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SINGULAR PERIODIC IMPULSE PROBLEMS

СИНГУЛЯРНІ ПЕРІОДИЧНІ ІМПУЛЬСНІ ЗАДАЧІ

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We obtain an existence principle for the impulsive periodic boundary-value problem $u'' + c\,u' = g(x) + e(t)$, $u(t_i+) = u(t_i) + J_i(u, u')$, $u'(t_i+) = u'(t_i) + M_i(u, u')$, $i = 1, \ldots, m$, $u(0) = u(T)$, $u'(0) = u'(T)$, *where* $g \in C(0, \infty)$ *can have a strong singularity at the origin. Furthermore, we assume that* $0 < t_1 < \ldots < t_m < T$, $e \in L_1[0, T]$, $c \in \mathbb{R}$ *and* J_i, M_i, $i = 1, 2, \ldots, m$, *are continuous mappings of* $G[0, T] \times G[0, T]$ *into* $\mathbb{R}$, *where* $G[0, T]$ *denotes the space of functions regulated on* $[0, T]$.

The presented principle is based on an averaging procedure similar to that introduced by Manásevich and Mawhin for singular periodic problems with p*-Laplacian.*

Отримано принцип існування розв'язку періодичної граничної задачі з імпульсною дією, $u'' + c\,u' = g(x) + e(t)$, $u(t_i+) = u(t_i) + J_i(u, u')$, $u'(t_i+) = u'(t_i) + M_i(u, u')$, $i = 1, \ldots, m$, $u(0) = u(T)$, $u'(0) = u'(T)$, *де* $g \in C(0, \infty)$ *може мати сильну особливість у нулі. Далі, припускається, що* $0 < t_1 < \ldots < t_m < T$, $e \in L_1[0, T]$, $c \in \mathbb{R}$ *і* J_i, M_i, $i = 1, 2, \ldots, m$, *— неперервні відображення з* $G[0, T] \times G[0, T]$ *в* $\mathbb{R}$, *де* $G[0, T]$ *— простір функцій, регульованих на* $[0, T]$.

Отримання принципу базується на процедурі усереднення, яка є аналогом процедури, запропонованої Менасевічем та Мавхіним, для сингулярних періодичних задач із p*-лапласіаном.*

1. Preliminaries. Starting with Hu and Lakshmikantham [1], periodic boundary-value problems for nonlinear second order impulsive differential equations of the form

$$u'' = f(t, u, u'), \tag{1.1}$$

$$\begin{aligned} u(t_i+) &= u(t_i) + J_i(u, u'), \\ u'(t_i+) &= u'(t_i) + M_i(u, u'), \quad i = 1, 2, \ldots, m, \end{aligned} \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T) \tag{1.3}$$

have been studied by many authors. Usually it is assumed that the function $f\colon [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ fulfils the Carathéodory conditions,

$$0 < t_1 < t_2 < \ldots < t_m < T \ \text{ are fixed points of the interval } [0, T] \tag{1.4}$$

and $J_i, M_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are continuous functions. A rather representative (however not complete) list of related papers is given in references. In particular, in [2–6] the existence results in terms of lower/upper functions obtained by the monotone iterative method can be found. All of these results impose monotonicity of the impulse functions and existence of an associated pair of well-ordered lower/upper functions. The papers [7] and [8] are based on the method of bound sets, however the effective criteria contained therein correspond to the situation when there is a well-ordered pair of constant lower and upper functions. The existence results which apply also to the case when there is a pair of lower and upper functions, which need not be well-ordered, were provided only by Rachůnková and Tvrdý, see [9–12]. Analogous results for impulsive problems with quasilinear differential operator were delivered by Rachůnková and Tvrdý in [13–15]. When no impulses are acting, periodic problems with singularities have been treated by many authors. For rather representative overview and references, see e.g. [16] or [17]. To our knowledge, up to now singular periodic impulsive problems have not been treated. For singular Dirichlet impulsive problems we refer to the papers by Rachůnková [18], Rachůnková and Tomeček [19] and Lee and Liu [20].

In this paper we establish an existence principle suitable for solving singular impulsive periodic problems.

