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ASYMPTOTIC APPROXIMATION FOR THE SOLUTION TO A BOUNDARY-VALUE PROBLEM WITH VARYING TYPE OF BOUNDARY CONDITIONS IN A THICK TWO-LEVEL JUNCTION

АСИМПТОТИЧНЕ НАБЛИЖЕННЯ РОЗВ'ЯЗКУ КРАЙОВОЇ ЗАДАЧІ ЗІ ЗМІНОЮ ТИПУ КРАЙОВИХ УМОВ У ГУСТОМУ ДВОРІВНЕВОМУ З'ЄДНАННІ

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We consider a mixed boundary-value problem for the Poisson equation in a plane two-level junction Ω_ε, which is the union of a domain Ω_0 and a large number $3N$ of thin rods with thickness of order $\varepsilon = \mathcal{O}(N^{-1})$. The thin rods are divided into two levels depending on their length. In addition, the thin rods from each level are ε-periodically alternated. The uniform Dirichlet conditions and the nonuniform Neumann conditions are given respectively on the sides of the thin rods from the first level and the second level. Using the method of matched asymptotic expansions and special junction-layer solutions, we construct the asymptotic approximation for the solution and prove the corresponding estimates in the Sobolev space $H^1(\Omega_\varepsilon)$ as $\varepsilon \to 0$ $(N \to +\infty)$.

Розглядається мішана крайова задача для рівняння Пуассона у плоскому дворівневому з'єднанні Ω_ε, яке є об'єднанням деякої області Ω_0 та великої кількості $3N$ тонких стержнів з товщиною порядку $\varepsilon = \mathcal{O}(N^{-1})$. Тонкі стержні розділено на два рівні в залежності від їх довжини, і стержні з кожного рівня ε-періодично чергуються. На сторонах тонких стержнів з першого рівня задано однорідні крайові умови Діріхле, а на сторонах стержнів другого рівня — неоднорідні крайові умови Неймана. З допомогою методу узгодження асимптотичних розвинень та спеціальних розв'язків типу примежового шару в зоні з'єднання побудовано асимптотичне наближення для розв'язку даної задачі та доведено відповідні асимптотичні оцінки у просторі Соболєва $H^1(\Omega_\varepsilon)$ при $\varepsilon \to 0$ $(N \to +\infty)$.

1. Introduction and statement of the problem. There are two ways for investigation of boundary-value problems in perturbed domains. The first one is the proof of the corresponding convergence theorem. The second way consists in construction of the asymptotic approximation for the solution and in proving the corresponding asymptotic estimate. For these two ways we need different assumptions for data of the investigated problem. The last way is more suitable for applied problems.

Boundary-value problems in thick one-level junctions (thick junctions) are very intensively investigated in recent time. As was shown in the papers [1, 2], such problems lose the coercitivity and compactness as $\varepsilon \to 0$. This creates special difficulties in the asymptotic investigation. In addition, thick junctions are non-convex domains with Lipschitz boundaries, therefore the solutions to boundary-value problems in such domains have only minimal H^1-smoothness. In [3–9], classification of thick one-level junctions was given and basic results (convergence theorems and asymptotic approximations) were obtained both for boundary-value and spectral problems in thick junctions of different types. It was shown that qualitative properties of solutions essentially depend on the junction type and on the conditions given on the boundaries of the attached thin domains. A survey of results obtained in this direction is presented in the papers [3–9]. Here we mention only the pioneer papers [10–12], where the asymptotic behaviour of Green's function of the homogeneous Neumann problem for the Helmholtz equation in unbounded thick junctions was studied.

In the present paper we continue the asymptotic research of boundary-value problems in thick multilevel junctions. Those thick junctions have a more complex structure and because of this the asymptotic study has own particularities and qualitative new results (see [13–18]). Here we construct the asymptotic approximation for a solution to a mixed boundary-value problem in a thick two-level junction and investigate influence of varying type of boundary conditions on the asymptotic behavior. In particular, the homogeneous Dirichlet boundary conditions are given on the lateral sides of the thin rods from the first level and the inhomogeneous Neumann boundary conditions $\partial_\nu u_\varepsilon = \varepsilon g_\varepsilon$ are given on the lateral sides of the thin rods from the second level. At first sight it seems that there is no difference between this inhomogeneous Neumann condition and the homogeneous Neumann condition since the term g_ε is multiplied by the factor ε. But, as we will see in this paper, this is quite false. The Fourier conditions or the inhomogeneous Neumann conditions make the process of homogenization and approximation more complicated and, to homogenize boundary-value problems in thick junctions with those conditions, the method of the integral identities was proposed in [7–9]. The convergence theorem for the solution of the investigated problem was proved in [16].

1.1. The statement of the problem. Let $a,\ d_1,\ d_2,\ b_1,\ h_1,\ h_2$ be positive real numbers and let $d_1 \le d_2$,

$$0 < b_1 < \frac{1}{2}, \qquad 0 < b_1 - \frac{h_1}{2}, \qquad b_1 + \frac{h_1}{2} < \frac{1}{2} - \frac{h_2}{2}.$$

The last restrictions mean that the intervals

$$I_{h_2}(s_2) := \left(s_2 - \frac{h_2}{2}\,,\, s_2 + \frac{h_2}{2} \right), \quad I_{h_1}(s_n) := \left(s_n - \frac{h_1}{2}\,,\, s_n + \frac{h_1}{2} \right),\ \ n = 1, 3,$$

belong to $(0,1)$ and don't intersect; here $s_1 = b_1,\ s_2 = 1/2,\ s_3 = 1 - b_1$.

Let us divide the segment $I_0 := \{x : x_1 \in [0,a],\ \ x_2 = 0\}$ into N equal segments

$$[\varepsilon j\,,\, \varepsilon(j+1)], \quad j = 0, \ldots, N-1 \quad \left(\varepsilon = \frac{a}{N} \right).$$

Here N is a large integer, therefore, the value ε is a small discrete parameter.

A model *thick two-level junction* Ω_ε consists of the junction's body

$$\Omega_0 = \{x \in \mathbb{R}^2 : \ 0 < x_1 < a, \quad 0 < x_2 < \gamma(x_1)\},$$

where $\gamma \in C^1([0,a])$, $\min_{[0,a]} \gamma =: \gamma_0 > 0$, and a large number of the thin rods

$$G_j^{(1)}(s_n, \varepsilon) = \left\{ x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + s_k)| < \frac{\varepsilon h_1}{2}, \quad x_2 \in (-d_1, 0] \right\}, \quad n = 1, 3,$$

$$G_j^{(2)}(s_2, \varepsilon) = \left\{ x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + s_2)| < \frac{\varepsilon h_2}{2}, \quad x_2 \in (-d_2, 0] \right\}, \quad j = 0, 1, \ldots, N-1.$$

Thus $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, where

$$G_\varepsilon^{(1)} = \bigcup_{j=0}^{N-1} \Big(G_j^{(1)}(s_1, \varepsilon) \cup G_j^{(1)}(s_3, \varepsilon) \Big), \qquad G_\varepsilon^{(2)} = \bigcup_{j=0}^{N-1} G_j^{(2)}(s_2, \varepsilon),$$

see Fig. 1.

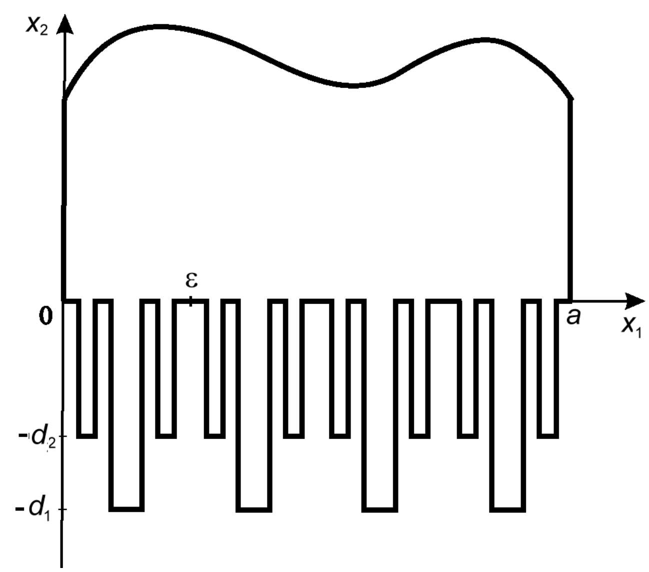

Fig. 1

We see that the number of the thin rods is equal to $3N$ and they are divided into two levels $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ depending on their length. The length of the rods from the first level is equal to d_1 and it is equal to d_2 for the rods from the second one. The parameter ε characterizes the distance between the neighboring thin rods and their thickness. These thin rods from each level are ε-periodically alternated along the segment I_0.

