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IMPLICIT DIFFERENCE METHODS FOR FIRST ORDER PARTIAL DIFFERENTIAL FUNCTIONAL EQUATIONS

НЕЯВНІ РІЗНИЦЕВІ МЕТОДИ ДЛЯ ДИФЕРЕНЦІАЛЬНО-ФУНКЦІОНАЛЬНИХ РІВНЯНЬ ПЕРШОГО ПОРЯДКУ

A. Kępczyńska

Gdańsk Univ. Technology
Gabriel Narutowicz Str., 11–12 80–952 Gdańsk, Poland

We present a new class of numerical methods for quasilinear first order partial functional differential equations. The numerical methods are difference schemes which are implicit with respect to time variable. We give a complete convergence analysis for the methods and we show by an example that the new methods are considerably better than the explicit schemes. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type for given operators with respect to the functional variable.

Розглянуто новий клас чисельних методів для квізілінійних функціонально-диференціальних рівнянь першого порядку з частинними похідними. Розглянуті чисельні методи є різницевими схемами, що задаються неявно відносно часової змінної. Наведено повний аналіз збіжності методів і приклад, що показує значну перевагу нових методів над явними схемами. Доведення стійкості базується на техніці порівняння з нелінійною оцінкою перронівського типу для заданого оператора відносно функціональної змінної.

1. Introduction. For any metric spaces X and Y we denote by $C(X, Y)$ the class of all continuous functions from X to Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

We consider the sets

$$e = [0, a] \times (-b, b), \quad D = [-d_0, 0] \times [-d, d]$$

where $a > 0$, $b = (b_1, \ldots, b_n)$, $d = (d_1, \ldots, d_n)$, and $d_i \geq 0$, $b_i > 0$ for $1 \leq i \leq n$, $d \in R^n_+$, $R_+ = (0, +\infty)$. Let $c = (c_1, \ldots, c_n) = b + d$ and

$$E_0 = [-d_0, 0] \times [-c, c], \quad \partial_0 E = [0, a] \times \big([-c, c] \setminus (-b, b)\big).$$

Put $\Omega = E_0 \cup E \cup \partial_0 E$. For a function $z : \Omega \to R$ and for a point $(t, x) \in [0, a] \times [-b, b]$ we define the function $z_{(t,x)} : D \to R$ as follows:

$$z_{(t,x)}(s, y) = z(t + s, x + y), \quad (s, y) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t - d_0, t] \times [x - d, x + d]$ and this restriction is shifted to the set D. For a function $w \in C(D, R)$ we write

$$\|w\|_D = \max\{|w(t, x)| : (t, x) \in D\}.$$

Suppose that

$$f : E \times C(D, R) \to R^n, \quad f = (f_1, \ldots, f_n),$$

$$g : E \times C(D, R) \to R, \quad \varphi : E_0 \cup \partial_0 E \to R$$

are given functions. We will deal with the quasilinear differential functional equation

$$\partial_t z(t,x) = \sum_{i=1}^{n} f_i(t,x,z_{(t,x)}) \partial_{x_i} z(t,x) + g(t,x,z_{(t,x)}) \tag{1}$$

and the initial boundary condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0 \cup \partial_0 E. \tag{2}$$

A function $v : \Omega \to R$ is a classical solution of (1), (2) provided:
(i) $v \in C(\Omega, R)$ and the partial derivatives $\partial_t v$, $\partial_x v = (\partial_{x_1} v, \ldots, \partial_{x_n} v)$ exist on E,
(ii) v satisfies (1) on E and condition (2) holds.

We are interested in establishing a method of approximation of solutions to problem (1), (2) by means of solutions of associated difference functional equations and in estimating the difference bet ween the exact and approximate solutions.

In resent years, a number of papers concerning numerical methods for initial or initial boundary-value problems related to first order partial functional differential equations have been published.

Difference approximations of nonlinear equations with initial boundary conditions were considered in [1, 2]. The convergence result for a general class of difference methods related to initial problems and solutions defined on unbounded domain can be found in [3]. Error estimates implying the convergence results for initial problems on the Haar pyramid were considered in [4–6].

