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CONTROL OF THE STRAIN FIELD NEAR THE CRACK IN ELASTIC LAYER

КЕРУВАННЯ ПОЛЕМ НАПРУГИ БІЛЯ ТРІЩИНИ В ПРУЖНОМУ ШАРІ

M. Ciarletta, G. Iovane

Univ. Salerno
Via Ponte don Melillo, 84084 Fisciano (SA), Italy
e-mail: iovane@diima.unisa.it

M. A. Sumbatyan

Rostov State Univ.
Zorge Str., 5, Rostov-on-Don 344090, Russia
e-mail: sumbat@math.rsu.ru

In the present paper we consider the strain analysis near the crack parallel to boundary surfaces of a linear isotropic elastic layer of constant thickness. We pose the problem to minimize opening of the crack's faces, when a normal force is applied to a boundary of the layer. The control secondary loading is a normal stress uniformly distributed over the boundary surfaces as to oppose the opening caused by the external applied force. We show that the most dangerous are tangential deformations, and the normal ones are always by many times less dangerous.

Проаналізовано пружність біля тріщини, яка паралельна межі поверхні лінійного ізотропного пружного шару сталої товщини. Ставиться проблема мінімізації ширини тріщини у випадку, коли на межі шару діє нормальна сила. Вторинним керуючим навантаженням є нормальне напруження, рівномірно розподілене по межі поверхонь, яке протидіє розширенню тріщини, що виникає під дією зовнішньо прикладеної сили. Показано, що найбільш загрозливими є тангенціальні деформації, а нормальні деформації завжди в багато разів менш загрозливі.

1. Introduction. Strain-stress control of cracked elastic materials is very important, both in theoretical and practical aspects, problems in civil engineering, soil mechanics, mechanics of composite and other inhomogeneous materials, in fracture mechanics, medicine, and some other fields of science and technology [1]. Typically, the researchers' main goal is to control stress concentration near the edges of the cracks arising in damaged materials, but to control the crash process and its consequence, including the control of the internal crack faces behavior is also very important in medicine and some other application [2].

From the mathematical point of view, crack analysis can be usually reduced to some integral equations, which admit explicit-form solutions only for simplest geometries [3]. If the elastic medium occupies an infinite layer of a constant thickness, then the problem admits an explicit analytical solution in the form of some Fourier integral quadratures. The most direct method reduces the crack problems in elastic media to hypersingular integral equations that require a nonstandard theoretical or numerical treatment.

In our recent work [4] we have developed and justified a numerical method, which allows one to directly treat hypersingular integral equations without any additional transformations (in

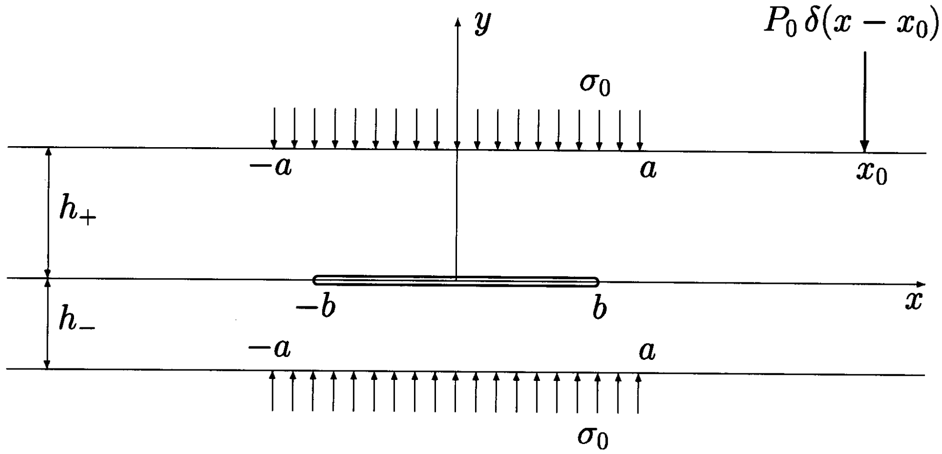

Fig 1. A crack dislocated in the elastic layer of a constant thickness $h = h_+ + h_-$, under the applied external load P_0 and the uniform controlling stress σ_0.

contrast with a standard approach reducing the problem to an infinite system of linear algebraic equations). In that work we demonstrate an application of the proposed technique to crack problems in unbounded classical elastic medium. Then we further develop these results for the crack problem in a porous elastic medium. In this work we consider the problem of cracks for the linear isotropic elastic layer, which is subjected to a process. As it can be seen below, in the case of a classical linear isotropic elastic layer, the problem is reduced to a system of hypersingular integral equations. So, the main purpose of the present work is to show that our approach is efficient not only for a single hypersingular equation but also for the system of such equations. Besides, we aim at a practical application in control of behavior of the free faces of the crack, under various load (i.e., the character of boundary conditions) applied to boundaries of the layer. We stress, when the applied load causes higher opening of the crack and a higher relative displacement of its faces, that is not desired in applications such as treatment and restoration of cracked bones, gluing together different materials during a technological process under the loading.