In Ω_ε we consider the problem

$$\begin{gathered} -\Delta u_\varepsilon(x) = f_\varepsilon(x), \quad x \in \Omega_\varepsilon, \\ u_\varepsilon(x) = 0, \quad x \in S_\varepsilon^{(1)} = \partial G_\varepsilon^{(1)} \setminus I_0, \\ \partial_\nu u_\varepsilon(x) = \varepsilon g_\varepsilon(x), \quad x \in S_\varepsilon^{(2)} = \partial G_\varepsilon^{(2)} \setminus \big(I_0 \cup \{x : \ x_2 = -d_2\} \big), \\ \partial_\nu u_\varepsilon(x) = 0, \quad x \in \Gamma_\varepsilon = \partial \Omega_\varepsilon \setminus \big(S_\varepsilon^{(1)} \cup S_\varepsilon^{(2)} \big). \end{gathered} \tag{1}$$

Here $\partial_\nu = \partial / \partial \nu$ is the outward normal derivative.

Without loss of generality, we can assume that $f_\varepsilon \in L^2(\Omega_2)$, where $\overline{\Omega}_2 = \overline{\Omega}_0 \cup \overline{D}_2$; $D_2 = I_0 \times (-d_2, 0)$ is the rectangle that is filled up by the thin rods from the second level in the limit passage as $\varepsilon \to 0$. Analogously, we define $D_1 = I_0 \times (-d_1, 0)$ and $\overline{\Omega}_1 = \overline{\Omega}_0 \cup \overline{D}_1$. We also suppose that the function g_ε and its generalized derivative with respect to x_1 belong to $L^2(D_2)$ and

$$\exists\, C_0 > 0 \quad \forall\, \varepsilon > 0 : \quad \|\partial_{x_1} g_\varepsilon\|_{L^2(D_2)} \le C_0. \tag{2}$$

A function $u_\varepsilon \in H_\varepsilon = \{u \in H^1(\Omega_\varepsilon) : \ u = 0 \text{ on } S_\varepsilon^{(1)}\}$ is called a weak solution of problem (1), if it satisfies the integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi \, dx = \int\limits_{\Omega_\varepsilon} f_\varepsilon \varphi \, dx + \varepsilon \int\limits_{S_\varepsilon^{(2)}} g_\varepsilon \varphi \, dx_2 \qquad \forall \varphi \in H_\varepsilon. \tag{3}$$

It follows from the fundamental statements of the theory of boundary-value problems that for every fixed value $\varepsilon > 0$ there exists a unique generalized solution to problem (1).

The aim of our research is to construct the asymptotic approximation of the solution to problem (1) as $\varepsilon \to 0$, i.e., when the number of the attached thin rods from each level infinitely increases and their thickness tends to 0, and to prove the corresponding asymptotic estimates.

2. Formal asymptotics for the solution. In this section to construct the leading terms of formal asymptotic expansions we assume that the right-hand sides in (1) are independent of ε, i.e., $f_\varepsilon = f_0, \ g_\varepsilon = g_0$ and f_0, g_0 are smooth.

2.1. Outer expansions. We seek the leading terms for the solution u_ε, restricted to Ω_0, in the form

$$u(x, \varepsilon) \approx v_0^+(x) + \sum_{k=1}^{\infty} \varepsilon^k v_k^+(x, \varepsilon), \tag{4}$$

and, restricted to each of the thin rods $G_j^{(1)}(s_1, \varepsilon), \ G_j^{(1)}(s_3, \varepsilon), \ G_j^{(2)}(s_2, \varepsilon)$, $j = 0, \ldots, N-1$, respectively in the form

$$u(x, \varepsilon) \approx v_0^{(i,-)}(x) + \sum_{k=1}^{\infty} \varepsilon^k v_k^{(i,-)}(x, \xi_1 - j), \quad \xi_1 = \frac{x_1}{\varepsilon}, \ i = 1, 2. \tag{5}$$

In (5) we don't indicate the dependence of $s_1 = b_1$ and $s_3 = 1 - b_1$ at $i = 1$, since the asymptotic expansions on these rods are the same.

Substituting the series (4) in the equation of problem (1) and in the boundary conditions on $\Gamma_0 = \partial\Omega_0 \setminus I_0$, and collecting coefficients of the same powers of ε, we get the following relations for the function v_0^+ :

$$\begin{aligned} -\Delta v_0^+(x) &= f_0(x), \qquad x \in \Omega_0, \\ \partial_\nu v_0^+(x) &= 0, \qquad\quad\ \ x \in \Gamma_0. \end{aligned}$$

Now let us find limit relations in the rectangles $D_i,\ i = 1, 2$. We write the Taylor series of $v_k^{(i,-)}$ with respect to the x_1 at the point $x_1 = \varepsilon(j+s_n)$ and pass to the "fast" variable $\xi_1 = x_1/\varepsilon$, where $s_1 = b_1,\ s_2 = 1/2,\ s_3 = 1 - b_1$. Then (5) takes the form

$$u(x,\varepsilon) \approx v_0^{(i,-)}(\varepsilon(j+s_n), x_2) + \sum_{k=1}^{\infty} \varepsilon^k V_k^{i,j}(s_n, \xi_1, x_2), \quad x \in G_j^{(i)}(s_n, \varepsilon), \tag{6}$$

where

$$V_k^{i,j} = v_k^{(i,-)}(\varepsilon(j+s_n), x_2, \xi_1 - j) + \sum_{m=1}^{k} \frac{(\xi_1 - j - s_n)^m}{m!} \frac{\partial^m v_{k-m}^{(i,-)}}{\partial x_1^m}\big(\varepsilon(j+s_n), x_2, \xi_1 - j\big). \tag{7}$$

Substituting the series (6) in the differential equation of problem (1) on the rod $G_j^{(2)}(s_2,\varepsilon)$ from the second level and into the Neumann condition and collecting the coefficients of the same power of ε, we obtain one dimensional boundary-value problems with respect to ξ_1.

The first problem is the following:

$$\begin{gathered} \partial^2_{\xi_1^2} V_1^{2,j}(s_2, \xi_1, x_2) = 0, \quad \xi_1 \in I_{h_2}(s_2), \\ \partial_{\xi_1} V_1^{2,j}\left(s_2, s_2 \pm \frac{h_2}{2}, x_2\right) = 0, \end{gathered} \tag{8}$$

where $\partial_{\xi_1} = \partial/\partial\xi_1,\ \partial^2_{\xi_1\xi_1} = \partial^2/\partial\xi_1^2$; the variable x_2 is regarded as a parameter in this problem. From (8) it follows that the function $V_1^{2,j}$ doesn't depend on ξ_1. We restrict ourselves to the leading terms of the asymptotics and set $V_1^{2,j} \equiv 0$. Then, due to (7), we have

$$v_1^{(2,-)}(\varepsilon(j+s_2), x_2, \xi - j) = (-\xi_1 + j + s_2)\partial_{x_1} v_0^{(2,-)}(\varepsilon(j+s_2), x_2).$$

The problem for the function $V_2^{2,j}$ is as follows:

$$\begin{gathered} -\partial^2_{\xi_1^2} V_2^{2,j}(s_2, \xi_1, x_2) = \partial^2_{x_2^2} v_0^{(2,-)}(\varepsilon(j+s_2), x_2) + f_0(\varepsilon(j+s_2), x_2), \quad \xi_1 \in I_{h_2}(s_2), \\ \partial_{\xi_1} V_2^{2,j}\left(s_2, s_2 \pm \frac{h_2}{2}, x_2\right) = \pm g_0(\varepsilon(j+s_2), x_2). \end{gathered} \tag{9}$$

The solvability condition for (9) is given by the differential equation

$$-h_2\, \partial^2_{x_2} v_0^{(2,-)}(\varepsilon(j+s_2), x_2) = h_2\, f_0(\varepsilon(j+s_2), x_2) + 2g_0(\varepsilon(j+s_2), x_2). \tag{10}$$

If we substitute (6) into the Neumann conditions on the lower base $Q_j^{(2)}(s_2,\varepsilon)$ of rod $G_j^{(2)}(s_2,\varepsilon)$, we get

$$\partial_{x_2} v_0^{(2,-)}(\varepsilon(j+s_2), -d_2) = 0. \tag{11}$$

Since the segments $\{x: \ x_1 = \varepsilon(j+s_2), \ x_2 \in [-d_2, 0]\}, \ \ j = 0, 1, \ldots, N-1$, fill out the rectangle $\overline{D}_2$ in the limit as $\varepsilon \to 0$ $(N \to +\infty)$, we can continue the differential equation (10) into the rectangle D_2 and the relations (11) into the segment $I_{d_2} = \{x: \ x_1 \in [0, a], \ x_2 = -d_2\}$.