All this considerations have the following property: the main question in the investigations of numerical methods is to find a difference functional equation which is stable and satisfies a consistency condition with respect to the original problem. The method of difference inequalities and theorems on recurrence inequalities are used in the investigation of the stability. The convergence results are based also on a general theorem on error estimates of approximate solutions to functional difference equations of the Volterra type with initial or initial boundary condition. The monograph [7] contains an exposition of the theory of difference methods for nonlinear hyperbolic functional differential problems.

In the paper we start the investigation of implicit difference methods for quasilinear functional differential equations. We prove that under natural assumptions on given functions and on the mesh there is a class of implicit difference schemes for (1), (2) which is convergent. The stability of the methods is investigated by using the comparison technique. It is important in our considerations that we assume the nonlinear estimates of the Perron type for given functions with respect to the functional variable. Our results are based on general ideas for finite difference equations which were introduced in [8, 9].

The paper is organized as follows. In Section 2 we formulate an implicit difference functional problem corresponding to (1), (2) and we prove that there is exactly one solution of a difference scheme. In Section 3 we prove a theorem on the error estimate for implicit difference functional problems. Section 4 deals with a convergence result and an error estimate for the method. A numerical example is given in Section 5.

We list below examples of equations which can be obtained from (1) by specializing the operators f and g.

Example 1. Suppose that

$$\widetilde{f} : E \times R \to R^n, \quad \widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_n), \quad \widetilde{g} : E \times R \to R$$

and

$$\alpha : E \to R^{1+n}, \quad \alpha = (\alpha_0, \alpha'), \quad \alpha' = (\alpha_1, \ldots, \alpha_n),$$

are given functions, and

$$-d_0 \le \alpha_0(t,x) - t \le 0, \quad -d \le \alpha'(t,x) - x \le d, \quad (t,x) \in E.$$

Write

$$f(t,x,w) = \widetilde{f}(t,x,w(\alpha_0(t,x) - t,\, \alpha'(t,x) - x)),$$

$$g(t,x,w) = \widetilde{g}(t,x,w(\alpha_0(t,x) - t,\, \alpha'(t,x) - x)),$$

where $(t,x,w) \in E \times C(D,R)$. Then equation (1) reduces to the equation with deviated variables,

$$\partial_t z(t,x) = \sum_{i=1}^{n} \widetilde{f}_i\big(t,x,z(\alpha(t,x))\big)\partial_{x_i} z(t,x) + \widetilde{g}(t,x,z(\alpha(t,x))).$$

Example 2. For the above $\widetilde{f}$ and $\widetilde{g}$ we put

$$f(t,x,w) = \widetilde{f}\left(t,x,\int_D w(s,y)dyds\right),$$

$$g(t,x,w) = \widetilde{f}\left(t,x,\int_D w(s,y)dyds\right).$$

Then (1) is the differential integral equation

$$\partial_t z(t,x) = \sum_{i=1}^{n} \widetilde{f}_i\left(\int_D z(t+s,x+y)dyds\right)\partial_{x_i} z(t,x) + \widetilde{g}\left(\int_D z(t+s,x+y)dyds\right).$$

Existence results for mixed problems (1), (2) can be found in [7, 10, 11].

2. Difference functional equations. We will denote by $F(X,Y)$ the class off all functions defined on X and taking values in Y where X and Y are arbitrary sets. For $x,\ y \in R^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ we write

$$\|x\| = \sum_{i=1}^{n} |x_i|, \quad x \diamond y = (x_1y_1, \ldots, x_ny_n).$$

We define a mesh on the set Ω in the following way. Let (h_0, h'), $h' = (h_1, \ldots, h_n)$, stand for steps of the mesh. Let us denote by H the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbf{Z}$ and $K = (K_1, \ldots, K_n) \in \mathbf{N}^n$ with the properties $K_0 h_0 = d_0$ and $K \diamond h' = d$. For $h \in H$ and $(r, m) \in \mathbf{Z}^{1+n}$, where $m = (m_1, \ldots, m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \quad x^{(m)} = m \diamond h', \quad x^{(m)} = \left(x^{(m_1)}, \ldots, x^{(m_n)}\right).$$