Notations. Throughout the paper we keep the following notation and conventions: for a real-valued function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup \mathrm{ess}_{t\in[0,T]}\, |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)|\, ds.$$

For a given interval $J \subset \mathbb{R}$, by $C(J)$ we denote the set of real-valued functions which are continuous on J. Furthermore, $C^1(J)$ is the set of functions having continuous first derivatives on J and $L_1(J)$ is the set of functions which are Lebesgue integrable on J.

Any function $x : [0,T] \to \mathbb{R}$ which possesses finite limits

$$x(t+) = \lim_{\tau\to t+} x(\tau) \quad \text{and} \quad x(s-) = \lim_{\tau\to s-} x(\tau)$$

for all $t \in [0,T)$ and $s \in (0,T]$ is said to be regulated on $[0,T]$. The linear space of functions regulated on $[0,T]$ is denoted by $G[0,T]$. It is well known that $G[0,T]$ is a Banach space with respect to the norm $x \in G[0,T] \to \|x\|_\infty$ (cf. [21], Theorem I.3.6).

Let $m \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$ be a division of the interval $[0,T]$. We denote $D = \{t_1, t_2, \ldots, t_m\}$ and define $C^1_D[0,T]$ as the set of functions $u\colon [0,T] \to \mathbb{R}$ such that

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \ldots & \ldots\ldots\ldots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]} \in C^1[t_i,t_{i+1}]$ for $i = 0,1,\ldots,m$. In particular, if $u \in C^1_D[0,T]$, then u' possesses finite one-sided limits

$$u'(t-) := \lim_{\tau\to t-} u(\tau) \quad \text{and} \quad u'(s+) := \lim_{\tau\to s+} u(\tau)$$

for each $t \in (0,T]$ and $s \in [0,T)$. Moreover, $u'(t-) = u'(t)$ for all $t \in (0,T]$ and $u'(0+) = u'(0)$. For $u \in C^1_D[0,T]$ we put

$$\|u\|_D = \|u\|_\infty + \|u'\|_\infty.$$

Then $C^1_D[0,T]$ becomes a Banach space when endowed with the norm $\|.\|_D$. Furthermore, by $AC^1_D[0,T]$ we denote the set of functions $u \in C^1_D[0,T]$ having first derivatives absolutely continuous on each subinterval (t_i, t_{i+1}), $i = 1, 2, \ldots, m+1$.

We say that $f : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies the *Carathéodory conditions* on $[0,T] \times \mathbb{R}^2$ if (i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(.,x,y)$ is measurable on $[0,T]$; (ii) for almost every $t \in [0,T]$ the function $f(t,.,.)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in L[0,T]$ such that $|f(t,x,y)| \le m_K(t)$ holds for a.e. $t \in [0,T]$ and all $(x,y) \in K$. The set of functions satisfying the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$ is denoted by $\mathrm{Car}([0,T] \times \mathbb{R}^2)$.

Given a subset Ω of a Banach space X, its closure is denoted by $\overline{\Omega}$. Finally, we will write $\bar{e}$ instead of $\dfrac{1}{T}\displaystyle\int_0^T e(s)\,ds$ and $\Delta^+ u(t)$ instead of $u(t+) - u(t)$.

If $f \in \mathrm{Car}([0,T]\times\mathbb{R}^2)$, problem (1.1) – (1.3) is said to be *regular* and a function $u \in AC^1_D[0,T]$ is its solutions if

$$u''(t) = f(t, u(t), u'(t)) \quad \text{holds for a.e. } t \in [0,T]$$

and conditions (1.2) and (1.3) are satisfied. If $f \notin \mathrm{Car}([0,T] \times \mathbb{R}^2)$, problem (1.1) – (1.3) is said to be *singular*.

In this paper we will deal with rather simplified, however the most typical, case of the singular problem with

$$f(t,x,y) = c\,y + g(x) + e(t) \text{ for } x \in (0,\infty),\ y \in \mathbb{R} \text{ and a.e. } t \in [0,T],$$

where

$$c \in \mathbb{R}, \quad g \in C(0,\infty), \quad e \in L_1[0,T]. \tag{1.5}$$

Definition 1.1. *A function $u \in AC^1_D[0,T]$ is called a solution of the problem*

$$u'' + c\,u' = g(u) + e(t), \quad (1.2),\ (1.3) \tag{1.6}$$

if $u > 0$ a.e. on $[0,T]$,

$$u''(t) + c\,u'(t) = g(u(t)) + e(t) \quad \textit{for a.e. } t \in [0,T],$$

and conditions (1.2) and (1.3) are satisfied.