2. Mathematical formulation of the problem. Let us consider the crack problem in the layer of thickness $h = h_+ + h_-$ (see Fig. 1). Then the basic partial differential equations for the displacement vector components $\bar{u} = \{u_x(x,y),\, u_y(x,y),\, 0\}$ in the (in-plane) 2-dimensional problem are ($c^2 = \mu/(\lambda + 2\mu) = c_s^2/c_p^2$, λ and μ-elastic moduli)[1]

$$\begin{aligned} &\frac{\partial^2 u_x}{\partial x^2} + c^2\,\frac{\partial^2 u_x}{\partial y^2} + (1 - c^2)\frac{\partial^2 u_y}{\partial x \partial y} = 0, \\ &\frac{\partial^2 u_y}{\partial y^2} + c^2\,\frac{\partial^2 u_y}{\partial x^2} + (1 - c^2)\frac{\partial^2 u_x}{\partial x \partial y} = 0. \end{aligned} \tag{2.1}$$

[1] Note that the constant c in these formulas is a ratio of transverse and longitudinal wave speeds.

As soon as the system (2.1) is solved, the components of the stress tensor (σ) can be calculated in terms of displacements, as follows:

$$\begin{aligned}
\frac{\sigma_{xx}}{\lambda+2\mu} &= \frac{\partial u_x}{\partial x} + (1-2c^2)\frac{\partial u_y}{\partial y},\\
\frac{\sigma_{yy}}{\lambda+2\mu} &= (1-2c^2)\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y},\\
\frac{\tau_{xy}}{\mu} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x},
\end{aligned} \tag{2.2}$$

where only representation for the most physically important components is given.

The system (2.1) is to be solved in the domain of the x-variable with $x \in [-\infty, \infty]$. So, we can apply the Fourier transform along this variable, so that any function $f(x, y)$ transforms to its Fourier image ($f \Longrightarrow F$) as follows:

$$\begin{aligned}
F(s,y) &= \int\limits_{-\infty}^{\infty} f(x,y)e^{isx}dx,\\
f(x,y) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(s,y)e^{-isx}ds,
\end{aligned} \tag{2.3}$$

with the following evident and well known properties: if $f \Longrightarrow F$, then

$$\begin{aligned}
\frac{\partial f(x,y)}{\partial x} &\Longrightarrow (-is)F(s,y), \quad \frac{\partial^2 f(x,y)}{\partial x^2} \Longrightarrow -s^2F(s,y),\\
\frac{\partial f(x,y)}{\partial y} &\Longrightarrow F'(s,y), \quad \frac{\partial^2 f(x,y)}{\partial y^2} \Longrightarrow F''(s,y).
\end{aligned} \tag{2.4}$$

Here primes denote respective derivatives applied to the variable y (in fact, the variable s can be considered as a parameter when solving the system with respect to y).

According to the Eqs. (2.3), the system (2.1) of partial differential equations in the Fourier transformed space becomes a system of ordinary differential equations with constant coefficients (all capital letters denote Fourier images for relevant original functions),

$$\begin{aligned}
c^2U_x'' - s^2U_x + (1-c^2)(-is)U_y' &= 0,\\
(1-c^2)(-is)U_x' + U_y'' - c^2s^2\,U_y &= 0,
\end{aligned} \tag{2.5}$$

where all derivatives here are again applied with respect to the variable y.

It is well known that exact analytical solution of such a system can be constructed in terms of a characteristic polynomial. Let us consider separately the upper (+) layer $0 < y < h_+$ and the lower (−) one $-h < y < 0$. Then it can be verified directly that a general solution to the system (2.5) is

$$
\begin{gathered}
U_x^{\pm} = A^{\pm} i \operatorname{ch}(sy) + B^{\pm} i \operatorname{sh}(sy) - C^{\pm} i \left[\frac{1+c^2}{s(1-c^2)} \operatorname{ch}(sy) + y \operatorname{sh}(sy) \right] - \\
- D^{\pm} i \left[\frac{1+c^2}{s(1-c^2)} \operatorname{sh}(sy) + y \operatorname{ch}(sy) \right], \\
U_y^{\pm} = -A^{\pm} \operatorname{sh}(sy) - B^{\pm} \operatorname{ch}(sy) + C^{\pm} y \operatorname{ch}(sy) + D^{\pm} y \operatorname{sh}(sy),
\end{gathered}
\tag{2.6}
$$

where sh and ch are hyperbolic sine and cosine, respectively.

Eight unknown constants $A^{\pm}$, $B^{\pm}$, $C^{\pm}$, $D^{\pm}$ arising in Eqs. (2.6) should be defined by satisfying the boundary conditions that we are now beginning to formulate. Let us assume that the both sides of the layer are free of tangential stress. Taking into account $(2.2)_3$ and (2.4), the first two boundary conditions are: $\tau_{xy} = 0$, $y = h_+$, $y = -h_-$: $U_x' - isU_y = 0$. Consequently by using (2.6) we obtain

$$
\begin{gathered}
A^+ s \operatorname{sh}(sh_+) + B^+ s \operatorname{ch}(sh_+) - C^+ \left[\frac{\operatorname{sh}(sh_+)}{1-c^2} + (sh_+) \operatorname{ch}(sh_+) \right] - \\
- D^+ \left[\frac{\operatorname{ch}(sh_+)}{1-c^2} + (sh_+) \operatorname{sh}(sh_+) \right] = 0, \\
- A^- s \operatorname{sh}(sh_-) + B^- s \operatorname{ch}(sh_-) + C^- \left[\frac{\operatorname{sh}(sh_-)}{1-c^2} + (sh_-) \operatorname{ch}(sh_-) \right] - \\
- D^- \left[\frac{\operatorname{ch}(sh_-)}{1-c^2} + (sh_-) \operatorname{sh}(sh_-) \right] = 0.
\end{gathered}
\tag{2.7}
$$