Let $N_0 \in N$ be defined by the relations $N_0 h_0 \leq a < (N_0 + 1)h_0$. Write

$$R_h^{1+n} = \left\{\left(t^{(r)}, x^{(m)}\right) : (r, m) \in \mathbf{Z}^{1+n}\right\}$$

and

$$E_h = E \cap R_h^{1+n}, \quad E_{h,0} = E_0 \cap R_h^{1+n}, \quad D_h = D \cap R_h^{1+n},$$

$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \quad \Omega_h = \Omega \cap R_h^{1+n}.$$

Moreover we put

$$E_{h,r} = E_h \cap \left(\left[-d_0, t^{(r)}\right] \times R^n\right)$$

where $0 \leq r \leq N_0$ and

$$E'_h = \left\{\left(t^{(r)}, x^{(m)}\right) \in E_h : 0 \leq r \leq N_0 - 1\right\},$$

$$I_h = \left\{t^{(r)} : 0 \leq r \leq N_0\right\}, \quad I'_h = I_h \setminus \left\{t^{(N_0)}\right\}.$$

For a function $z : \Omega_h \to R$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$. For the above z and for a point $(t^{(r)}, x^{(m)}) \in E_h$ we define the function $z_{[r,m]} : D_h \to R$ by the formula

$$z_{[r,m]}(\tau, s) = z\left(t^{(r)} + \tau,\ x^{(m)} + s\right), \quad (\tau, s) \in D_h.$$

The function $z_{[r,m]}$ is the restriction of z to the set

$$\left([t^{(r)} - d_0] \times [x^{(m)} - d,\ x^{(m)} d]\right) \cap R_h^{1+n}$$

and this restriction is shifted to the set D_h. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$, 1 standing on the i-th place, $1 \leq i \leq n$. Suppose that the functions

$$f_h : E'_h \times F(D_h, R) \to R^n, \quad f_h = (f_{h,1}, \ldots, f_{h,n}),$$

$$g_h : E'_h \times F(D_h, R) \to R, \quad \varphi_h : E_{h,0} \cup \partial_0 E_h \to R$$

are given. We consider the difference functional equation

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_{h,i}\left(t^{(r)}, x^{(m)}, z_{[r,m]}\right) \delta_i z^{(r+1,m)} + g_h\left(t^{(r)}, x^{(m)}, z_{[r,m]}\right) \tag{3}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{h,0} \cup \partial_0 E_h. \tag{4}$$

The difference operators δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ are defined in the following way. Put

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\left(z^{(r+1,m)} - z^{(r,m)}\right) \tag{5}$$

and

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\left(z^{(r,m+e_i)} - z^{(r,m)}\right) \quad \text{if} \quad f_{h,i}\left(t^{(r)}, x^{(m)}, z_{[r,m]}\right) \geq 0, \tag{6}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\left(z^{(r,m)} - z^{(r,m-e_i)}\right) \quad \text{if} \quad f_{h,i}\left(t^{(r)}, x^{(m)}, z_{[r,m]}\right) < 0 \tag{7}$$

where $1 \leq i \leq n$.

Let us consider the explicit difference equation corresponding to (3)

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_{h,i}\left(t^{(r)}, x^{(m)}, z_{[r,m]}\right)\delta_i z^{(r+1,m)} + g_h\left(t^{(r)}, x^{(m)}, z_{[r,m]}\right). \tag{8}$$

It is clear that the difference problem (4), (8) with δ_0 and δ given (5)–(7) has exactly one solution $\widetilde{u}_h : \Omega_h \to R$.

We prove that under natural assumptions on f_h and g_h there exists exactly one solution $u_h : \Omega_h \to R$ of the implicit difference functional problem (3), (4). We will approximate classical solutions of (1), (2) with solutions of (3), (4).