Substitute now the series (6) in the differential equation of problem (1) on the rods $G_j^{(1)}(s_1, \varepsilon)$ and $G_j^{(1)}(s_3, \varepsilon)$ from the first level. Since these asymptotic expansions are the same, we consider the rod $G_j^{(1)}(s_1, \varepsilon)$. Due to the Dirichlet conditions on the vertical sides, we have $v_0^{(1,-)} \equiv 0$. The problem for the function $V_1^{1,j}(s_1, \xi_1, x_2)$ is as follows:

$$\partial^2_{\xi_1^2} V_1^{1,j}(s_1, \xi_1, x_2) = 0, \quad \xi_1 \in I_{h_1}(s_1),$$

$$V_1^{1,j}\left(s_1, s_1 \pm \frac{h_1}{2}, x_2\right) = 0,$$

from which $V_1^{1,j} \equiv 0$. Then, due to (7), we have $v_1^{(1,-)}(\varepsilon(j+s_1), x_2, \xi - j) = 0$. The problem for $V_2^{1,j}$ is the following:

$$-\partial^2_{\xi_1^2} V_2^{1,j}(s_1, \xi_1, x_2) = f_0(\varepsilon(j+s_1), x_2), \quad \xi_1 \in I_{h_1}(s_1),$$

$$V_2^{1,j}\left(s_1 \pm \frac{h_1}{2}, x_2\right) = 0.$$

It is easy to verify that

$$V_2^{(1,j)}(s_1, \xi_1, x_2) = h_1^{-1}\left(\xi_1 - s_1 - \frac{h_1}{2}\right) \int\limits_{s_1 - h_1/2}^{\xi_1} \left(s_1 - \frac{h_1}{2} - t\right) f_0(t, x_2)\, dt +$$

$$+ h_1^{-1}\left(s_1 - \frac{h_1}{2} - \xi_1\right) \int\limits_{\xi_1}^{s_1 + h_1/2} \left(t - s_1 - \frac{h_1}{2}\right) f_0(t, x_2) dt.$$

According to (7), we obtain

$$v_2^{(1,-)}(\varepsilon(j+s_1), x_2, \xi_1 - j) = V_2^{(1,j)}(s_1, \xi_1, x_2).$$

Thus, the asymptotic expansions on the rods from the first level have the form

$$\varepsilon^2 v_2^{(1,-)}(\varepsilon(j+s_n), x_2) + \sum_{k=3}^{\infty} \varepsilon^k V_k^{1,j}(s_n, \xi_1, x_2), \quad x \in G_j^{(1)}(s_n, \varepsilon), \ n = 1, 3. \qquad (12)$$

It is evident that the first terms of the asymptotic expansions must coincide on the joint zone I_0. Therefore, from (12) and (4) it follows that $v_0^+(\varepsilon(j+s_1), 0) = v_0^+(\varepsilon(j+s_3), 0) = 0$,

$j = 0, 1, \ldots, N-1$. On the other hand, $v_0^+(\varepsilon(j+s_2), 0)$ must be equal to $v_0^{(2,-)}(\varepsilon(j+s_2), 0)$ at $j = 0, 1, \ldots, N-1$. This is possible if

$$v_0^+(x_1, 0) = v_0^{(2,-)}(x_1, 0) = 0, \quad x_1 \in I_0.$$

As a result, the first terms of the asymptotic expansions (4) and (5) on $G_\varepsilon^{(2)}$ have to be solutions of the following problems:

$$\begin{aligned} -\Delta v_0^+(x) &= f_0(x), \quad x \in \Omega_0, \\ \partial_\nu v_0^+(x) &= 0, \quad x \in \Gamma_0, \\ v_0^+(x_1, 0) &= 0, \quad x_1 \in I_0; \end{aligned} \tag{13}$$

$$\begin{aligned} -h_2\, \partial_{x_2}^2 v_0^{(2,-)}(x) &= h_2\, f_0(x) + 2g_0(x), \quad x \in D_2, \\ v_0^{(2,-)}(x_1, 0) &= 0, x_1 \in I_0, \\ \partial_{x_2} v_0^{(2,-)}(x_1, -d_2) &= 0, \quad x_1 \in I_0. \end{aligned} \tag{14}$$

Obviously, there is a unique weak solution to problem (13) that belongs to the Sobolev space $H^1(\Omega_0, I_0) = \{u \in H^1(\Omega_0) : \ u = 0 \text{ on } I_0\}$. It is easy to calculate that

$$v_0^{(2,-)}(x) = -x_2 \int\limits_{-d_2}^{x_2} \big(f_0(x_1, t) + 2h_2^{-1} g_0(x_1, t)\big)\, dt - \int\limits_{x_2}^{0} t\big(f_0(x_1, t) + 2h_2^{-1} g_0(x_1, t)\big)\, dt. \tag{15}$$

Thus, the asymptotic expansion on the rod $G_j^{(2)}(s_2, \varepsilon)$ has the following form:

$$v_0^{(2,-)}(\varepsilon(j+s_2), x_2) + \varepsilon(-\xi_1 + j + s_2)\partial_{x_1} v_0^{(2,-)}(\varepsilon(j+s_2), x_2) + \sum_{k=2}^{\infty} \varepsilon^k V_k^{2,j}(s_2, \xi_1, x_2).$$

If we consider the function

$$R(x) = \begin{cases} v_0^+(x), & x \in \Omega_0, \\ v_0^{(2,-)}(x) + \varepsilon Y\left(\dfrac{x_1}{\varepsilon}\right) \partial_{x_1} v_0^{(2,-)}(x), & x \in G_\varepsilon^{(2)}, \\ 0, & x \in G_\varepsilon^{(1)}, \end{cases} \tag{16}$$

as a first approximation for the solution to problem (1), then it leaves the remainder

$$\int\limits_{I_0 \cap G_\varepsilon^{(2)}} \big(\partial_{x_2} v_0^+(x_1, 0) - h_2 \partial_{x_2} v_0^{(2,-)}(x_1, 0)\big)\varphi(x_1, 0)\, dx_1, \quad \varphi \in H_\varepsilon, \tag{17}$$

in the corresponding integral identity. In (16), $Y(t) = -t + \frac{1}{2} + [t]$, where $[t]$ is the integral part of t. To neutralize this remainder, we should construct a special inner expansion in a neighborhood of the joint zone I_0.

2.2. Inner expansion. Near the joint zone I_0 we introduce the "rapid"coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = x_1/\varepsilon$, $\xi_2 = x_2/\varepsilon$. Passing to $\varepsilon = 0$, we see that the rods $G_0^{(1)}(s_1, \varepsilon)$, $G_0^{(2)}(s_2, \varepsilon)$, $G_0^{(1)}(s_3, \varepsilon)$ transform into the semiinfinite strips

$$\Pi_1^- = I_{h_1}(s_1) \times (-\infty, 0), \quad \Pi_2^- = I_{h_2}(s_2) \times (-\infty, 0), \quad \Pi_3^- = I_{h_1}(s_3) \times (-\infty, 0).$$

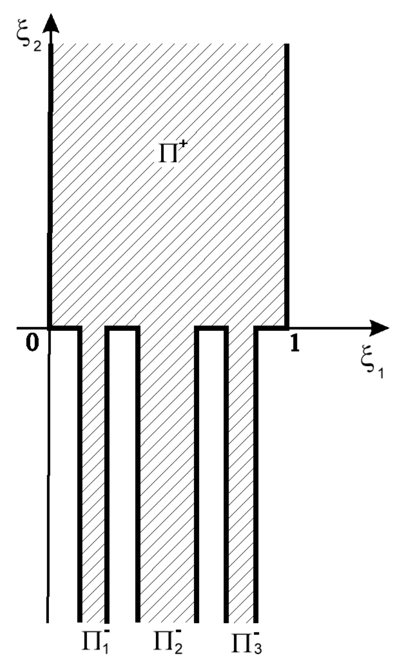

Fig. 2

The domain Ω_0 transforms into the first quadrant $\{\xi : \xi_1 > 0, \xi_2 > 0\}$. Taking into account the periodicity of the thin rods we can regard the union $\Pi = \Pi^+ \cup \Pi_1^- \cup \Pi_2^- \cup \Pi_3^-$, where $\Pi^+ = (0,1) \times (0, +\infty)$, as the base domain, see Fig. 2, in which junction-layer problems have to be considered. Obviously, solutions of these junction-layer problems must be 1-periodic in ξ_1, i.e.,

$$Z(0, \xi_2) = Z(1, \xi_2), \qquad \partial_{\xi_1} Z(0, \xi_2) = \partial_{\xi_1} Z(1, \xi_2), \quad \xi_2 > 0. \tag{18}$$

We seek the first term of the inner expansion in a neighborhood of I_0 in the form

$$u_\varepsilon(x) \approx \varepsilon \left(Z_1\left(\frac{x}{\varepsilon}\right) \partial_{x_2} v_0^+(x_1, 0) + \Xi_1\left(\frac{x}{\varepsilon}\right) \left(h_2^{-1} \partial_{x_2} v_0^+(x_1, 0) - \partial_{x_2} v_0^{(2,-)}(x_1, 0)\right)\right) + \mathcal{O}(\varepsilon^2), \tag{19}$$

where the functions $Z_1(\xi)$, $\Xi_1(\xi)$, $\xi \in \Pi$, are 1-periodic with respect to ξ_1 (see (18)). The last summand in (19) to neutralize the remainder (17).