Further, we assume that the normal load applied to the upper boundary surface $y = h_+$ of the layer is $\sigma^+(x)$, and the one applied to the lower boundary $y = -h_-$ is $\sigma^-(x)$. By considering $(2.2)_2$ and (2.4) the second pair of the boundary conditions is

$$
\sigma_{yy} = \sigma^{\pm}(x), \quad y = h_+, \quad y = -h_- : (1-2c^2)(-is)U_x + U_y' = \frac{\widetilde{\sigma}^{\pm}(s)}{\lambda + 2\mu}. \tag{2.8}
$$

By using (2.6) we get

$$
\begin{gathered}
- A^+ s \operatorname{ch}(sh_+) - B^+ s \operatorname{sh}(sh_+) + C^+ \left[\frac{c^2 \operatorname{ch}(sh_+)}{1-c^2} + (sh_+) \operatorname{sh}(sh_+) \right] + \\
+ D^+ \left[\frac{c^2 \operatorname{sh}(sh_+)}{1-c^2} + (sh_+) \operatorname{ch}(sh_+) \right] = \frac{\widetilde{\sigma}^+(s)}{2\mu}, \\
- A^- s \operatorname{ch}(sh_-) + B^- s \operatorname{sh}(sh_-) + C^- \left[\frac{c^2 \operatorname{ch}(sh_-)}{1-c^2} + (sh_-) \operatorname{sh}(sh_-) \right] - \\
- D^- \left[\frac{c^2 \operatorname{sh}(sh_-)}{1-c^2} + (sh_-) \operatorname{ch}(sh_-) \right] = \frac{\widetilde{\sigma}^-(s)}{2\mu},
\end{gathered}
\tag{2.9}
$$

where $\widetilde{\sigma}^{\pm}(s)$ denote the Fourier transform of the functions $\sigma^{\pm}(s)$.

We thus have already formulated four boundary conditions (2.7), (2.9) for eight constants. The remaining four conditions should be put over the junction line $y = 0$, where we obviously have the mixed boundary conditions. If the faces of the crack are free of load, then at $y = 0$,

$$\begin{aligned} \tau_{xy}^{+} &= \tau_{xy}^{-}, \quad |x| > b, \\ \tau_{xy}^{+} &= \tau_{xy}^{-} = 0, \quad |x| \le b, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \sigma_{yy}^{+} &= \sigma_{yy}^{-}, \quad |x| > b, \\ \sigma_{yy}^{+} &= \sigma_{yy}^{-} = 0, \quad |x| \le b, \end{aligned} \tag{2.11}$$

where the first relations in the Eqs. (2.10), (2.11) mean continuity of the stress.

The last pair of the junction boundary conditions should take into account condition of the displacements,

$$u_x^{+} = u_x^{-}, \quad u_y^{+} = u_y^{-}, \quad |x| > b, \quad y = 0. \tag{2.12}$$

It should be noted that (2.10), (2.11) imply continuity of both stresses τ_{xy} and σ_{yy} along the total line $y = 0$,

$$\tau_{xy}^{+} = \tau_{xy}^{-}, \quad \sigma_{yy}^{+} = \sigma_{yy}^{-}, \quad |x| < \infty, \tag{2.13}$$

which with the use of the Eqs. (2.2) gives the (third) pair of relations for the eight unknown coefficients,

$$\begin{aligned} B^{+}s - D^{+}\frac{1}{1-c^2} &= B^{-}s - D^{-}\frac{1}{1-c^2}, \\ -A^{+}s + C^{+}\frac{c^2}{1-c^2} &= -A^{-}s + C^{-}\frac{c^2}{1-c^2}. \end{aligned} \tag{2.14}$$

In order to construct the 7-th and the 8-th conditions as to finally complete the 8 x 8 system, let us introduce two auxiliary unknown functions, in accordance with Eq. (2.12),

$$u_y^{+} - u_y^{-} = \begin{cases} g_y(x), & |x| \le b, \\ 0, & |x| > b, \end{cases} \qquad y = 0, \tag{2.15}$$

$$u_x^{+} - u_x^{-} = \begin{cases} g_x(x), & |x| \le b, \\ 0, & |x| > b, \end{cases} \qquad y = 0. \tag{2.16}$$

Obviously, the physical meaning of these functions is the relative displacement (vertical and horizontal, respectively) of the upper and the lower faces of the crack. If one considers the functions $g_y(x)$ and $g_x(x)$ which are defined over the total interval $|x| \le b$, then (2.15) and (2.16) become the last pair of the required conditions in Fourier's transformed space,

$$U_y^+ - U_y^- = G_y(s), \quad U_x^+ - U_x^- = G_x(s), \quad y = 0. \tag{2.17}$$

Consequently by using (2.6) and (2.14) we obtain

$$\begin{gathered} -B^+ + B^- = G_y(s), \\ \left[A^+ - C^+ \frac{1+c^2}{s(1-c^2}\right] - \left[A^- - C^- \frac{1+c^2}{s(1-c^2)}\right] = -iG_x(s). \end{gathered} \tag{2.18}$$

So, we have finally arrived at the 8 x 8 system for the uknown coefficients $A^\pm$, $B^\pm$, $C^\pm$, $D^\pm$ that is given by Eqs. (2.7), (2.9), (2.14), (2.18).