2. Green's functions and operator representations for impulsive two-point boundary-value problems. For our purposes an appropriate choice of the operator representation of (1.1) – (1.3) is important. To this aim, let us consider the following impulsive problem with nonlinear two-point boundary conditions,

$$u'' + a_2(t)\,u' + a_1(t)\,u = f(t,u,u') \text{ a.e. on } [0,T], \tag{2.1}$$

$$\Delta^+ u(t_i) = J_i(u,u'), \quad \Delta^+ u'(t_i) = M_i(u,u'), \quad i = 1,2,\ldots,m, \tag{2.2}$$

$$P\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q\begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = R(u,u'), \tag{2.3}$$

and its linearized version

$$u'' + a_2(t)\,u' + a_1(t)\,u = h(t) \text{ a.e. on } [0,T], \tag{2.4}$$

$$\Delta^+ u(t_i) = d_i, \quad \Delta^+ u'(t_i) = d'_i, \quad i = 1,2,\ldots,m, \tag{2.5}$$

$$P\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q\begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = c, \tag{2.6}$$

where

$$\begin{cases} a_1,\, h \in L[0,T],\ a_2 \in C[0,T],\ f \in \mathrm{Car}([0,T]\times\mathbb{R}^2), \\ J_i \text{ and } M_i:\ G[0,T]\times G[0,T] \to \mathbb{R},\ i=1,2,\ldots,m, \text{ are continuous mappings}, \\ c \in \mathbb{R}^2,\ d_i,\ d'_i \in \mathbb{R},\ i = 1,2,\ldots,m, \\ P,\ Q \text{ are real } 2\times 2\text{-matrices},\ \mathrm{rank}(P,Q) = 2, \\ R:\ G[0,T]\times G[0,T] \to \mathbb{R}^2 \text{ is a continuous mapping.} \end{cases} \tag{2.7}$$

Solutions of problems (2.1) – (2.3) and (2.4) – (2.6) are defined in a natural way quite analogously to the above mentioned definition of regular periodic problems. Problem (2.4) – (2.6) is equivalent to the two-point problem for a special case of generalized linear differential systems of the form

$$x(t) - x(0) - \int_0^t A(s)\,x(s)\,ds = b(t) - b(0) \quad \text{on } [0,T], \tag{2.8}$$

$$P\,x(0) + Q\,x(T) = c, \tag{2.9}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -a_1(t) & -a_2(t) \end{pmatrix}, \tag{2.10}$$

$$b(t) = \int_0^t \begin{pmatrix} 0 \\ h(s) \end{pmatrix} ds + \sum_{i=1}^m \begin{pmatrix} d_i \\ d'_i \end{pmatrix} \chi_{(t_i,T]}(t), \quad t \in [0,T],$$

and $\chi_{(t_i,T]}(t) = 1$ if $t \in (t_i,T]$, $\chi_{(t_i,T]}(t) = 0$ otherwise. Solutions of (2.8), (2.9) are 2-vector functions of bounded variation on $[0,T]$ satisfying the two-point condition (2.9) and fulfilling the integral equation (2.8) for all $t \in [0,T]$, cf. e.g. [22]. Assume that the homogeneous problem

$$u'' + a_2(t)\,u' + a_1(t)\,u = 0, \quad P\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q\begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = 0 \tag{2.11}$$

has only the trivial solution. Then, obviously, the problem

$$x' - A(t)\,x = 0, \quad P\,x(0) + Q\,x(T) = 0 \tag{2.12}$$

has also only the trivial solution. In view of [23] (Theorems 4.2 and 4.3) (see also [24], Theorem 4.1), problem (2.8), (2.9) has a unique solution x and it is given by

$$x(t) = \int_0^T \Gamma(t,s)\, d[b(s)] + x_0(t), \quad t \in [0,T], \tag{2.13}$$

where x_0 is the uniquely determined solution of

$$x' - A(t)\, x = 0, \;\; P\, x(0) + Q\, x(T) = c,$$

and

$$\Gamma(t,s) = (\gamma_{i,j}(t,s))_{i,j=1,2}$$

is Green's matrix for (2.12). Recall that, for each $s \in (0,T)$, the matrix function $t \to \Gamma(t,s)$ is absolutely continuous on $[0,T] \setminus \{s\}$ and