Substituting (19) in the differential equation of problem (1) and in the corresponding boundary conditions, taking into account that the Laplace operator takes the form $\varepsilon^{-2}\Delta_\xi$ in the coordinates ξ and collecting the coefficients of the same power of ε, we obtain that the function Z_1 must be a nontrivial solution to the following homogeneous problem:

$$
\begin{aligned}
-\Delta Z_1(\xi) &= 0, && \xi \in \Pi,\\
\partial^p_{\xi_1} Z_1(\xi)|_{\xi_1=0} &= \partial^p_{\xi_1} Z_1(\xi)|_{\xi_1=1}, && \xi_2 > 0,\ p = 0,1,\\
\partial_{\xi_2} Z_1(\xi_1, 0) &= 0, && \xi_1 \in (0,1) \setminus (I_{h_1}(s_1) \cup I_{h_2}(s_2) \cup I_{h_1}(s_3)),\\
[Z_1]_{|_{\xi_2=0}} = [\partial_{\xi_2} Z_1]_{|_{\xi_2=0}} &= 0, && \xi_1 \in I_{h_1}(s_1) \cup I_{h_2}(s_2) \cup I_{h_1}(s_3),\\
Z_1(\xi) &= 0, && \xi \in S_1^- \cup S_3^-,\\
\partial_{\xi_1} Z_1(\xi) &= 0, && \xi \in S_2^-,
\end{aligned}
\tag{20}
$$

where $S_n^- = \partial\Pi_n^- \setminus I_{h_1}(s_n),\ n = 1,3,\ S_2^- = \partial\Pi_2^- \setminus I_{h_2}(s_2),$ the brackets denote the jump of the enclosed quantities. The function Ξ_1 must be a solution to the following problem:

$$
\begin{aligned}
-\Delta \Xi_1(\xi) &= 0, && \xi \in \Pi,\\
\partial^p_{\xi_1} \Xi_1(\xi)|_{\xi_1=0} &= \partial^p_{\xi_1} \Xi_1(\xi)|_{\xi_1=1}, && \xi_2 > 0,\ p = 0,1,\\
\partial_{\xi_2} \Xi_1(\xi_1, 0) &= 0, && \xi_1 \in (0,1) \setminus (I_{h_1}(s_1) \cup I_{h_2}(s_2) \cup I_{h_1}(s_3)),\\
[\Xi_1]_{|_{\xi_2=0}} &= 0, && \xi_1 \in I_{h_1}(s_1) \cup I_{h_2}(s_2) \cup I_{h_1}(s_3),\\
[\partial_{\xi_2} \Xi_1]_{|_{\xi_2=0}} &= 0, && \xi_1 \in I_{h_1}(s_1) \cup I_{h_1}(s_3),\\
[\partial_{\xi_2} \Xi_1]_{|_{\xi_2=0}} &= 1, && \xi_1 \in I_{h_2}(s_2),\\
\Xi_1(\xi) &= 0, && \xi \in S_1^- \cup S_3^-,\\
\partial_{\xi_1} \Xi_1(\xi) &= 0, && \xi \in S_2^-.
\end{aligned}
\tag{21}
$$

The main asymptotic relations for a solution to problem (20) can be obtained from general results about the asymptotic behaviour of solutions to elliptic problems in domains with different exits to infinity [19, 20]. The proofs simplify substantially if the polynomial property of the corresponding sesquilinear forms is employed [21]. However, using symmetry of the domain Π, we can define more exactly the asymptotic relations and detect other properties of the junction-layer solution Z_1 similarly as in the papers [4, 5].

Statement 1. *There exits a unique solution Z_1 to problem (20), which has the following differentiable asymptotics:*

$$
Z_1(\xi) = \begin{cases}
\xi_2 + \alpha^+ + \mathcal{O}(\exp(-\delta_1\xi_2)), & \xi_2 \to +\infty,\ \xi \in \Pi^+,\\
h_2^{-1}\xi_2 + \alpha^- + \mathcal{O}(\exp(\delta_1\xi_2)), & \xi_2 \to -\infty,\ \xi \in \Pi_2^-,\\
\mathcal{O}(\exp(\delta_1\xi_2)), & \xi_2 \to -\infty,\ \xi \in \Pi_1^- \cup \Pi_3^-,
\end{cases}
\tag{22}
$$

where $\alpha^\pm$ are some fixed constants; δ_1 is an arbitrary fixed number from the interval $(0,\pi)$. In addition, the function Z_1 is even in ξ_2 with respect to $1/2$.

In order to find the constant $\alpha^{\pm}$ in (22), it is necessary to substitute the function Z_1 and ξ_2 into the second Green's formula in the domains $\Pi^+ \cap \{\xi : \ 0 < \xi_2 < R\}$, $\Pi_1^- \cap \{\xi : \ -R < < \xi_2 < 0\}$, $\Pi_2^- \cap \{\xi : \ -R < \xi_2 < 0\}, \Pi_3^- \cap \{\xi : \ -R < \xi_2 < 0\}$, and to pass to the limit as $R \to \infty$. As a result, we obtain

$$\alpha^+ = \int\limits_{[0,1]\setminus(I_{h_1}(s_1)\cup I_{h_1}(s_3))} Z_1(\xi_1, 0)\, d\xi_1, \quad \alpha^- = \frac{1}{h_2} \int\limits_{I_{h_2}(s_2)} Z_1(\xi_1, 0)\, d\xi_1. \tag{23}$$

Now let us investigate the solvability of problem (21). Let $\widehat{C}_0^\infty(\overline{\Pi}, S_1^- \cup S_3^-)$ be a space of infinitely differentiable functions that are equal to zero on $S_1^- \cup S_3^-$, satisfy the periodic conditions (18), and are finite in ξ_2, i.e.,

$$\forall \varphi \in \widehat{C}_0^\infty(\overline{\Pi}, S_1^- \cup S_3^-) \quad \exists R > 0 \quad \forall \xi \in \overline{\Pi} \quad |\xi_2| \geq R : \ \varphi(\xi) = 0.$$

Let $\mathcal{H}$ be the completion of the space $\widehat{C}_0^\infty(\overline{\Pi}, S_1^- \cup S_3^-)$ with respect to the norm

$$\|u\|_{\mathcal{H}} = \|\nabla_\xi u\|_{L^2(\Pi)}.$$

A function $\Xi \in \mathcal{H}$ is called a weak solution of problem (21), if it satisfies the integral identity

$$\int\limits_{\Pi} \nabla_\xi \Xi \cdot \nabla_\xi \varphi \, d\xi = - \int\limits_{I_{h_2}(s_2)} \varphi(\xi_1, 0)\, d\xi_1 \qquad \forall\, \varphi \in \mathcal{H}.$$

Lemma 1. *There exits a unique weak solution Ξ_1 to problem (21), which has the following differentiable asymptotics:*

$$\Xi_1(\xi) = \begin{cases} \beta^+ + \mathcal{O}(\exp(-\delta_2\xi_2)), & \xi_2 \to +\infty, \ \xi \in \Pi^+, \\ \beta^- + \mathcal{O}(\exp(\delta_2\xi_2)), & \xi_2 \to -\infty, \ \xi \in \Pi_2^-, \\ \mathcal{O}(\exp(\delta_2\xi_2)), & \xi_2 \to -\infty, \ \xi \in \Pi_1^- \cup \Pi_3^-, \end{cases} \tag{24}$$

where $\beta^{\pm}$ are some fixed constants; δ_2 is an arbitrary fixed number from the interval $(0, \pi)$. In addition, the function Ξ_0 is even in ξ_2 with respect to $1/2$.