The principal determinant of the considered 8 x 8 system is

$$\Delta = \frac{s^2 \left[2\,\mathrm{ch}(sh_-)\,\mathrm{ch}(sh_+)\,\mathrm{ch}(sh) - ch^2(sh_+) - \mathrm{ch}^2(sh_-) - (sh)^2\right]}{(1-c^2)^2}. \tag{2.19}$$

Other determinants are such that application of the Kramer's rule gives for the most important stress tensor components the following expressions ($|x| \le b$):

$$\begin{aligned} \frac{\widetilde{\tau}^+_{xy}(s,0)}{2\mu} &= \frac{\widetilde{\sigma}^+(s)}{2\mu} L_\tau^+(s) + \frac{\widetilde{\sigma}^-(s)}{2\mu} L_\tau^-(s) - G_y(s) L_\tau^y(s) + iG_x(s) L_\tau^x(s), \\ \frac{\widetilde{\sigma}^+_{yy}(s,0)}{2\mu} &= \frac{\widetilde{\sigma}^+(s)}{2\mu} L_\sigma^+(s) + \frac{\widetilde{\sigma}^-(s)}{2\mu} L_\sigma^-(s) + G_y(s) L_\sigma^y(s) + iG_x(s) L_\sigma^x(s), \end{aligned} \tag{2.20}$$

where

$$L_\tau^+ = -\frac{s^2}{\Delta(1-c^2)^2} \left\{(sh_+)\,\mathrm{sh}(sh_-)\,\mathrm{sh}(sh) + (sh_-)(sh)\,\mathrm{sh}(sh_+)\right\},$$

$$L_\tau^- = \frac{s^2}{\Delta(1-c^2)^2} \left\{(sh_-)\,\mathrm{sh}(sh_+)\,\mathrm{sh}(sh) + (sh_+)(sh)\,\mathrm{sh}(sh_-)\right\},$$

$$L_\tau^y = \frac{s^3}{\Delta(1-c^2)} \left\{(sh_+)^2\,\mathrm{sh}^2(sh_-) - (sh_-)^2\,\mathrm{sh}^2(sh_+)\right\}, \quad L_\sigma^x = L_\tau^y,$$

$$\begin{aligned} L_\tau^x = -\frac{s^3}{\Delta(1-c^2)} \Big\{ & \left[1+(sh_+)^2\right] \mathrm{ch}(sh_-)\,\mathrm{sh}(sh_-) + \left[1+(sh_-)^2\right] \mathrm{ch}(sh_+)\,\mathrm{sh}(sh_+) - \\ & - \mathrm{ch}(sh_+)\,\mathrm{ch}(sh_-)\,\mathrm{sh}(sh) - (sh_-)\,\mathrm{sh}^2(sh_+) - (sh_+)\,\mathrm{sh}^2(sh_-) + (sh_+)(sh_-)(sh) \Big\}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} L_\sigma^+ = -\frac{s^2}{\Delta(1-c^2)^2} \{ & [1 + (sh_-)(sh)]\,\mathrm{ch}(sh_+) - (sh_+)\,\mathrm{ch}(sh_-)\,\mathrm{sh}(sh) - \\ & - \mathrm{ch}(sh_-)\,\mathrm{ch}(sh) + (sh)\,\mathrm{sh}(sh_+)\}, \end{aligned}$$

$$L_\sigma^- = -\frac{s^2}{\Delta(1-c^2)^2}\left\{[1+(sh_+)(sh)]\operatorname{ch}(sh_-) - (sh_-)\operatorname{ch}(sh_+)\operatorname{sh}(sh) - \right.$$

$$\left. - \operatorname{ch}(sh_+)\operatorname{ch}(sh) + (sh)\operatorname{sh}(sh_-)\right\},$$

$$L_\sigma^y = -\frac{s^3}{\Delta(1-c^2)}\left\{[1+(sh_+)(sh_-)](sh) - \left[1+(sh_-)^2\right]\operatorname{ch}(sh_+)\operatorname{sh}(sh_+) - \right.$$

$$- [\operatorname{ch}(sh_-) + (sh_+)\operatorname{sh}(sh_-)](sh+)\operatorname{ch}(sh_-) +$$

$$\left. + \operatorname{sh}(sh_+)\operatorname{ch}(sh_-)\operatorname{ch}(sh) - (sh_-)\operatorname{ch}^2(sh_+)\right\}.$$