$$\frac{\partial}{\partial t}\Gamma(t,s) - A(t)\,\Gamma(t,s) = 0 \quad \text{for a.e. } t \in [0,T],$$

$$P\,\Gamma(0,s) + Q\,\Gamma(T,s) = 0,$$

$$\Gamma(t+,t) - \Gamma(t-,t) = I,$$

where I stands for the identity 2×2-matrix. In particular, the component $\gamma_{1,2}$ of Γ is absolutely continuous on $[0,T]$ for each $s \in (0,T)$ and

$$\frac{\partial}{\partial t}\gamma_{1,2}(t,s) = \gamma_{2,2}(t,s) \quad \text{for a.e. } t \in [0,T].$$

Denote $G(t,s) = \gamma_{1,2}(t,s)$. Then $G(t,s)$ is Green's function of (2.11). Furthermore, we have

$$\frac{\partial}{\partial s}\Gamma(t,s) = -\Gamma(t,s)\, A(s) \quad \text{for all } t \in (0,T) \text{ and a.e. } s \in [0,T].$$

In particular,

$$\gamma_{1,1}(t,s) = -\frac{\partial}{\partial s} G(t,s) + a_1(s)\, G(t,s) \quad \text{for all } t \in [0,T] \text{ and a.e. } s \in [0,T].$$

Inserting (2.10) into (2.13) we get that, for each $h \in L[0,T]$, $c, d_i, d_i' \in R$, $i = 1,2,\ldots,m$, the unique solution u of problem (2.4) – (2.6) is given by

$$u(t) = u_0(t) + \int_0^t G(t,s)\, h(s)\, ds +$$
$$+ \sum_{i=1}^m \left(-\frac{\partial}{\partial s} G(t,t_i) + a_1(t)\, G(t,t_i)\right) d_i + \sum_{i=1}^m G(t,t_i)\, d_i' \quad \text{for } t \in [0,T],$$

where u_0 is the uniquely determined solution of the problem

$$u'' + a_2(t)\, u' + a_1(t)\, u = 0, (2.6). \tag{1}$$

Now, choose an arbitrary $w \in C^1_D[0,T]$ and put

$$h(t) = f(t, w(t), w'(t)) \quad \text{for a.e. } t \in [0,T],$$

$$d_i = J_i(w, w'),\ d'_i = M_i(w, w'),\ i = 1, 2, \ldots, m,$$

$$c = R(w, w').$$

Then $h \in L[0,T]$, c, d_i, $d'_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, and there is a unique $u \in AC^1_D[0,T]$ fulfilling (2.4) – (2.6) and it is given by (2.12). Therefore, assuming, in addition, that the problem

$$u'' + a_2(t)\, u' + a_1(t)\, u = 0, \tag{2.14}$$

$$P \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q \begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = R(u, u') \tag{2.15}$$

has a unique solution u_0, we conclude that $u \in C^1_D[0,T]$ is a solution to (2.1) – (2.3) if and only if

$$u(t) = u_0(t) + \int\limits_0^t G(t,s)\, f(s, u(s), u'(s))\, ds +$$

$$+ \sum_{i=1}^m \left(-\frac{\partial}{\partial s} G(t, t_i) + a_1(t)\, G(t, t_i) \right) J_i(u, u') + \sum_{i=1}^m G(t, t_i)\, M_i(u, u') \quad \text{for } t \in [0,T].$$

Let us define operators F_1 and $F_2 : C^1_D[0,T] \to C^1_D[0,T]$ by

$$(F_1\, u)(t) = \int\limits_0^T G(t,s)\, f(s, u(s), u'(s))\, ds, \quad t \in [0,T],$$

and

$$(F_2\, u)(t) = u_0(t) + \sum_{i=1}^m \left(-\frac{\partial}{\partial s} G(t, t_i) + a_1(t)\, G(t, t_i) \right) J_i(u, u') +$$

$$+ \sum_{i=1}^m G(t, t_i)\, M_i(u, u'), \quad t \in [0,T],$$

respectively. The former one, F_1, is a composition of the Green type operator

$$h \in L_1[0,T] \to \int\limits_0^T G(t,s)\, h(s)\, ds \in C^1[0,T],$$

which is known to map equiintegrable subsets[1] of $L_1[0,T]$ onto relatively compact subsets of $C^1[0,T] \subset C^1_D[0,T]$, and the superposition operator generated by $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, which