Proof. To prove the first part of this lemma it is enough to show that the linear functional

$$l(\varphi) := \int\limits_{I_{h_2}(s_2)} \varphi(\xi_1, 0)\, d\xi_1, \quad \varphi \in \mathcal{H},$$

is bounded. With help of the Friedrichs inequality and the cut-off function $\chi_0 \in C^\infty(\mathbb{R}), \ 0 \leq$

$\leq \chi_0 \leq 1$, $\chi_0(\xi_2) = 1$ if $|\xi_2| \leq 1/2$, and $\chi_0(\xi_2) = 0$ if $|\xi_2| \geq 1$, we deduce

$$|l(\varphi)|^2 \leq h_2 \int\limits_{I_{h_2}(s_2)} \varphi^2(\xi_1, 0)\, d\xi_1 \leq h_2 \int\limits_{I_{h_2}(s_2)\times(0,1)} \big(\partial_{\xi_2}(\chi_0(\xi_2)\,\varphi(\xi_1,\xi_2)\,)\big)^2\, d\xi_1 d\xi_2 \leq$$

$$\leq 2h_2 \left(\int\limits_{I_{h_2}(s_2)\times(0,1)} (\chi_0')^2 \varphi^2\, d\xi + \int\limits_{I_{h_2}(s_2)\times(0,1)} \big(\partial_{\xi_2}\varphi\big)^2\, d\xi \right) \leq$$

$$\leq c_1 \left(\int\limits_{\Pi\cap\{\xi:-1<\xi_2<1\}} \varphi^2\, d\xi + \|\varphi\|^2_{\mathcal{H}} \right) \leq$$

$$\leq c_2 \left(\int\limits_{\Pi\cap\{\xi:-1<\xi_2<1\}} |\nabla\varphi|^2\, d\xi + \|\varphi\|^2_{\mathcal{H}} \right) \leq c_2 \|\varphi\|^2_{\mathcal{H}}.$$

Taking into account the properties of solutions to elliptic problems in semiinfinite domains, we can state that the solution Ξ_1 has the asymptotics (24).

Due to the symmetry of the domain ω and using the substitution $\overline{\xi}_1 = 1 - \xi_1$ in problem (21), we obtain that the difference $\Xi_1(\xi_1, \xi_2) - \Xi_1(1 - \xi_1, \xi_2)$ is a solution of the homogeneous problem (20) and it belongs to $\mathcal{H}$. By virtue of the uniqueness of such a solution, it follows that this difference vanishes.

By the same way as we obtained (23), we get

$$\beta^+ = \int\limits_{[0,1]\setminus(I_{h_1}(s_1)\cup I_{h_1}(s_3))} \Xi_1(\xi_1, 0)\, d\xi_1, \qquad \beta^- = \frac{1}{h_2} \int\limits_{I_{h_2}(s_2)} \Xi_1(\xi_1, 0)\, d\xi_1.$$

If we apply the second Green's formula to the functions Z_1 and Ξ_1 in the domain $\Pi\cap\{\xi : -R < < \xi_2 < R\}$ and then pass to the limit as $R \to \infty$, we deduce the relation $\beta^+ - \beta^- = h_2\alpha^-$.

The lemma is proved.

3. Asymptotic approximations. Here we construct an approximation function R_ε using the terms v_0^+, $v_0^{(2,-)}$, Z_1, Ξ_1 defined in the previous section and the following cut-off function $\chi_1 \in C^\infty(\mathbb{R})$, $0 \leq \chi_1 \leq 1$, $\chi_1(x_2) = 1$ if $|x_2| \leq \lambda_1/2$, and $\chi_1(x_2) = 0$ if $|x_2| \geq \lambda_1$, where $\lambda_1 = 2^{-1}\min\{\gamma_0, d_1, d_2\}$. It is equal to

$$R_\varepsilon^+(x) := R_\varepsilon(x) = v_0^+(x) + \varepsilon\chi_1(x_2) N^+(\xi, x_1)|_{\xi=\frac{x}{\varepsilon}}, \qquad x \in \Omega_0,$$

$$R_\varepsilon^{(2,-)}(x) := R_\varepsilon = v_0^{(2,-)}(x) + \varepsilon\big(Y(\xi_1)\partial_{x_1} v_0^{(2,-)}(x) + \chi_1(x_2) N^{(2,-)}(\xi, x_1)\big)|_{\xi=\frac{x}{\varepsilon}}, \quad x \in G_\varepsilon^{(2)}, \tag{25}$$

$$R_\varepsilon^{(1,-)}(x) := R_\varepsilon(x) = \varepsilon\, \chi_1(x_2)\, N^{(1,-)}(\xi, x_1)|_{\xi=\frac{x}{\varepsilon}}, \qquad x \in G_\varepsilon^{(1)}.$$

Here Y is defined in (16); the functions $N^+(\xi, x_1)$, $N^{(2,-)}(\xi, x_1)$, $N^{(1,-)}(\xi, x_1)$ are 1-periodic with respect to ξ_1 and

$$N^+(\xi, x_1) = (Z_1(\xi) - \xi_2)\partial_{x_2} v_0^+(x_1, 0) + \Xi_1(\xi)\big(\partial_{x_2} v_0^{(2,-)}(x_1, 0) - h_2^{-1}\partial_{x_2} v_0^+(x_1, 0)\big), \quad \xi_1 > 0,$$

$$N^{(2,-)} = (Z_1(\xi) - h_2^{-1}\xi_2)\partial_{x_2} v_0^+(x_1, 0) + \Xi_1(\xi)\big(\partial_{x_2} v_0^{(2,-)}(x_1, 0) - h_2^{-1}\partial_{x_2} v_0^+(x_1, 0)\big),$$
$$\xi_1 < 0,\ \xi \in \Pi_2^-,$$

$$N^{(1,-)} = Z_1(\xi)\partial_{x_2} v_0^+(x_1, 0) + \Xi_1(\xi)\big(\partial_{x_2} v_0^{(2,-)}(x_1, 0) - h_2^{-1}\partial_{x_2} v_0^+(x_1, 0)\big),$$
$$\xi_1 < 0,\ \xi \in \Pi_1^- \cup \Pi_3^-.$$

It is easy to verify that $R_\varepsilon^+(x_1, 0) = R_\varepsilon^{(1,-)}(x_1, 0)$, $x_1 \in I_0 \cap G_\varepsilon^{(1)}$, and $R_\varepsilon^+(x_1, 0) = R_\varepsilon^{(2,-)}(x_1, 0)$, $x_1 \in I_0 \cap G_\varepsilon^{(2)}$. Thus, $R_\varepsilon \in H_\varepsilon$. In addition, $[\partial_{x_2} R_\varepsilon]_{|x_2=0} = 0$, $x_1 \in I_0 \cap G_\varepsilon^{(1)}$; and

$$[\partial_{x_2} R_\varepsilon]_{|x_2=0} = -\varepsilon Y(\xi_1)\partial^2_{x_2 x_1} v_0^{(2,-)}(x_1, 0), \quad x_1 \in I_0 \cap G_\varepsilon^{(2)}. \tag{26}$$

Theorem 1. *Suppose that $f_0 \in C_0^2(\Omega_2)$, $g_0 \in C_0^2(D_2)$. Then for any $\delta_0 \in (0,1)$ there exist positive constants C_0, ε_0 such that for all values $\varepsilon \in (0, \varepsilon_0)$ the difference between the solution u_ε to problem (1) and the approximation function R_ε defined by (25) satisfies the following estimate:*

$$\|u_\varepsilon - R_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_0 \left(\varepsilon^{1-\delta_0} + \varepsilon + \|f_\varepsilon - f_0\|_{L^2(\Omega_0 \cup G_\varepsilon^{(2)})} + \|g_0 - g_\varepsilon\|_{L^2(G_\varepsilon^{(2)})}\right). \tag{27}$$

Proof. 1. *Discrepancies in the domain* Ω_0. Taking into account the properties of the functions Z_1, Ξ_1, v_0^+ and f_0, g_0, we conclude that R_ε^+ satisfies all boundary conditions for problem (1) on $\partial\Omega_0 \cap \partial\Omega_\varepsilon$. Putting R_ε^+ in the equation of problem (1), we get

$$-\Delta_x R_\varepsilon^+(x) - f_\varepsilon(x) = -\chi_1'(x_2)\left(\partial_{\xi_2} N^+(\xi, x_1)\right)|_{\xi = x/\varepsilon} -$$
$$- \chi_1(x_2)\left(\partial^2_{x_1 \xi_1} N^+(\xi, x_1)\right)|_{\xi = x/\varepsilon} - \varepsilon \partial_{x_2}\left(\chi_1'(x_2)\, N^+\left(\frac{x}{\varepsilon}, x_1\right)\right) -$$
$$- \varepsilon \chi_1(x_2)\,\partial_{x_1}(\partial_{x_1} N^+(\xi, x_1)|_{\xi = x/\varepsilon}) - (f_\varepsilon(x) - f_0(x)), \quad x \in \Omega_0. \tag{28}$$