3. Reduction to a system of hypersingular equations and its properties. In this section we consider the above problem in the context of the applied mechanics, engineering, and biomedical science. Let us now come back to engineering aspects of the considered problem, as it is discussed in Introduction. Recall that we plan to arrange an efficient control of the strain field near the crack. More precisely, we are interested in a minimization of relative displacements of the upper and lower crack's faces, when some external force $P_0\delta(x-x_0)$ is applied in the point x_0 of the (for instance) upper boundary of the layer. Then we try to suppress the introduced relative displacements, i.e., our functions $g_y(x)$ and $g_x(x)$, $|x| \le b$, by arranging a couple of symmetric uniformly distributed normal loads $\sigma_{yy} \equiv \sigma_0$, $y = h$, $y = -h_-$. The main aspect of the present study is to establish relations between σ_0, P_0, $\max|g_x(x)|$, $\max|g_y(x)|$, and the variation of other physical and geometrical parameters. Strictly speaking, the problem posed in such away is a classical inverse problem. However we do not aim at its full investigation, and restrict our goal to a statement of some helpful relations between all these quantities.

First of all, let us note that for the chosen type of the applied load we have

$$\sigma_+ = \left\{\begin{array}{cc} -\sigma_0, & |x| \le a \\ 0\,, & |x| > a \end{array}\right\} - P_0\,\delta(x-x_0), \tag{3.1}$$

$$\sigma_- = \left\{\begin{array}{cc} -\sigma_0\,, & |x| \le a \\ 0, & |x| > a \end{array}\right\}, \tag{3.2}$$

so the Fourier transforms $\widetilde{\sigma}^{\pm}$, which are present in the system (2.20), now can be calculated explicitly as follows:

$$\begin{aligned} \sigma^+(x) \quad &\Longrightarrow \quad \widetilde{\sigma}^+(s) = -2\sigma_0\frac{\sin(as)}{s} - P_0\,e^{i\,s\,x_0}, \\ \sigma^-(x) \quad &\Longrightarrow \quad \widetilde{\sigma}^-(s) = -2\sigma_0\frac{\sin(as)}{s}. \end{aligned} \tag{3.3}$$

Then application of the inverse Fourier transform to Eqs. (2.20) reduces the problem to the following system of two integral equations with respect to the functions $g_y(x)$, $g_x(x)$, $|x| \le b$:

$$\begin{aligned} \int_{-b}^{b} K_{11}(x-\xi)g_y(\xi)d\xi + \int_{-b}^{b} K_{12}(x-\xi)g_x(\xi)d\xi &= \frac{\sigma_0}{\mu} f_1^{\sigma}(x) + \frac{P_0}{2\mu} f_1^P(x), \\ \int_{-b}^{b} K_{21}(x-\xi)g_y(\xi)d\xi + \int_{-b}^{b} K_{22}(x-\xi)g_x(\xi)d\xi &= \frac{\sigma_0}{\mu} f_2^{\sigma}(x) + \frac{P_0}{2\mu} f_2^P(x), \end{aligned} \tag{3.4}$$

where

$$K_{11}(x) = \int_0^\infty L_\sigma^y(s)\cos(sx)ds, \quad K_{12}(x) = K_{21}(x) = \int_0^\infty L_\sigma^x(s)\sin(sx)ds,$$

$$K_{22}(x) = \int_0^\infty L_\tau^x(s)\cos(sx)ds, \quad f_1^P(x) = \int_0^\infty L_\sigma^+(s)\cos[s(x-x_0)]ds,$$

$$f_1^\sigma(x) = \int_0^\infty \left[L_\sigma^+(s) + L_\sigma^-(s)\right]\frac{\sin(as)}{s}\cos(sx)ds, \tag{3.5}$$

$$f_2^P(x) = \int_0^\infty L_\tau^+(s)\sin[s(x-x_0)]ds,$$

$$f_2^\sigma(x) = \int_0^\infty \left[L_\tau^+(s) + L_\tau^-(s)\right]\frac{\sin(as)}{s}\sin(sx)ds.$$

Here we have used the evident properties that functions L_σ^y, L_σ^+, L_σ^-, L_τ^x are even, and L_σ^x, L_τ^y, L_τ^+, L_τ^- are odd.

In order to evaluate qualitative properties of this system, let us note that all Fourier images of the kernels (2.21) in the Eqs. (3.5) are continuos over $0 \le s < \infty$. So,we only need to estimate their behavior at infinity. It can be directly estimated that at $s \to +\infty$,

$$\begin{gathered} L_\sigma^y(s) = -\frac{1-c^2}{2}\,s + L_{11}^0(s), \quad L_\tau^x(s) = \frac{1-c^2}{2}\,s + L_{22}^0(s), \\ L_{11}^0(s),\ L_{22}^0(s),\ L_\sigma^x(s),\ L_\tau^y(s),\ L_\sigma^+(s),\ L_\sigma^-(s),\ L_\tau^+(s),\ L_\tau^-(s) = O\left(e^{-\varepsilon s}\right), \end{gathered} \tag{3.6}$$

where ε is some positive constant. So, it becomes clear that all off-diagonal kernels in (3.3), (3.4) are regular, and such are all right-hand sides. To give an estimate of the qualitative properties of the diagonal kernels, we should use the theory of generalized functions [5], from which it follows that these kernels represent hypersingular integral operators (see also [4]), since

$$\int_0^\infty s\cos(sx)ds = \lim_{\delta\to+0}\int_0^\infty e^{-\delta s}s\cos(sx)ds = \lim_{\delta\to+0}\left[-\frac{x^2}{(\delta^2+x^2)^2}\right] = -\frac{1}{x^2}. \tag{3.7}$$

Therefore, our system (3.4) is a system of two hypersingular integral equations. In our previous work [4] we developed a theory and proposed a numerical method to investigate a single hypersingular equation. Thus, we have shown that a single characteristic equation (i.e., the one with the kernel (3.7)) admits an explicit-form exact analytical solution. Then we constructed also a collocation technique to solve the full hypersingular equation numerically when the kernel is a sum of the characteristic part (3.7) and arbitrary regular function.