[1] I.e., sets of functions having a common integrable majorant.

similarly to the classical setting maps bounded subsets of $C^1_D[0,T]$ to equiintegrable subsets of $L_1[0,T]$. Therefore, it is easy to see that F_1 is completely continuous. Furthermore, since J_i, M_i, $i = 1,2,\dots,m$, are continuous mappings, the operator F_2 is continuous as well. Having in mind that F_2 maps bounded sets onto bounded sets and its values are contained in a $(2m{+}1)$-dimensional subspace[2] of $C^1_D[0,T]$, we conclude that the operators F_2 and $F = F_1 + F_2$ are completely continuous as well.

So, we have the following assertion.

Proposition 2.1. *Assume (1.4) and (2.7). Furthermore, let problem (2.11) have Green's function $G(t,s)$ and let $u_0 \in AC^1_D[0,T]$ be a uniquely defined solution of problem (2.14), (2.15). Then $u \in AC^1_D$ is a solution to (2.1)–(2.3) if and only if $u = Fu$, where $F : C^1_D[0,T] \to \to C^1_D[0,T]$ is the completely continuous operator given by*

$$(Fu)(t) = u_0(t) + \int_0^T G(t,s)\,\big(f(t,u(s),u'(s)) - a_1(s)\,u(s) - a_2(s)\,u'(s)\big)\,ds +$$

$$+ \sum_{i=1}^m \left(-\frac{\partial}{\partial s}G(t,t_i) + a_1(t)G(t,t_i)\right) J_i(u,u') + \sum_{i=1}^m G(t,t_i)\,M_i(u,u'),\ t \in [0,T].$$

In particular, if $a_1(t) = a_2(t) = 0$ on $[0,T]$,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad\text{and}\quad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then problem (2.11) reduces to the simple Dirichlet problem

$$u'' = 0, \quad u(0) = u(T) = 0$$

and its Green's function is well-known,

$$G(t,s) = \begin{cases} \dfrac{s\,(t-T)}{T} & \text{if } 0 \le s < t \le T, \\[2mm] \dfrac{t\,(s-T)}{T} & \text{if } 0 \le t \le s \le T \end{cases} \tag{2.16}$$

and

$$\frac{\partial}{\partial s}G(t,s) = \begin{cases} \dfrac{T-t}{T} & \text{if } 0 \le s < t \le T, \\[2mm] -\dfrac{t}{T} & \text{if } 0 \le t \le s \le T. \end{cases}$$

Furthermore, let us notice that the periodic boundary conditions (1.3) can be reformulated as

$$u(0) = u(T) = u(0) + u'(0) - u'(T),$$

i.e., in the form (2.15), where

$$R(u,v) = u(0) + v(0) - v(T) \quad \text{for } u,v \in G[0,T].$$

It is easy to see that, in such a case, for any $c \in \mathbb{R}$ the only solution to (2.14), (2.15) is $u_0(t) \equiv c$. Therefore, we have the following corollary of Proposition 2.1.

[2] I.e., spanned over the set $\left\{u_0, G(.,t_i), \left(-\dfrac{\partial}{\partial s}G(.,t_i) + a_1\,G(.,t_i)\right), i = 1,2,\dots,m\right\}$.

Corollary 2.1. *Assume (1.4) and (2.7) and let the function $G(t,s)$ be given by (2.16). Then $u \in AC^1_D$ is a solution to (1.1)–(1.3) if and only if $u = Fu$, where $F : C^1_D[0,T] \to C^1_D[0,T]$ is the completely continuous operator given by*

$$(Fu)(t) = u(0) + u'(0) - u'(T) + \int_0^T G(t,s)\, f(t, u(s), u'(s))\, ds -$$

$$- \sum_{i=1}^m \frac{\partial}{\partial s} G(t,t_i)\, J_i(u,u') + \sum_{i=1}^m G(t,t_i)\, M_i(u,u'),\ \ t \in [0,T].$$