Further, the arguments of functions involved in calculations are indicated only if their absence may cause confusion. We multiply the identify (28) by a test function $\psi \in H_\varepsilon$ and integrate by parts in Ω_0,

$$\int\limits_{I_0 \cap \Omega_\varepsilon} \partial_{x_2} R_\varepsilon^+(x_1, 0)\, \psi\, dx_1 + \int\limits_{\Omega_0} \nabla_x R_\varepsilon^+ \cdot \nabla_x \psi dx - \int\limits_{\Omega_0} f_\varepsilon \psi dx =$$
$$= I_1^+(\varepsilon, \psi) + \ldots + I_5^+(\varepsilon, \psi), \tag{29}$$

where

$$I_1^+(\varepsilon,\psi) = -\int\limits_{\Omega_0} \chi_1'(x_2)\,(\partial_{\xi_2} N^+(\xi,x_1))|_{\xi=x/\varepsilon}\psi\,dx,$$

$$I_2^+(\varepsilon,\psi) = -\int\limits_{\Omega_0} \chi_1(x_2)\,(\partial^2_{x_1\xi_1} N^+(\xi,x_1))|_{\xi=x/\varepsilon}\psi\,dx,$$

$$I_3^+(\varepsilon,\psi) = \varepsilon\int\limits_{\Omega_0} \chi_1'(x_2)\,N^+\left(\frac{x}{\varepsilon},x_1\right)\,\partial_{x_2}\psi dx,$$

$$I_4^+(\varepsilon,\psi) = \varepsilon\int\limits_{\Omega_0} \chi_1(x_2)\,\big(\partial_{x_1} N^+(\xi,x_1)\big)|_{\xi=x/\varepsilon}\,\partial_{x_1}\psi\,dx,$$

$$I_5^+(\varepsilon,\psi) = -\int\limits_{\Omega_0} (f_\varepsilon(x)-f_0(x))\,\psi\,dx.$$

2. *Discrepancies in the thin rings from the second level.* It is easy to calculate that $\partial_{x_2} R_\varepsilon^{(2,-)}(x_1,-d_1) = 0$, and

$$\partial_\nu R_\varepsilon^{1,-} = \pm\varepsilon\big(Y(\xi_1)\partial^2_{x_1x_1} v_0^{(2,-)}(x) + \chi_1(x_2)(\partial_{x_1} N^{(2,-)}(\xi,x_1))|_{\xi=x/\varepsilon}\big), \qquad x\in S_\varepsilon^{(2)}, \tag{30}$$

where we take " $+$ " or " $-$ " depending on the right or left side of the thin rod.

Putting $R_\varepsilon^{(2,-)}$ in the differential equation of problem (1), we obtain

$$\begin{aligned}
-\Delta_x R_\varepsilon^{(2,-)}(x) - f_\varepsilon(x) =& \\
&= -\chi_1'(x_2)\,(\partial_{\xi_2} N^{(2,-)}(\xi,x_1))|_{\xi=x/\varepsilon} - \chi_1(x_2)\,(\partial^2_{x_1\xi_1} N^{(2,-)}(\xi,x_1))|_{\xi=x/\varepsilon} - \\
&- \varepsilon\partial_{x_2}\left(\chi_1'(x_2)\,N^{(2,-)}\left(\frac{x}{\varepsilon},x_1\right)\right) - \varepsilon\chi_1(x_2)\,\partial_{x_1}\left(\partial_{x_1} N^{(2,-)}(\xi,x_1)|_{\xi=x/\varepsilon}\right) - \\
&- (f_\varepsilon(x)-f_0(x)) - \varepsilon\,\mathrm{div}\left(Y_1\left(\frac{x_1}{\varepsilon}\right)\nabla_x\big(\partial_{x_1} v_0^{(2,-)}\big)\right) + 2h_2^{-1}g_0(x), \qquad x\in G_\varepsilon^{(2)}.
\end{aligned} \tag{31}$$

Using the following integral identity:

$$\varepsilon 2^{-1}h_2\int\limits_{S_\varepsilon^{(2)}} v\,dx_2 = \int\limits_{G_\varepsilon^{(2)}} v\,dx - \varepsilon\int\limits_{G_\varepsilon^{(2)}} Y\left(\frac{x_1}{\varepsilon}\right)\partial_{x_1} v\,dx \quad \forall\, v\in H_\varepsilon, \tag{32}$$

which was proved in [8], and taking into account (26) and the boundary values of $\partial_\nu R_\varepsilon^-$ (see (30)), we multiply (31) by a test function $\psi \in H_\varepsilon$ and integrate by parts in $G_\varepsilon^{(2)}$. This yields

$$- \int\limits_{I_0 \cap G_\varepsilon^{(2)}} \partial_{x_2} R_\varepsilon^+(x_1,0)\,\psi(x_1,0)\,dx_1 + \int\limits_{G_\varepsilon^{(2)}} \nabla_x R_\varepsilon^{(2,-)} \cdot \nabla_x \psi\,dx -$$

$$- \int\limits_{G_\varepsilon^{(2)}} f_\varepsilon \psi dx - \varepsilon \int\limits_{S_\varepsilon^{(2)}} g_\varepsilon\,\psi\,dx_2 = I_1^{2,-}(\varepsilon,\psi) + \ldots + I_7^{2,-}(\varepsilon,\psi), \qquad (33)$$

where

$$I_1^{2,-}(\varepsilon,\psi) = - \int\limits_{G_\varepsilon^{(2)}} \chi_1'(x_2)\left(\partial_{\xi_2} N^{(2,-)}(\xi,x_1)\right)|_{\xi=x/\varepsilon}\,\psi\,dx,$$

$$I_2^{2,-}(\varepsilon,\psi) = - \int\limits_{G_\varepsilon^{(2)}} \chi_1(x_2)\left(\partial^2_{x_1\xi_1} N^{(2,-)}(\xi,x_1)\right)|_{\xi=x/\varepsilon}\,\psi\,dx,$$

$$I_3^{2,-}(\varepsilon,\psi) = \varepsilon \int\limits_{G_\varepsilon^{(2)}} \chi_1'(x_2)\,N^{(2,-)}\left(\frac{x}{\varepsilon}, x_1\right)\,\partial_{x_2}\psi\,dx,$$

$$I_4^{2,-}(\varepsilon,\psi) = \varepsilon \int\limits_{G_\varepsilon^{(2)}} \chi_1(x_2)\left(\partial_{x_1} N^{(2,-)}(\xi,x_1)\right)|_{\xi=x/\varepsilon}\,\partial_{x_1}\psi\,dx,$$

$$I_5^{2,-}(\varepsilon,\psi) = - \int\limits_{G_\varepsilon^{(2)}} \left(f_\varepsilon(x) - f_0(x)\right)\psi\,dx,$$

$$I_6^{2,-}(\varepsilon,\psi) = \varepsilon \int\limits_{G_\varepsilon^{(2)}} Y_1\left(\frac{x_1}{\varepsilon}\right) \nabla_x\left(\partial_{x_1} v_0^{(2,-)}\right) \cdot \nabla_x \psi\,dx,$$

$$I_7^{2,-}(\varepsilon,\psi) = \varepsilon \int\limits_{S_\varepsilon^{(2)}} \left(g_0(x) - g_\varepsilon(x)\right)\,\psi(x)\,dx_2 \;-\; 2\varepsilon h_2^{-1} \int\limits_{G_\varepsilon^{(2)}} Y\left(\frac{x_1}{\varepsilon}\right)\,\partial_{x_2}(g_0\,\psi)\,dx.$$

3. *Discrepancies in the thin rings from the first level.* It is easy to verify that $\partial_{x_2} R_\varepsilon^{(1,-)}(x_1, -d_1) = 0$ and $R_\varepsilon^{(1,-)}(x) = 0$ on $S_\varepsilon^{(1)}$. Putting $R_\varepsilon^{(1,-)}$ in the differential equation of problem (1), we obtain

$$-\Delta_x R_\varepsilon^{(1,-)}(x) - f_\varepsilon(x) = -\chi_1'(x_2)\,(\partial_{\xi_2} N^{(1,-)}(\xi, x_1))|_{\xi=x/\varepsilon} -$$

$$- \chi_1(x_2)\,(\partial^2_{x_1\xi_1} N^{(1,-)}(\xi, x_1))|_{\xi=x/\varepsilon} - \varepsilon\partial_{x_2}\left(\chi_1'(x_2)\,N^{(1,-)}\left(\frac{x}{\varepsilon}, x_1\right)\right) -$$

$$- \varepsilon\chi_1(x_2)\,\partial_{x_1}\big(\partial_{x_1} N^{(1,-)}(\xi, x_1)|_{\xi=x/\varepsilon}\big) - f_\varepsilon(x), \quad x \in G_\varepsilon^{(1)}. \tag{34}$$