It can be proved that the same method is applicable to the system of hypersingular equations. Here we demonstrate this by an example which admits an exact analytical solution.

A model example. Let us study the 2 x 2 system

$$\begin{aligned}&\int_{-1}^{1} K_{11}(x-\xi)\varphi(\xi)d\xi + \int_{-1}^{1} K_{12}(x-\xi)\psi(\xi)d\xi = f_1(x)),\\ &\int_{-1}^{1} K_{21}(x-\xi)\varphi(\xi)d\xi + \int_{-1}^{1} K_{22}(x-\xi)\psi(\xi)d\xi = f_2(x),\end{aligned} \tag{3.8}$$

with

$$\begin{aligned}&K_{11} = K_{22} = \frac{1}{(x-\xi)^2}, \quad K_{12} = K_{21} = x\xi,\\ &f_1(x) = -\pi + \frac{\pi x}{16}, \quad f_2(x) = -2\pi x.\end{aligned} \tag{3.9}$$

It can be directly verified that a solution to this system is

$$\varphi(x) = \sqrt{1-x^2}, \quad \psi(x) = x\sqrt{1-x^2}. \tag{3.10}$$

Our collocation method consists of the following steps.

Let us first consider a single full hypersigular equation of the following type:

$$\int_a^b \left[\frac{1}{(x-\xi)^2} + K_0(x-\xi)\right] g(\xi)d\xi = f(x), \quad x\in(a,b), \tag{3.11}$$

and arrange a mesh with two sets of nodes $x_i = -b + (i-1/2)\,\Delta x$, $i = 1,\ldots,n$, and $t_j = -b + j\Delta x$, $j = 0,1,\ldots,n$, taken with the constant step $x = (b-a)/n$. Then we prove that if

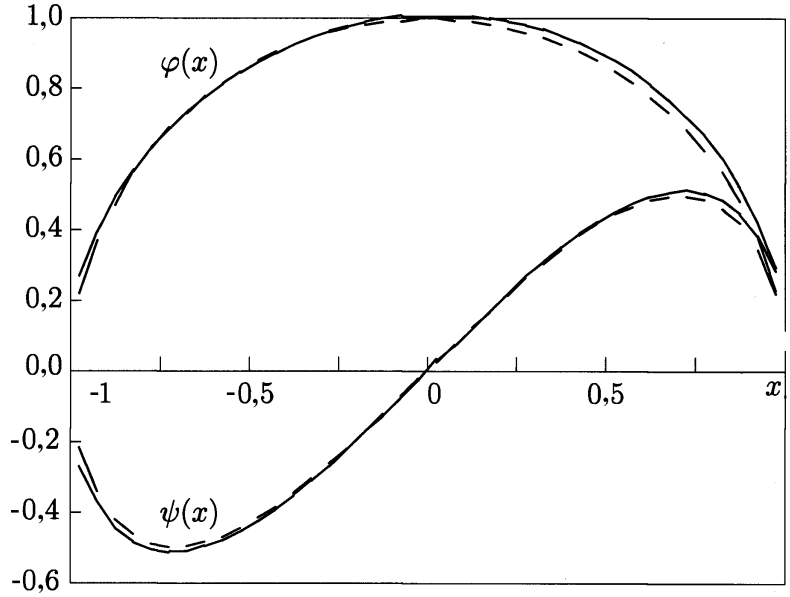

Fig 2. Comparison between the explicit-form exact solution and the numerical collocation method for the model system (3.8), (3.9): the solid line is an exact solution; the dashed line is an numerical solution, $n = 40$.

$K_0(x) \in C_1(a-b, b-a)$, then for any $x \in (a,b)$ the difference between a solution $g(x)$ of the linear algebraic system

$$\sum_{j=1}^{n} \left[\left(\frac{1}{x_i - t_{j-1}} - \frac{1}{x_i - t_j} \right) + K_0(x_i - t_j)\Delta x \right] g(t_j) = f(x_i), \qquad i = 1, 2, \ldots, n, \tag{3.12}$$

and the solution of the Eq. (3.11) vanishes with $\Delta x \to 0$ (i.e., $n \to \infty$).

It should be noted that the (second) regular term is a simple quadrature formula for numerical integration of regular functions. It can be also shown that any solution of the Eq. (3.11) bounded at the both edges $x = \pm b$, vanishes near these edges. This property is confirmed by a direct numerical solution of the linear algebraic system (3.12).