We first prove a maximum principle for difference inequalities generated by (3), (4). Write

$$B = (-b, b), \quad B^\star = [-c, c],$$

and $R_h^n = \{x^{(m)} : m \in \mathbf{Z}^n\}$. We consider the sets

$$B_h = B \cap R_h^n, \quad B_h^\star = B^\star \cap R_h^n, \quad \partial_0 B_h = B_h^\star \backslash B_h.$$

Theorem 1. *Suppose that $h \in H$, $f_h : E'_h \times F(D_h, R) \to R^n$ and $0 \leq r \leq N_0 - 1$, is fixed.*
1. If $z_h : E_{h,r+1} \to R$ satisfies the implicit difference inequality

$$z_h^{(r+1,m)} \leq h_0 \sum_{i=1}^{n} f_{h,i}\left(t^{(r)}, x^{(m)}, (z_h)_{[r,m]}\right)\delta_i z_h^{(r+1,m)}$$

for $x^{(m)} \in B_h$ and $\mu \in \mathbf{Z}^n$, $\mu = (\mu_1, \ldots, \mu_n)$, is such that $z_h^{(r+1,\mu)} = M$, where

$$M = \max\left\{z_h^{(r+1,m)} : x^{(m)} \in B_h^*\right\} \quad \textit{and} \quad M > 0, \tag{9}$$

then $x^{(\mu)} \in \partial_0 B_h$.

2. If $z_h : E_{h,r+1} \to R$ *satisfies the implicit difference inequality*

$$z_h^{(r+1,m)} \geq h_0 \sum_{i=1}^{n} f_{h,i}\left(t^{(r)}, x^{(m)}, (z_h)_{[r,m]}\right) \delta_i z_h^{(r+1,m)}$$

for $x^{(m)} \in B_h$ *and* $\widetilde{\mu} \in \mathbf{Z}^n$ *is such that* $z_h^{(r+1,\widetilde{\mu})} = \widetilde{M}$, *where*

$$\widetilde{M} = \min\left\{z_h^{(r+1,m)} : x^{(m)} \in B_h^*\right\} \quad \textit{and} \quad \widetilde{M} < 0,$$

then $x^{(\widetilde{\mu})} \in \partial_0 B_h$.

Proof. Consider the case 1. Write

$$J_+^{(r,m)}[z_h] = \left\{i : 1 \leq i \leq n \quad \text{and} \quad f_{h,i}\left(t^{(r)}, x^{(m)}, (z_h)_{[r,m]}\right) \geq 0\right\}, \tag{10}$$

$$J_-^{(r,m)}[z_h] = \{1, \ldots, n\} \setminus J_+^{(r,m)}[z_h]. \tag{11}$$

Suppose that $x^{(\mu)} \in B_h$. Then

$$z_h^{(r+1,\mu)} \leq h_0 \sum_{i \in J_+^{(r,m)}[z_h]} \frac{1}{h_i} f_{h,i}\left(t^{(r)}, x^{(\mu)}, (z_h)_{[r,m]}\right)\left[z_h^{(r+1,\mu+e_i)} - z_h^{(r+1,\mu)}\right] +$$

$$+ h_0 \sum_{i \in J_-^{(r,m)}[z_h]} \frac{1}{h_i} f_{h,i}\left(t^{(r)}, x^{(\mu)}, (z_h)_{[r,m]}\right)\left[z_h^{(r+1,\mu)} - z_h^{(r+1,\mu-e_i)}\right].$$