Multiplying (34) by a test function $\psi \in H_\varepsilon$ and integrating by parts in $G_\varepsilon^{(1)}$, we get

$$- \int\limits_{I_0 \cap G_\varepsilon^{(1)}} \partial_{x_2} R_\varepsilon^{+}(x_1, 0)\,\psi(x_1, 0)\,dx_1 + \int\limits_{G_\varepsilon^{(1)}} \nabla_x R_\varepsilon^{(1,-)} \cdot \nabla_x \psi\,dx - \int\limits_{G_\varepsilon^{(1)}} f_\varepsilon \psi dx =$$

$$= I_1^{1,-}(\varepsilon, \psi) + \ldots + I_5^{1,-}(\varepsilon, \psi), \tag{35}$$

where

$$I_1^{1,-}(\varepsilon, \psi) = - \int\limits_{G_\varepsilon^{(1)}} \chi_1'(x_2)\,\big(\partial_{\xi_2} N^{(1,-)}(\xi, x_1)\big)|_{\xi=x/\varepsilon}\,\psi\,dx,$$

$$I_2^{1,-}(\varepsilon, \psi) = - \int\limits_{G_\varepsilon^{(1)}} \chi_1(x_2)\,\big(\partial^2_{x_1\xi_1} N^{(1,-)}(\xi, x_1)\big)|_{\xi=x/\varepsilon}\,\psi\,dx,$$

$$I_3^{1,-}(\varepsilon, \psi) = \varepsilon \int\limits_{G_\varepsilon^{(1)}} \chi_1'(x_2)\,N^{(1,-)}\left(\frac{x}{\varepsilon}, x_1\right)\,\partial_{x_2}\psi\,dx,$$

$$I_4^{1,-}(\varepsilon, \psi) = \varepsilon \int\limits_{G_\varepsilon^{(1)}} \chi_1(x_2)\,\big(\partial_{x_1} N^{(1,-)}(\xi, x_1)\big)|_{\xi=x/\varepsilon}\,\partial_{x_1}\psi\,dx,$$

$$I_5^{1,-}(\varepsilon, \psi) = - \int\limits_{G_\varepsilon^{(1)}} f_\varepsilon(x)\,\psi\,dx.$$

4. *Asymptotic estimates.* Summing (29), (33) and (35), we see that the function R_ε constructed by formulas (25) satisfies the following integral identity:

$$\int\limits_{\Omega_\varepsilon} \nabla_x R_\varepsilon \cdot \nabla_x \psi\,dx = \int\limits_{\Omega_\varepsilon} f_\varepsilon \psi dx + \varepsilon \int\limits_{S_\varepsilon^{(2)}} g_\varepsilon\,\psi\,dx_2 + F_\varepsilon(\psi) \quad \forall\,\psi \in H_\varepsilon,$$

where $F_\varepsilon(\psi) = I_1^{\pm}(\varepsilon,\psi) + \ldots + I_5^{\pm}(\varepsilon,\psi) + I_6^{2,-}(\varepsilon,\psi) + I_7^{2,-}(\varepsilon,\psi),\ I_j^{\pm}(\varepsilon,\psi) = I_j^{+}(\varepsilon,\psi) + {}+I_j^{1,-}(\varepsilon,\psi) + I_j^{2,-}(\varepsilon,\psi),\ j = 1,\ldots,5$. Subtracting the integral identity (3) from (32), we get

$$\int\limits_{\Omega_\varepsilon} \nabla_x\big(R_\varepsilon - u_\varepsilon\big)\cdot\nabla_x\psi\,dx = F_\varepsilon(\psi) \quad \forall\,\psi\in H_\varepsilon. \tag{36}$$

Now we should estimate the value $F_\varepsilon(\psi)$. At first we note that due to Lemma 1 [16] the usual norm $\|u\|_{H^1(\Omega_\varepsilon)}$ and the norm $\|\nabla u\|_{L^2(\Omega_\varepsilon)}$ are uniformly equivalent with respect to ε.

Integrals in $I_1^{\pm}(\varepsilon,\psi)$ and $I_3^{\pm}(\varepsilon,\psi)$ are taken, in fact, over

$$\operatorname{supp}\big(\chi_1'(x_2)\big)\,\cap\,\Omega_\varepsilon = \left\{x:\ \frac{\lambda_1}{2} < |x_2| < \lambda_1\right\}\cap\Omega_\varepsilon,$$

respectively. In this domain, by virtue of Statement 1 and Lemma 1, the functions

$$\big(\partial_{\xi_2}N^{+}(\xi,x_1)\big)|_{\xi=x/\varepsilon},\quad \big(\partial_{\xi_2}N^{(1,-)}(\xi,x_1)\big)|_{\xi=x/\varepsilon},\quad \big(\partial_{\xi_2}N^{(2,-)}(\xi,x_1)\big)|_{\xi=x/\varepsilon}$$

are exponentially small and the functions N^{+}, $N^{(1,-)}$, $N^{(2,-)}$ are uniformly bounded with respect to ε. Therefore,

$$|I_1^{\pm}(\varepsilon,\psi) + I_3^{\pm}(\varepsilon,\psi)| \le \varepsilon\,C_1\|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Here and in what follows all constants in asymptotic inequalities are independent of ε.

In order to estimate the term $I_2^{\pm}(\varepsilon,\psi)$ we use the following statement.

Statement 2 [4, 5]. *Assume that a function $\mathcal{N}$ is 1-periodic in ξ_1, belongs to the space $L_2(\Pi)$ and is exponentially decreasing at infinity as $\xi_2\to\infty$, i.e., there exist positive constants c, R, σ such that for any $|\xi_2|\ge R$*

$$\big|\mathcal{N}(\xi)\big| \le c\,\exp(-\sigma|\xi_2|).$$

Then for any $\delta > 0$ there exist positive constants c_1,ε_0 such that for all values $\varepsilon\in(0,\varepsilon_0)$ the following inequality holds:

$$\left|\int\limits_{\Omega_\varepsilon}\mathcal{N}\left(\frac{x}{\varepsilon}\right)\,\psi(x)\,dx\right| \le c_1\varepsilon^{1-\delta}\|\psi\|_{H^1(\Omega_\varepsilon)} \quad \forall\psi\in\mathcal{H}_\varepsilon.$$

Since all functions of ξ entering in $\partial^2_{x_1\xi_1}N^{+}(\xi,x_1)$, $\partial^2_{x_1\xi_1}N^{(1,-)}(\xi,x_1)$, $\partial^2_{x_1\xi_1}N^{(2,-)}(\xi,x_1)$ exponentially decrease as $|\xi_2|\to+\infty$, on the basis of Statement 2 we get that

$$|I_2^{\pm}(\varepsilon,\psi)| \le \varepsilon^{1-\delta_0}\,C_2(\delta_0)\|\psi\|_{H^1(\Omega_\varepsilon)},$$

where δ_0 is an arbitrary fixed positive number.

Integrals in $I_4^{\pm}(\varepsilon,\psi)$ are over $\{x\ :\ |x_2|\ <\ \lambda_1\}\cap\Omega_\varepsilon$ and they can be estimated in the following way. Consider, for example, the integral

$$\varepsilon^{-1}|I_4^{+}(\varepsilon,\psi)| = \left|\int\limits_{\Omega_0}\chi_1(x_2)\,\partial_{x_1}\psi\,\Big((Z_1(\xi)-\xi_2)\partial^2_{x_1x_2}v_0^{+}(x_1,0)+\right.$$

$$\left.+\ \Xi_1(\xi)\big(\partial^2_{x_1x_2}v_0^{(2,-)}(x_1,0)-h_2^{-1}\partial^2_{x_1x_2}v_0^{+}(x_1,0)\big)\Big)\big|_{\xi=x/\varepsilon}\,dx\right|\le$$

$$\le\int\limits_{\Omega_0}\chi_1\,|\partial_{x_1}\psi|\,\Big(|Z_1(\xi)-\xi_2-\alpha^{+}|+|\Xi_1(\xi)-\beta^{+}|\Big)\big|_{\xi=x/\varepsilon}\Big(|\partial^2_{x_1x_2}v_0^{(2,-)}(x_1,0)|+$$

$$+\,h_2^{-1}|\partial^2_{x_1x_2}v_0^{+}(x_1,0)|\Big)\,dx+$$

$$+\int\limits_{\Omega_0}\chi_1\,|\partial_{x_1}\psi|\,\big(|\alpha^{+}|+|\beta^{+}|\big)\big(|\partial^2_{x_1x_2}v_0^{(2,-)}(x_1,0)|+h_2^{-1}|\partial^2_{x_1x_2}v_0^{+}(x_1,0)|\big)\,dx\le$$

$$\le c_1\|\partial_{x_1}\psi\|_{L_2(\Omega_0)}\left(\sqrt{\int\limits_{\Omega_0}\chi_1\,\big|Z_1(\xi)-\xi_2-\alpha^{+}\big|^2_{\xi=x/\varepsilon}\,dx}\,+\right.$$

$$\left.+\sqrt{\int\limits_{\Omega_0}\chi_1\,\big|\Xi_1(\xi)-\beta^{+}\big|^2_{\xi=x/\varepsilon}\,dx}+1\right)\le$$

$$\le c_2\|\partial_{x_1}\psi\|_{L_2(\Omega_0)}\big(\sqrt{\varepsilon}\,\|Z_1(\xi)-\xi_2-\alpha^{+}\|_{L^2(\Pi^{+})}\,+\,\sqrt{\varepsilon}\,\|\Xi_1(\xi)-\beta^{+}\|_{L^2(\Pi^{+})}+1\big)\,,$$

where $|\Omega_0|$ is measure of Ω_0. The values $\|Z_1(\xi)-\xi_2-\alpha^{+}\|_{L^2(\Pi^{+})}$ and $\|\Xi_1(\xi)-\beta^{+}\|_{L^2(\Pi^{+})}$ are bounded because of (22) and (24). As a result, we have

$$|I_4^{\pm}(\varepsilon,\psi)|\ \le\ \varepsilon\,C_4\|\psi\|_{H^1(\Omega_\varepsilon)}. \tag{37}$$