Remark 2.1. Similarly, $u \in AC^1_D$ is a solution to the impulsive Dirichlet problem (1.1), (1.2), $u(0) = u(T) = c$ if and only if $u = F_{\mathrm{dir}}\, u$, where

$$(F_{\mathrm{dir}} u)(t) = c + \int_0^T G(t,s)\, f(t, u(s), u'(s))\, ds -$$

$$- \sum_{i=1}^m \frac{\partial}{\partial s} G(t,t_i)\, J_i(u,u') + \sum_{i=1}^m G(t,t_i)\, M_i(u,u'),\ \ t \in [0,T].$$

3. Existence principle.

Theorem 3.1. *Let assumptions (1.4) and (1.5) hold. Furthermore, assume that there exist $r \in (0,\infty)$, $R \in (r,\infty)$ and $R' \in (0,\infty)$ such that*

(i) $r < v < R$ on $[0,T]$ and $\|v'\|_\infty < R'$ for each $\lambda \in (0,1]$ and for each positive solution v of the problem

$$v''(t) = \lambda\, \big(-c\, v'(t) + g(v(t)) + e(t)\big) \quad \text{for a.e. } t \in [0,T], \tag{3.1}$$

$$\Delta^+ v(t_i) = \lambda\, J_i(v,v'), \qquad i = 1,2,\ldots,m, \tag{3.2}$$

$$\Delta^+ v'(t_i) = \lambda\, M_i(v,v'), \quad i = 1,2,\ldots,m, \tag{3.3}$$

$$v(0) = v(T), \quad v'(0) = v'(T); \tag{3.4}$$

(ii) $(g(x) + \bar{e} = 0) \implies r < x < R$;
(iii) $(g(r) + \bar{e})\, (g(R) + \bar{e}) < 0$.
Then problem (1.6) has a solution u such that

$$r < u < R \ \ \text{on } [0,T] \text{ and } \ \|u'\|_\infty < R'.$$

Proof. *Step 1.* For $\lambda \in [0,1]$ and $v \in C^1_D[0,T]$ denote

$$\Xi_\lambda(v) = \int_0^T g(v(s))\, ds + T\bar{e} + \sum_{i=1}^m M_i(v,v') + \lambda\, c \sum_{i=1}^m J_i(v,v'). \tag{3.5}$$

Notice that

$$\Xi_\lambda(v) = 0 \quad \text{holds for all solutions } v \in C^1_D[0,T] \text{ of } (3.1)-(3.4). \tag{3.6}$$

Indeed, let $v \in C^1_D[0,T]$ be a solution to (3.1) – (3.4). Then

$$\int_0^T v''(s)\,ds = \sum_{i=0}^m \int_{t_i}^{t_{i+1}} v''(s)\,ds = \sum_{i=0}^m \big[v'(t_{i+1}) - v'(t_i+)\big] =$$

$$= v'(T) - v'(0) - \sum_{i=1}^m \Delta^+ v'(t_i) = -\lambda \sum_{i=1}^m M_i(v,v')$$

and

$$\int_0^T c\,v'(s)\,ds = c\sum_{i=0}^m \int_{t_i}^{t_{i+1}} v'(s)\,ds = c\sum_{i=0}^m \big[v(t_{i+1}) - v(t_i+)\big] =$$

$$= c\left[v(T) - v(0) - \sum_{i=1}^m \Delta^+ v(t_i)\right] = -\lambda\, c \sum_{i=1}^m J_i(v,v').$$

Thus, integrating (3.1) over $[0,T]$ gives (3.6).

Step 2. Consider system (3.7), (3.2), (3.4), where (3.7) is the functional-differential equation:

$$v'' = \lambda\big[-c\,v' + g(v) + e(t)\big] + (1-\lambda)\,\frac{1}{T}\,\Xi_\lambda(v). \tag{3.7}$$

Due to (3.6), we can see that for each $\lambda \in [0,1]$ the problems (3.1) – (3.4) and (3.7), (3.2) – (3.4) are equivalent. Moreover, for $\lambda = 1$, problem (3.7), (3.2), (3.4) reduces to the given problem (1.6) (with u replaced by v).