This gives

$$z_h^{(r+1,\mu)}\left[1 + h_0 \sum_{i=1}^{n} \frac{1}{h_i}\left|f_{h,i}\left(t^{(r)}, x^{(\mu)}, (z_h)_{[r,\mu]}\right)\right|\right] \leq$$

$$\leq h_0 \sum_{i \in J_+^{(r,m)}[z_h]} \frac{1}{h_i} f_{h,i}\left(t^{(r)}, x^{(\mu)}, (z_h)_{[r,\mu]}\right) z_h^{(r+1,\mu+e_i)} -$$

$$- h_0 \sum_{i \in J_-^{(r,m)}[z_h]} \frac{1}{h_i} f_{h,i}\left(t^{(r)}, x^{(\mu)}, (z_h)_{[r,\mu]}\right) z_h^{(r+1,\mu-e_i)} \leq$$

$$\leq h_0 M \sum_{i=1}^{n} \frac{1}{h_i}\left|f_{h,i}\left(t^{(r)}, x^{(\mu)}, (z_h)_{[r,\mu]}\right)\right|.$$

We thus get $z_h^{(r+1,\mu)} \leq 0$ which contradicts (15). Then $x^{(\mu)} \in \partial_0 B_h$ which is our claim. In a similar way we prove that $x^{(\widetilde{\mu})} \in \partial_0 B_h$ in the case 2. This completes proof.

Lemma 1. *If* $h \in H$, $f_h : E'_h \times F(D_h, R) \to R^n$ *and* $g_h : E_h \times F(D_h, R) \to R^n$ *then difference functional problem (3), (4) with* δ_0 *and* δ *defined by (5)–(7) has exactly one solution* $u_h : \Omega_h \to R$.

Proof. Suppose that $0 \le r \le N_0 - 1$ is fixed and $u_h : E_{h,r} \to R$ is known. Then (3), (4) is a linear system, which allows to calculate $u_h^{(r+1,m)}$ for $x^{(m)} \in B_h$. The homogeneous problem corresponding to (3), (4) has the from

$$z^{(r+1,m)} = h_0 \sum_{i=1}^{n} f_{h,i} \left(t^{(r)}, x^{(m)}, z_{[r,m]} \right) \delta_i z^{(r+1,m)}, \tag{12}$$

$$z^{(r+1,m)} = 0 \quad \text{on} \quad E_{h,0} \cup \partial_0 E_h. \tag{13}$$

It follows from Theorem 1 that system (12), (13) has exactly one zero solution. Therefore the problem (3), (4) has exactly one solution for any choice of the function $g_h : E_h \times F(D_h, R) \to$ $\to R$. Then the numbers $u_h^{(r+1,m)}$, $x^{(m)} \in B_h$, exist and they are unique. Since u_h is given on $E_{h,0}$, the proof is completed by induction.

3. Approximate solutions of difference functional equations. Let us denote by F_h the Niemycki operator corresponding to (3), i.e.,

$$F_h[z]^{(r,m)} = \sum_{i=1}^{n} f_{h,i} \left(t^{(r)}, x^{(m)}, z_{[r,m]} \right) \delta_i z^{(r+1,m)} + g_h \left(t^{(r)}, x^{(m)}, z_{[r,m]} \right).$$

Then we consider the difference functional equation

$$\delta_0 z^{(r,m)} = F_h[z]^{(r,m)} \tag{14}$$

with initial boundary condition (4). Suppose that $v_h : \Omega_h \to R$ and $\gamma, \alpha_0 : H \to R_+$ are such functions that

$$\left| \delta_0 v_h^{(r,m)} - F_h[v_h]^{(r,m)} \right| \le \gamma(h) \quad \text{on} \quad E'_h, \tag{15}$$

$$\left| \varphi_h^{(r,m)} - v_h^{(r,m)} \right| \le \alpha_0(h) \quad \text{on} \quad E_{h,0} \cup \partial_0 E_h \tag{16}$$

and

$$\lim_{h \to 0} \gamma(h) = 0, \quad \lim_{h \to 0} \alpha_0(h) = 0.$$

The function v_h satisfying the relations (15), (16) is considered as an approximate solution of problem (4), (14). We give a theorem on the estimate of the difference between the exact and approximate solutions of (4), (14).

