1} < q < +\infty$ and $N \geq 1$.

From Lemma 3.7, we can easily get the following result.

Proposition 3.6. *Let $f_1(x) \in L^p(\Omega)$, $f_2(x) \in L^q(\Omega)$ satisfy (3.1). Then, we have $(f_1, f_2) \in \operatorname{int} R(A_{p,q})$.*

Proposition 3.7. *Let $(f_1, f_2) \in L^p(\Omega) \times L^q(\Omega)$, $(u, v) \in L^p(\Omega) \times L^q(\Omega)$ and $(f_1, f_2) \in A_{p,q}(u, v)$. Then, we have the following:*

(a) $-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + \varepsilon |u|^{p-2} u = f_1(x)$ *a.e.* $x \in \Omega$;

(b) $-\left\langle \vartheta, |\nabla u|^{p-2} \nabla u \right\rangle \in \beta_x(u(x))$ *a.e.* $x \in \Gamma$;

(c) $-\operatorname{div}(|\nabla v|^{q-2} \nabla v) + \varepsilon |v|^{q-2} v = f_2(x)$ *a.e.* $x \in \Omega$;

(d) $-\left\langle \vartheta, |\nabla v|^{q-2} \nabla v \right\rangle \in \beta_x(v(x))$ *a.e.* $x \in \Gamma$.

Proof. The proof is similar to that of Proposition 2.2 in [9].

Definition 3.3. *Define $g_+(x) = \liminf_{s,t \to +\infty} g(x, s, t)$ and $g_-(x) = \limsup_{s,t \to -\infty} g(x, s, t)$. Further, define a function $g_1 : \Omega \times R \times R \to R$ by*

$$g_1(x, s, t) = \begin{cases} (\inf_{a \geq s, b \geq t} g(x, a, b)) \wedge (s - T(x)) \wedge (t - T(x)) & \forall\, s, t \geq T(x), \\[2ex] \left(\sup_{a \leq s, b \leq t} g(x, a, b)\right) \vee (s + T(x)) \vee (t + T(x)) & \forall\, s, t \leq -T(x), \\[2ex] 0, & \textit{for the rest of } s \textit{ and } t. \end{cases}$$

We note that for each $x \in \Omega$, $g_1(x,s,t)$ is increasing in t if $s \in R$ is fixed; and $g_1(x,s,t)$ is also increasing in s if $t \in R$ is fixed. Moreover, $\lim_{s,t\to\pm\infty} g_1(x,s,t) = g_\pm(x)$ for $x \in \Omega$. If we define $g_2(x,s,t) = g(x,s,t) - g_1(x,s,t)$, then $g_2(x,s,t)s \geq 0$ for $|s| \geq T(x)$, $x \in \Omega$ and for fixed $t \in R$; and $g_2(x,s,t)t \geq 0$ for $|t| \geq T(x)$, $x \in \Omega$ and for fixed $s \in R$.

Lemma 3.8. *The mapping $g_1 : \Omega \times R \times R \to R$ satisfies Carathéodory's condition and the functions $g_\pm(x)$ are measurable on Ω.*

Proof. Let Q denote the set of rational numbers. For $s,t \in R$, $g_1(\cdot,s,t)$ is measurable on Ω since

$$\{x \mid g_1(x,s,t) < \alpha\} = \{x \mid s \leq T(x)\} \cup \{x \mid t \leq T(x)\} \cup \{x \mid 0 < t - T(x) < \alpha\} \cup$$
$$\cup \{x \mid 0 < s - T(x) < \alpha\} \cup \{x \mid \exists r_1, r_2 \in Q,\ r_1 > s,\ r_2 > t,\ g_1(x,r_1,r_2) < \alpha\},$$

when $\alpha \geq 0$, and

$$\{x \mid g_1(x,s,t) < \alpha\} = \{x \mid s + T(x) < \alpha\} \cap \{x \mid t \leq T(x)\} \cap \{x \mid 0 < t - T(x) < \alpha\} \cap$$
$$\cap \{x \mid \exists r_1, r_2 \in Q,\ r_1 < s,\ r_2 < t,\ g_1(x,r_1,r_2) < \alpha\},$$

when $\alpha \leq 0$.