Now, notice that in view of (2.16) we have

$$\int_0^T G(t,s)\,ds = \frac{1}{2}\,t\,(t-T) \quad \text{for } t \in [0,T]$$

and define, for $\lambda \in [0,1]$, $u \in C^1_D[0,T]$, $u > 0$ on $[0,T]$, and $t \in [0,T]$,

$$F_\lambda(u)(t) = u(0) + u'(0) - u'(T) + \lambda \int_0^T G(t,s)\,\big[-cu'(s) + g(u(s)) + e(s)\big]\,ds +$$

$$+ (1-\lambda)\,\frac{t\,(t-T)}{2\,T}\,\Xi_\lambda(u) - \lambda \sum_{i=1}^m \frac{\partial}{\partial s} G(t,t_i)\,J_i(u,u') +$$

$$+ \lambda \sum_{i=1}^m G(t,t_i)\,M_i(u,u'). \tag{3.8}$$

In particular, if $\lambda = 0$, then

$$F_0(u)(t) = u(0) + u'(0) - u'(T) + \frac{t\,(t-T)}{2\,T}\,\Xi_0(u) \quad \text{for } t \in [0,T].$$

Let us put

$$\Omega = \{u \in C^1_D[0,T]:\ r < u < R \text{ and } |u'| < R' \text{ on } [0,T]\}.$$

Arguing similarly to the regular case (see Corollary 2.1), we can conclude that for each $\lambda \in [0,1]$ the operator $F_\lambda : \overline{\Omega} \subset C^1_D[0,T] \to C^1_D$ is completely continuous and a function $v \in \overline{\Omega}$ is a solution of (3.7), (3.2) – (3.4) if and only if it is a fixed point of F_λ. In particular,

$$u \in \overline{\Omega} \text{ is a solution to (1.6) if and only if } F_1(u) = u. \tag{3.9}$$

Step 3. We will show that

$$F_\lambda(u) \neq u \quad \text{for all } u \in \partial\Omega \quad \text{and} \quad \lambda \in [0,1]. \tag{3.10}$$

Indeed, for $\lambda \in (0,1]$ relation (3.10) follows immediately from assumption (i), while for $\lambda = 0$ it is a corollary of assumption (ii) and of the following claim.

Claim. *$u \in \overline{\Omega}$ is a fixed point of F_0 if and only if there is $x \in \mathbb{R}$ such that $u(t) \equiv x$ on $[0,T]$, $x \in (r,R)$, and*

$$g(x) + \bar{e} = 0. \tag{3.11}$$

Proof of Claim. Let $u \in \overline{\Omega}$ be a fixed point of $F_0(v)$, i.e.,

$$u(t) = u(0) + u'(0) - u'(T) + \frac{t\,(t-T)}{2\,T}\,\Xi_0(u) \quad \text{for all } t \in [0,T]. \tag{3.12}$$

Inserting $t = 0$ into (3.12), we get $u(0) = u(0) + u'(0) - u'(T)$, which implies that $u'(0) = u'(T)$. Similarly, inserting $t = T$ we get $u(T) = u(0)$. Furthermore,

$$u'(t) = \frac{2\,t - T}{2T}\,\Xi_0(u) \quad \text{for } t \in [0,T].$$

Since $u'(0) = u'(T)$, it follows that $\Xi_0(u) = 0$. This means that u is constant on $[0,T]$. Denote $x = u(0)$. Then $0 = \Xi_0(u) = T\,(g(x) + \bar{e})$, i.e., (3.11) is true. On the other hand, it is easy to see that if $x \in \mathbb{R}$ is such that (3.11) holds and $u(t) \equiv x$ on $[0,T]$, then $u \in \overline{\Omega}$ is a fixed point of F_0. This completes the proof of the claim.

Step 4. By Step 3 and by the invariance under the homotopy property of the topological degree, we have

$$\deg(I - F_1, \Omega) = \deg(I - F_0, \Omega). \tag{3.13}$$

Step 5. Let us denote

$$\mathbb{X} = \{u \in C^1_D[0,T]\ :\ u(t) \equiv u(0) \text{ on } [0,T]\} \quad \text{and} \quad \Omega_0 = \Omega \cap \mathbb{X}.$$