Assumption $H[f_h, g_h]$. 1. The function $\sigma_h : I'_h \times R_+ \to R_+$ satisfies the following conditions:
(i) $\sigma_h(t, \cdot) : R_+ \to R_+$ is continuous and nondecreasing for each $t \in I'_h$;
(ii) $\sigma_h(t, 0) = 0$ for $t \in I'_h$, and for each $\overline{c} \geq 1$ the difference problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \overline{c} \sigma_h \left(t^{(r)}, \eta^{(r)} \right), \quad 0 \leq r \leq N_0 - 1, \tag{17}$$

$$\eta^{(0)} = 0 \tag{18}$$

is stable in the following sense: if γ, $\alpha_0 : H \to R_+$ are such functions that

$$\lim_{h \to 0} \gamma(h) = 0, \quad \lim_{h \to 0} \alpha_0(h) = 0,$$

and $\eta_h : I_h \to R_+$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \overline{c} \sigma_h \left(t^{(r)}, \eta^{(r)} \right) + h_0 \gamma(h), \quad 0 \leq r \leq N_0 - 1,$$

$$\eta^{(0)} = \alpha_0(h),$$

then there is $\alpha : H \to R_+$ such that $\eta_h^{(r)} \leq \alpha(h)$ for $0 \leq r \leq N_0$ and $\lim\limits_{h \to 0} \alpha_0(h) = 0$.

2. The functions $f_h : E'_h \times F(D, R) \to R^n$, $g_h : E'_h \times F(D_h, R) \to R$ satisfy the estimates

$$\| f_h(t, x, w) - f_h(t, x, \overline{w}) \| \leq \sigma_h(t, \| w - \overline{w} \|_{D_h}),$$

$$| g_h(t, x, w) - g_h(t, x, \overline{w}) | \leq \sigma_h(t, \| w - \overline{w} \|_{D_h})$$

on Ω_h.

Theorem 2. *Suppose that the Assumption $H[f_h, g_h]$ is satisfied and*
1) $\varphi_h : E_{h,0} \cup \partial_0 E_h \to R$ and the function $u_h : \Omega_h \to R$ is a solution of the problem (4), (14);
2) $h \in H$ and the functions $v_h : \Omega_h \to R$, are such that the estimates (15), (16) are satisfied;
3) there is $c_0 \in R_+$ such that the estimate

$$\left| \delta_i v_h^{(r,m)} \right| \leq c_0, \quad \left(t^{(r)}, x^{(m)} \right) \in E_h, \tag{19}$$

is satisfied for $1 \leq i \leq n$.

Then there exists a function $\alpha : H \to R_+$ such that

$$\left| u_h^{(r,m)} - v_h^{(r,m)} \right| \leq \alpha(h) \quad \textit{on} \quad E_h \tag{20}$$

and

$$\lim_{h \to 0} \alpha(h) = 0.$$

Proof. Let $\Gamma_h : E'_h \to R$ be the function defined in the relation

$$\delta_0 v_h^{(r,m)} = F_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)}. \tag{21}$$

It follows from (15) that $|\Gamma_h^{(r,m)}| \le \gamma(h)$ on E'_h. Since u_h satisfies (4), (14), we have

$$(u_h - v_h)^{(r+1,m)} \left[1 + h_0 \sum_{i=1}^{n} \frac{1}{h_i} \left| f_{h,i}\left(t^{(r)}, x^{(m)}, (u_h)_{[r,m]}\right)\right|\right] = (u_h - v_h)^{(r,m)} +$$

$$+ h_0 \sum_{i \in J_+^{(r,m)}[u_h]} \frac{1}{h_i} f_{h,i}\left(t^{(r)}, x^{(m)}, (u_h)_{[r,m]}\right) (u_h - v_h)^{(r+1,m+e_i)} -$$

$$- h_0 \sum_{i \in J_-^{(r,m)}[u_h]} \frac{1}{h_i} f_{h,i}\left(t^{(r)}, x^{(m)}, (u_h)_{[r,m]}\right) (u_h - v_h)^{(r+1,m-e_i)} +$$