Next, let $x \in \Omega$ be such that $g(x,\cdot,\cdot)$ is continuous on $R \times R$. We shall show that

(i) for fixed $s \in R$ and $\forall t \in R$ such that $t > T(x)$, $t_n \uparrow t$, we have $\lim_{n\to\infty} g_1(x,s,t_n) = g_1(x,s,t)$;

(ii) for fixed $s \in R$ and $\forall t \in R$ such that $t \geq T(x)$, $t_n \downarrow t$, we have $\lim_{n\to\infty} g_1(x,s,t_n) = g_1(x,s,t)$.

In fact, to prove (i), we notice that $g(x,s,t)$ satisfies Carathéodory's condition and $t_n \uparrow t$, then

$$\lim_{n\to\infty} g(x,s,t_n) = g(x,s,t).$$

Therefore, for all $\varepsilon > 0$, there exists N_1 such that for fixed $s \in R$,

$$g(x,s,b) > g(x,s,t) - \frac{\varepsilon}{2},$$

when $t_{N_1} \leq b \leq t$ and $n \geq N_1$.

From the definition of $g_1(x,s,t)$, we also know that there exists N_2 such that for $n \geq N_2$,

$$|g_1(x,s,t_n) - g_1(x,s,t) \wedge \inf_{t_n \leq b \leq t} g(x,s,b)| \leq \frac{\varepsilon}{2}.$$

Therefore, if $n \geq \max\{N_1, N_2\}$, then

$$g_1(x,s,t_n) > g_1(x,s,t) - \varepsilon,$$

or

$$0 < g_1(x,s,t) - g_1(x,s,t_n) < \varepsilon.$$

This implies that (i) is true.

Next, to show (ii), we note the fact that $g_1(x,s,t)$ is increasing in t if $s \in R$ is fixed and $x \in \Omega$, it follows from $t_n \downarrow t$ that for fixed $s \in R$ and $x \in \Omega$,

$$g_1(x,s,t) \le g_1(x,s,t_n).$$

From the definition of $g_1(x,s,t)$, we also know that there exists N such that for all $\varepsilon > 0$ and $n \ge N$,

$$|g_1(x,s,t_n) - g_1(x,s,t) \wedge \inf_{t_n \le b \le t} g(x,s,b)| < \varepsilon.$$

Therefore,

$$g_1(x,s,t) \le g_1(x,s,t_n) < g_1(x,s,t) + \varepsilon,$$

for $n \ge N$, which implies that (ii) is true.

Similarly, we can show that for fixed $s \in R$, $\forall t \in R$ such that $t < -T(x)$, $g_1(x,s,t)$ is still continuous for t. Likewise, $g_1(x,s,t)$ is continuous for $s > T(x)$ or $s < -T(x)$. Combining the previous results that for each $x \in \Omega$, $g_1(x,s,t)$ is increasing in t for each fixed $s \in R$, and is also increasing in s for each fixed $t \in R$, we see that $g_1(x,s,t)$ is continuous for $(s,t) \in R \times R$.

Hence g_1 satisfies Caratheodory's conditions. The measurability of $g_\pm(x)$ on Ω is obvious from its definition.

This completes the proof.

Remark 3.2. Compared to our previous work, a new technique has been used in the constructi-on of the mapping $g_1 : \Omega \times R \times R \to R$.

Based on the assumption of $g(x,s,t)$ and Lemma 3.8, and using similar arguments as in the proof of Proposition 3.5 in [1], we obtain the following two results.