Now let us come back to the system (3.8), (3.9). Our collocation method reduces the Eqs. (3.8), (3.9) to a $2n \times 2n$ system of linear algebraic equations

$$\sum_{j=1}^{n} \left(\frac{1}{x_i - t_{j-1}} - \frac{1}{x_i - t_j} \right) \varphi(t_j) + \Delta x \sum_{j=1}^{n} x_i t_j \psi(t_j) = f_1(x_i),$$
$$\Delta x \sum_{j=1}^{n} x_i t_j \varphi(t_j) + \sum_{j=1}^{n} \left(\frac{1}{x_i - t_{j-1}} - \frac{1}{x_i - t_j} \right) \psi(t_j) = f_2(x_i), \qquad i = 1, 2, \ldots, n. \tag{3.13}$$

The Fig. 2 demonstrates comparison between analytical (Eq. (3.10), solid line) and numerical solutions (from the system (3.13), dashed line) when the number of nodes is $n = 40$.

The same numerical method has been applied to the main system:

$$\sum_{j=1}^{n}\left[\left(\frac{1}{x_i - t_{j-1}} - \frac{1}{x_i - t_j}\right) + \Delta x K_{11}^0(x_i - \xi_j)\right] g_y(t_j) +$$

$$+ \Delta x \sum_{j=1}^{n} K_{12}(x_i - t_j) g_x(t_j) = \frac{\sigma_0}{\mu} f_1^{\sigma}(x_i) + \frac{P_0}{2\mu} f_1^P(x_i),$$

$$\Delta x \sum_{j=1}^{n} K_{21}(x_i - t_j) g_y(t_j) + \sum_{j=1}^{n}\left[-\left(\frac{1}{x_i - t_{j-1}} - \frac{1}{x_i - t_j}\right) +\right.$$

$$\left. + \Delta x K_{22}^0(x_i - \xi_j)\right] g_x(t_j) = \frac{\sigma_0}{\mu} f_2^{\sigma}(x_i) + \frac{P_0}{2\mu} f_2^P(x_i),$$

$$i = 1, 2, \ldots, n,$$

where all kernels K_{11}^0, $K_{12} = K_{21}$, K_{22}^0 are regular (more precisely, infinitely differentiable).

4. Results of numerical treatment and physical conclusions. First of all, let us evaluate the number of independent dimensionless parameters in the considered problem: 1) b/h, 2) a/h, 3) h_+/h, 4) x_0/h, 5) σ_0/P_0. We thus operate with a five-parametric problem, where the calculation for any fixed combination of these five parameters requires solving a system of hypersingular singular equations.

Physically, to provide a perfect contact between the crack's faces (for medical or technological purposes), we need to arrange such an optimization when the solution of the main system of hypersingular equations gives the nonpositive function $g_y(x)$, $|x| \le b$, and the function $g_x(x)$, $|x| \le b$, is as small (in modulus) as possible.

A relatively unexpected result is that even a small applied normal load σ_0 provides vertical opening of the crack negative, i.e., closed in vertical direction. Numerous computations show that this conclusion remains valid uniformly over changes of all five parameters of the problem.

A typical example is presented in Fig. 3. As it can be seen from this figure, in the case when the controlling stress σ_0 is absent the force P_0 makes the vertical opening to be of alternating sign, so that it is closed in the central part of the interval $(-b, b)$ and open near the crack's edges. However, even a small controlling pressure provides the crack to be closed over its total length.

Fig. 4 shows that in the case of asymmetric disposition of the crack in the layer, this arrangement of closed interior space between the crack's faces requires a higher controlling pressure σ_0. We also outline that the strain near the crack faces in the asymmetric case is around 7 times larger than the one in the symmetric case.

Among other results, we stress the interesting property that the relative displacement $|g_x(x)|$ of the crack's faces in the horizontal direction is always considerably larger than the one in the vertical direction, $g_y(x)$, when the latter is positive (i.e., in the dangerous case of really positive vertical opening). This feature of the strain-stress process, under an external loading, indicates that the real obstacle for a perfect conglutination (for medical or technological aims) consists of the impossibility to provide a small relative tangential crack faces' displacement