Remark 1. The constant C_4 in (37) depends on the quantities

$$\sup_{x_1\in I_0}\Big|\frac{\partial^2v_0^{+}}{\partial x_1\partial x_2}(x_1,0)\Big|,\qquad \sup_{x_1\in I_0}\Big|\frac{\partial^2v_0^{(2,-)}}{\partial x_1\partial x_2}(x_1,0)\Big|. \tag{38}$$

Since $f_0\in C_0^2(\Omega_2)$ the function v_0^{+} and its derivatives have no singularities at the points $(0,0)$ and $(a,0)$. Therefore, by virtue of classical results on the smoothness of solutions to boundary-value problems, the first quantity in (38) is bounded. The boundedness of the second quantity follows from (15) and the condition $g_0 \in C_0^2(D_2)$.

With the help of the Cauchy–Buniakovsky inequality we estimate the summand $I_5^{\pm}(\varepsilon, \psi)$,

$$|I_5^{\pm}(\varepsilon, \psi)| = \left| \int\limits_{\Omega_0 \cup G_\varepsilon^{(2)}} (f_\varepsilon(x) - f_0(x))\, \psi\, dx \,+ \int\limits_{G_\varepsilon^{(1)}} f_\varepsilon(x)\, \psi\, dx \right| \le$$

$$\le \|f_\varepsilon - f_0\|_{L^2(\Omega_0 \cup G_\varepsilon^{(2)})} \|\psi\|_{L^2(\Omega_0 \cup G_\varepsilon^{(2)})} \,+\, \|f_\varepsilon\|_{L^2(G_\varepsilon^{(1)})}\, \varepsilon\, \|\partial_{x_1}\psi\|_{L^2(G_\varepsilon^{(1)})} \le$$

$$\le \left(\|f_\varepsilon - f_0\|_{L^2(\Omega_0 \cup G_\varepsilon^{(2)})} + \varepsilon C_5\right) \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Since $\partial_{x_1} v_0^{(2,-)} \in H^1(D_2)$, we have

$$|I_6^{2,-}(\varepsilon, \psi)| \;\le\; \varepsilon\, C_6\, \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Now it remains to estimate $I_7^{2,-}$. It is obvious that the last summand in $I_7^{2,-}$ is not greater than $c_1 \varepsilon \|\psi\|_{H^1(\Omega_\varepsilon)}$. Using identity (32), assumptions (2) and $g_0 \in C_0^2(D_2)$, we have

$$\left| \varepsilon \int\limits_{S_\varepsilon^{(2)}} \big(g_0(x) - g_\varepsilon(x)\big)\, \psi(x)\, dx_2 \right| \le c_2 \left(\int\limits_{G_\varepsilon^{(2)}} \big|g_0 - g_\varepsilon\big|\, |\psi|\, dx + \varepsilon \int\limits_{G_\varepsilon^{(2)}} |\partial_{x_1}\big((g_0 - g_\varepsilon)\,\psi\big)|\, dx \right) \le$$

$$\le c_3 \left(\|g_0 - g_\varepsilon\|_{L^2(G_\varepsilon^{(2)})} + \varepsilon \right) \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Thus, $|I_7^{2,-}(\varepsilon, \psi)| \;\le\; C_7 \left(\|g_0 - g_\varepsilon\|_{L^2(G_\varepsilon^{(2)})} + \varepsilon \right) \|\psi\|_{H^1(\Omega_\varepsilon)}$.

With regard to the inequalities obtained, we conclude that for the right-hand side in (36) the following inequality holds:

$$|F_\varepsilon(\psi)| \le C_0 \left(\varepsilon^{1-\delta_0} + \varepsilon + \|f_\varepsilon - f_0\|_{L^2(\Omega_0 \cup G_\varepsilon^{(2)})} + \|g_0 - g_\varepsilon\|_{L^2(G_\varepsilon^{(2)})} \right) \|\psi\|_{H^1(\Omega_\varepsilon)} \quad \forall \psi \in H_\varepsilon, \tag{39}$$

where δ_0 is a positive fixed number. From (36) and (39) it follows the inequality (27).

Theorem is proved.

Corollary 1. *From (27) it follows that*

$$\|u_\varepsilon - v_0^{+}\|_{L^2(\Omega_0)} + \|u_\varepsilon - v_0^{(2,-)}\|_{L^2(G_\varepsilon^{(2)})} + \|u_\varepsilon\|_{L^2(G_\varepsilon^{(1)})} \;\le$$

$$\le\; c_2 \left(\varepsilon^{1-\delta_0} + \varepsilon + \|f_\varepsilon - f_0\|_{L^2(\Omega_0 \cup G_\varepsilon^{(2)})} + \|g_0 - g_\varepsilon\|_{L^2(G_\varepsilon^{(2)})} \right).$$

Corollary 2. *Assume that*

$$f_\varepsilon(x) = f_0(x) + \varepsilon f_1(x,\varepsilon), \quad x \in \Omega_\varepsilon, \quad \textit{where} \quad \|f_1(\cdot,\varepsilon)\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}(1) \quad \textit{as} \quad \varepsilon \to 0,$$

$$g_\varepsilon(x) = g_0(x) + \varepsilon g_1(x,\varepsilon), \quad x \in G_\varepsilon^{(2)}, \quad \textit{where} \quad \|g_1(\cdot,\varepsilon)\|_{L^2(G_\varepsilon^{(2)})} = \mathcal{O}(1) \quad \textit{as} \quad \varepsilon \to 0.$$

Then for any $\delta_0 > 0$ there exist positive constants c_3, ε_0 such that for all values $\varepsilon \in (0, \varepsilon_0)$

$$\|u_\varepsilon - R_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c_3\,\varepsilon^{1-\delta_0}, \quad \|u_\varepsilon - v_0^+\|_{L^2(\Omega_0)} + \|u_\varepsilon - v_0^{(2,-)}\|_{L^2(G_\varepsilon^{(2)})} + \|u_\varepsilon\|_{L^2(G_\varepsilon^{(1)})} \le c_3\,\varepsilon^{1-\delta_0}.$$

Conclusion. An important problem for existing multiscale methods is their stability and accuracy. The proof of the error estimate between the constructed approximation and the exact solution is a general principle that has been applied to the analysis of the efficiency of a multiscale method. In our paper we have constructed the asymptotic approximation for the solution to problem (1) and proved such estimates in Theorem 1 and Corollaries 1 and 2.

It was shown that, due to the homogeneous Dirichlet boundary conditions on the vertical sides of the rods from of first level, the original boundary-value problem (1) decomposes in the limit into two independent problems (13) and (14). Thanks to the inhomogeneous Neumann boundary conditions on the vertical sides of the rods the second level, we have obtained the corresponding term in the right-hand side of the ordinary differential equation of problem (14). This fact was noted in [22], where elliptic boundary-value problems that describe processes in strongly inhomogeneous thin perforated domains with rapidly varying thickness were studied.

It follows from these results that for applied problems or for numerical calculations in thick two-level junctions we can use the corresponding limit problems, which are more simple, instead of the initial problem with a sufficient validity. Furthermore, owing to the convergence theorem for the solution to problem (1) proved in [16], we can use the limit problems (13) and (14) with minimal conditions for the right-hand sides f_0 and g_0.

In this paper we consider the thick two-level junction Ω_ε, which has more dense packing of the thin rods from the first level on the cell of the joining. As a result we have a more complex domain (the domain Π, see Fig. 2), where special junction-layer problems are considered. The junction-layer solutions behave as powers at infinity and do not decrease exponentially. Therefore, they influence directly the leading terms of the asymptotics. In addition, due to the homogeneous Dirichlet boundary conditions we had to modify the form of the inner expansion.

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