Lemma 3.9. *Define $C_1^{(1)} : L^p(\Omega) \to L^p(\Omega)$ by*

$$(C_1^{(1)}u)(x) = g_1(x,u(x),v(x))$$

for fixed $v(x) \in L^q(\Omega)$, $x \in \Omega$, and $u \in L^p(\Omega)$. $C_1^{(1)}$ is bounded, continuous and m-accretive. Also $C_1^{(2)} : L^q(\Omega) \to L^q(\Omega)$ defined by

$$(C_1^{(2)}v)(x) = g_1(x,v(x),u(x))$$

for fixed $u(x) \in L^p(\Omega)$, $x \in \Omega$ and $v \in L^q(\Omega)$, is bounded, continuous and m-accretive.

Lemma 3.10. *The mapping $C_2^{(1)} : L^p(\Omega) \to L^p(\Omega)$ defined by*

$$(C_2^{(1)}u)(x) = g_2(x,u(x),v(x)) = g(x,u(x),v(x)) - g_1(x,u(x),v(x))$$

satisfies the condition

$$(C_2^{(1)}(u+y), J_p u) \ge -C(y), \tag{3.2}$$

for any $u, y \in L^p(\Omega)$, *where* $C(y)$ *is a constant depending on* y. *The mapping* $C_2^{(2)} : L^q(\Omega) \to \to L^q(\Omega)$ *defined by* $(C_2^{(2)}v)(x) = g_2(x, v(x), u(x)) = g(x, v(x), u(x)) - g_1(x, v(x), u(x))$ *also satisfies (3.2), i.e.,*

$$(C_2^{(2)}(v + y), J_q v) \geq -C'(y)$$

for any $v, y \in L^q(\Omega)$, *where* $C'(y)$ *is a constant depending on* y.

Proposition 3.8. *The mapping* $C_1 : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ *defined by*

$$C_1(u, v) = (C_1^{(1)}u, C_1^{(2)}v)$$

for $(u, v) \in L^p(\Omega) \times L^q(\Omega)$ *is bounded, continuous and* m*-accretive.*

The mapping $C_2 : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ *defined by*

$$C_2(u, v) = (C_2^{(1)}u, C_2^{(2)}v)$$

for $(u, v) \in L^p(\Omega) \times L^q(\Omega)$ *satisfies the condition*

$$(C_2((u, v) + (w, y)), J(u, v)) \geq -C(w, y)$$

for any $u, w \in L^p(\Omega)$, $v, y \in L^q(\Omega)$, *where* $C(w, y)$ *is a constant depending on* w *and* y, *and* J *is the duality mapping from* $L^p(\Omega) \times L^q(\Omega)$ *to* $L^{p'}(\Omega) \times L^{q'}(\Omega)$ *defined in Proposition 3.1.*

Proof. The result follows from Lemmas 3.9 and 3.10.

Proposition 3.9. *The mapping* $C_1 : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ *defined in Proposition 3.8 satisfies condition (2.2).*

Proof. By Lemma 2.2, we know that both $C_1^{(1)} : L^p(\Omega) \to L^p(\Omega)$ and $C_1^{(2)} : L^q(\Omega) \to L^q(\Omega)$ satisfy Condition (2.1). Thus, in view of the definition of C_1, it is not difficult to check that $C_1 : L^p(\Omega) \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ satisfies condition (2.2).

This completes the proof.

Theorem 3.1. *Let* $(f_1, f_2) \in L^p(\Omega) \times L^q(\Omega)$ *satisfy*

$$\int_\Gamma \beta_-(x)d\Gamma(x) + \int_\Omega g_-(x)dx < \int_\Omega f_1(x)dx < \int_\Gamma \beta_+(x)d\Gamma(x) + \int_\Omega g_+(x)dx \tag{3.3}$$

and

$$\int_\Gamma \beta_-(x)d\Gamma(x) + \int_\Omega g_-(x)dx < \int_\Omega f_2(x)dx < \int_\Gamma \beta_+(x)d\Gamma(x) + \int_\Omega g_+(x)dx. \tag{3.4}$$

Then, system (1.6) has a solution in $L^p(\Omega) \times L^q(\Omega)$.