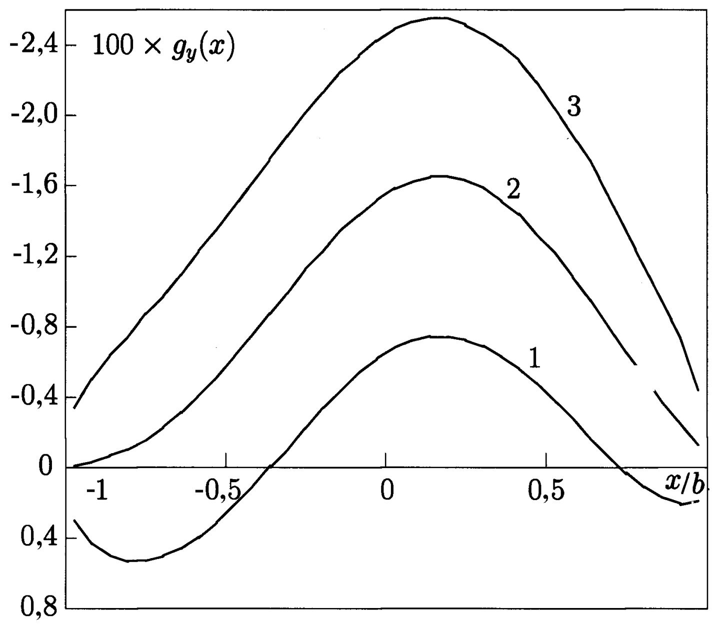

Fig 3. The vertical opening of the crack: $c^2 = 0,3,\ P_0/2\mu = 1,\ h_+/h = = h_-/h = 0,5, b/h = 0,5, a/h = 1, x_0/h = 3,5$. Line 1: $\sigma_0/P_0 = = 0$, Line 2: $\sigma_0/P_0 = 0,01$, Line 3: $\sigma_0/P_0 = 0,02$.

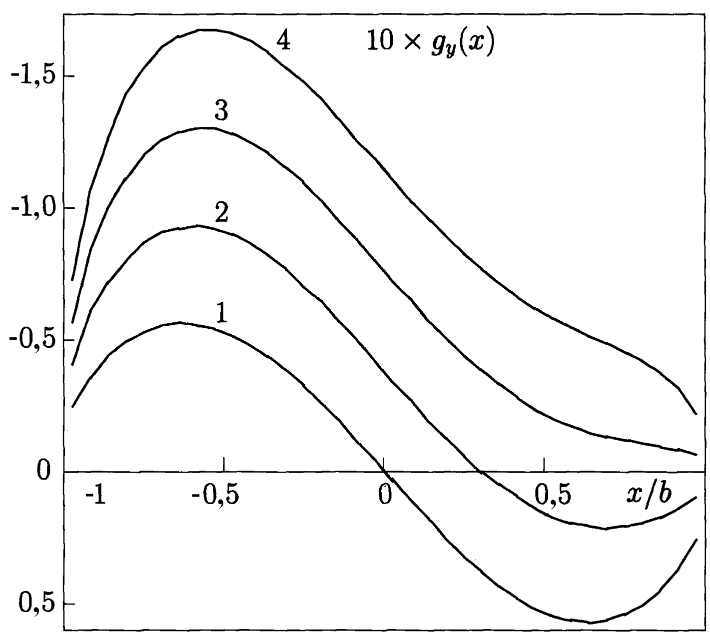

Fig 4. The vertical opening of the crack: $c^2 = 0,3, P_0/2\mu = 1, h_+/h = = 0,8, h_-/h = 0,2, b/h = 0,5, a/h = 1, x_0/h = 3,5$. Line 1: $\sigma_0/P_0 = 0$, Line 2: $\sigma_0/P_0 = 0,05$, Line 3: $\sigma_0/P_0 = 0,1$, Line 4: $\sigma_0/P_0 = 0,15$.

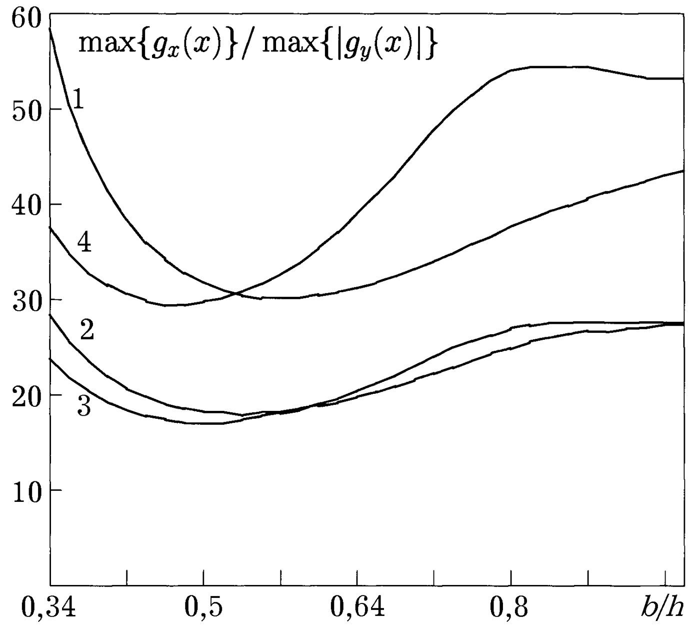

Fig 5. The ratio of the horizontal relative displacement to the vertical opening versus the length of the crack: $c^2 = 0,3l, P_0/2\mu = 1$, $\sigma_0 = 0, a/h = 1, x_0/h = 3,5$. Line 1: $h_+/h = 0,6$, Line 2: $h_+/h = 0,7$, Line 3: $h_+/h = 0,8$, Line 4: $h_+/h = 0,9$.

under the controlling normal pressure σ_0. Some striking examples are given in Fig. 5, where the ratio of the maximums $\max\{g_x(x)\}/\max\{|g_y(x)|\}$ over the interval $x \in (-b, b)$ is shown, which always happens when the controlling pressure $\sigma_0 = 0$ (cf. Figs. (3), (4)). Many other calculated examples show the same qualitative behavior like in Fig. 5, namely: the tangential function $|g_x(x)|$ is always greater by many times than the vertical opening $g_y(x)$, regardless the values of other parameters, including the controlling stress σ_0.

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