Notice that $\Omega_0 = \{u \in \mathbb{X}:\ r < u(0) < R\}$ and $\overline{\Omega}_0 = \{u \in \mathbb{X}:\ r \le u(0) < R\}$. By Claim in Step 3, all fixed points of F_0 belong to Ω_0. Hence, by the excision property of the topological degree we have

$$\deg(I - F_0, \Omega) = \deg(I - F_0, \Omega_0). \tag{3.14}$$

Step 6. Define

$$\widetilde{F}_\mu(u)(t) = u(0) + \Big[1 - \mu + \frac{\mu}{2}\, t\,(t - T)\Big]\; \big(g(u(0) + \bar{e}\big) \tag{3.15}$$

$$\text{for } t \in [0,T],\ u \in \overline{\Omega}_0 \text{ and } \mu \in [0,1].$$

We have

$$\widetilde{F}_0(u) = u(0) + g(u(0)) + \bar{e} \quad \text{and} \quad \widetilde{F}_1(u) = F_0(u) \quad \text{for each } u \in \mathbb{X}.$$

Similarly to F_λ, the operators $\widetilde{F}_\mu$, $\mu \in [0,1]$, are also completely continuous and, by the Claim in Step 3, we have

$$\widetilde{F}_1(u) \neq u \quad \text{for all} \quad u \in \partial\,\Omega_0.$$

Let i and i_{-1} be respectively the natural isometrical isomorphism $\mathbb{R} \to \mathbb{X}$ and its inverse, i.e.,

$$i(x)(t) \equiv u \text{ for } x \in \mathbb{R} \qquad \text{and} \qquad i_{-1}(u) = u(0) \text{ for } u \in \mathbb{X},$$

and assume that $\mu \in [0,1)$, $x \in (0,\infty)$, $u = i(x)$ and $\widetilde{F}_\mu(u) = u$. Then

$$\Big[1 - \mu + \frac{\mu}{2}\, t\,(T - t)\Big]\; \big(g(x) + \bar{e}\big) = 0 \quad \text{for all} \quad t \in [0,T].$$

If $t = 0$, this relation reduces to $g(x) + \bar{e} = 0$, which is, due to assumption (ii) possible only if $x \in (r,R)$. To summarize, we have

$$\widetilde{F}_\mu(u) \neq u \quad \text{for all } u \in \partial\,\Omega_0 \quad \text{and all } \mu \in [0,1].$$

Hence, using the homotopy invariance property of the topological degree and taking into account that $\dim \mathbb{X} = 1$, we conclude that

$$\deg(I - \widetilde{F}_1, \Omega_0) = \deg(I - \widetilde{F}_1, \Omega_0) = d_B(I - \widetilde{F}_0, \Omega_0), \tag{3.16}$$

where $d_B(I - \widetilde{F}_0, \Omega_0)$ stands for the Brouwer degree of $I - \widetilde{F}_0$ with respect to the set Ω_0 (and the point 0).

Step 7. Define $\Phi :\ x \in (0,\infty) \to g(x) + \bar{e} \in \mathbb{R}$. Then

$$(I - \widetilde{F}_0)(i(x)) = i(\Phi(x)) \quad \text{for each } x \in (0,\infty).$$

In other words, $\Phi = i_{-1} \circ (I - \widetilde{F}_0) \circ i$ on $(0,\infty)$. Consequently,

$$d_B(I - \widetilde{F}_0, \Omega_0) = d_B(\Phi, (r,R)). \tag{3.17}$$

Now, put

$$\Psi(x) = \Phi(r)\,\frac{R - x}{R - r} + \Phi(R)\,\frac{x - r}{R - r}\,.$$

We can see that Ψ has a unique zero $x_0 \in (r,R)$ and

$$\Psi'(x_0) = \frac{\Phi(R) - \Phi(r)}{R - r}.$$

Hence, by the definition of the Brouwer degree in $\mathbb{R}$, we have

$$d_B(\Psi, (r, R)) = \operatorname{sign} \Psi'(x_0) = \operatorname{sign} (\Phi(R) - \Phi(r)) .$$

By the homotopy property and thanks to our assumption (iii), we conclude that

$$d_B(\Phi, (r, R)) = d_B(\Psi, (r, R)) = \operatorname{sign} (\Phi(R) - \Phi(r)) \neq 0. \tag{3.18}$$

Step 8. To summarize, by (3.13) – (3.18) we have

$$\deg(I - F_1, \Omega) \neq 0,$$

which, in view of the existence property of the topological degree, shows that F_1 has a fixed point $u \in \Omega$. By Step 1 this means that problem (1.6) has a solution.

The theorem is proved.

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