Proof. Let $A_{p,q}$ be the m-accretive mapping as in Definition 3.1 and let $C_i : L^p(\Omega) \times \times L^q(\Omega) \to L^p(\Omega) \times L^q(\Omega)$ be as in Proposition 3.8, i.e., $(C_i(u, v))(x) = (g_i(x, u(x), v(x)), g_i(x, v(x), u(x)))$ for $x \in \Omega$ and $i = 1, 2$. We need to prove that $A_{p,q} + C_1$ is boundedly-inversely-compact. In fact, we only need to show that if $(w, y) \in A_{p,q}(u, v) + C_1(u, v)$ with $\{(w, y)\}$ and $\{(u, v)\}$ being bounded in $L^p(\Omega) \times L^q(\Omega)$, then $\{(u, v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$.

For this, we need to discuss the following two cases.

(i) Suppose $\frac{2N}{N+1} < p, q \leq 2$, for $N \geq 1$. From above we see that $(w,y) - C_1(u,v) \in A_{p,q}(u,v)$ with $\{(w,y) - C_1(u,v)\}$ and $\{(u,v)\}$ bounded in $L^p(\Omega) \times L^q(\Omega)$ which gives that $\{(u,v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$ since $A_{p,q}$ is m-accretive and has a compact resolvent from Proposition 3.4.

(ii) Suppose $p, q \geq 2$, or $\frac{2N}{N+1} < p \leq 2$ and $q \geq 2$, or $\frac{2N}{N+1} < q \leq 2$ and $p > 2$. From $(w,y) \in A_{p,q}(u,v) + C_1(u,v)$, it follows that $w \in A_p u + C_1^{(1)} u$ and $y \in A_q u + C_1^{(2)} v$. Since $\{(u,v)\}$ is bounded in $L^p(\Omega) \times L^q(\Omega)$, we have $\{u\}$ is bounded in $L^p(\Omega)$ and $\{v\}$ is bounded in $L^q(\Omega)$. From the proof of Theorem 3.1 in [12], we know that $\{u\}$ is relatively compact in $L^p(\Omega)$ and $\{v\}$ is relatively compact in $L^q(\Omega)$, therefore $\{(u,v)\}$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$.

Noting Propositions 3.1, 3.2, 3.3, 3.8 and 3.9, it is easy to show that all the conditions of Theorem 2.1 are satisfied. Further, from Propositions 3.5 and 3.6, we have $(f_1, f_2) \in \operatorname{int}[R(A_{p,q}) + R(C_1)]$. Therefore, Proposition 3.7 implies that Theorem 3.1 holds.

Remark 3.3. If $p \equiv q$, our result reduces to the work of [8–12]; if moreover, $\varepsilon_1, \varepsilon_2 \equiv 0$, our results reduce to those in [3–7].

Remark 3.4. The nonlinear elliptic system (1.6) can be extended to the following general form:

$$
\begin{aligned}
&-\operatorname{div}\left[(C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\right] + \varepsilon_1 |u|^{p-2} u + g(x, u(x), v(x)) = f_1(x) \quad \text{a.e. in} \quad \Omega, \\
&-\operatorname{div}\left[(C(x) + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v\right] + \varepsilon_2 |v|^{q-2} v + g(x, v(x), u(x)) = f_2(x) \quad \text{a.e. in} \quad \Omega, \\
&-\left\langle \vartheta, (C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right\rangle \in \beta_x(u(x)) \quad \text{a.e. on} \quad \Gamma, \\
&-\left\langle \vartheta, (C(x) + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v \right\rangle \in \beta_x(v(x)) \quad \text{a.e. on} \quad \Gamma.
\end{aligned}
\tag{3.5}
$$

By following the proof of this paper and using the results of paper [12], we can obtain the result that (3.5) has a solution $(u,v) \in L^p(\Omega) \times L^q(\Omega)$ if both (3.3) and (3.4) are satisfied.

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