$$+ h_0 \sum_{i=1}^{n} \frac{1}{h_i} \left(f_{h,i}\left(t^{(r)}, x^{(m)}, (u_h)_{[r,m]}\right) - f_{h,i}\left(t^{(r)}, x^{(m)}, (v_h)_{[r,m]}\right)\right) \delta_i v_h^{(r+1,m)} +$$

$$+ h_0 \left(g_h\left(t^{(r)}, x^{(m)}, (u_h)_{[r,m]}\right) - g_h\left(t^{(r)}, x^{(m)}, (v_h)_{[r,m]}\right)\right) - h_0 \Gamma_h^{(r,m)},$$

where $J_+^{(r,m)}[u_h]$ and $J_-^{(r,m)}[u_h]$ are given by (10), (11). Write

$$\varepsilon_h^{(r)} = \max\left\{ |(u_h - v_h)^{(r,m)} : -K \le m \le K \right\} \quad \text{for} \quad r = 0, \ldots, N_0.$$

Then we get the following difference inequality:

$$\varepsilon_h^{(r+1)} \le \max\left\{ \varepsilon_h^{(r)} + h_0(1 + c_0)\sigma_h\left(t^{(r)}, \varepsilon_h^{(r)}\right) + h_0\gamma(h), \alpha_0(h)\right\}.$$

Let us consider now the difference problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0(1 + c_0)\sigma_h\left(t^{(r)}, \eta_h^{(r)}\right) + h_0\gamma(h), \quad 0 \le r \le N_0 - 1,$$

$$\eta^{(r)} = \alpha_0(h),$$

and its solution η_h. It follows from the above considerations that

$$\varepsilon_h^{(r)} = \eta_h^{(r)}, \quad 0 \le r \le N_0.$$

Now we obtain the assertion of Theorem 2 from the stability of problem (17), (18).

4. Convergence of implicit difference methods. We consider a class of difference problems (3), (4) where f_h and g_h are superpositions of f and g with a suitable interpolating operator.

Assumption $H[T_h]$. Suppose that the operator $T_h : F(D_h, R) \to F(D, R)$ satisfies the following conditions:

1) if $w, \widetilde{w} \in F(D_h, R)$ then $T_h[w], T_h[\widetilde{w}] \in C(D, R)$ and

$$\|T_h[w] - T_h[\widetilde{w}]\|_D \le \|w - \widetilde{w}\|_{D_h},$$

2) if $w : D \to R$ is of class C^1 then there is $\gamma : H \to R_+$ such that

$$\|T_h[w] - w\|_D \le \gamma(h) \quad \text{and} \quad \lim_{h\to 0} = 0,$$

where w_h is the restriction of w to the set D_h.

We will approximate solutions of (1), (2) with solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_i\left(t^{(r)}, x^{(m)}, T_h z_{[r,m]}\right)\delta_i z^{(r+1,m)} + g\left(t^{(r)}, x^{(m)}, T_h z_{[r,m]}\right) \tag{22}$$

with initial boundary condition (4).

Assumption $H[f,g]$. 1. The functions $f : E \times F(D,R) \to R^n$ and $g : E \times F(D,R) \to R$ are continuous.

2. There is $\sigma : [0,a] \times R_+ \to R_+$ such that

(i) σ is continuous and nondecreasing with respect to both variables;

(ii) $\sigma(t,0) = 0$ for $t \in [0,a]$ and each $\bar{c} \ge 1$, $\varepsilon \ge 0$, and the maximal solution $\omega(\cdot,\varepsilon)$ of the Cauchy problem

$$w'(t) = \bar{c}\sigma(t, w(t)) + \varepsilon, \quad w(0) = \varepsilon, \tag{23}$$

is defined on $[0,a]$ and $w(t,0) = 0$ for $t \in (0,a)$;

(iii) the estimates

$$\|f(t,x,w) - f(t,x,\overline{w})\| \le \sigma(t, \|w - \overline{w}\|_D),$$

$$|g(t,x,w) - g(t,x,\overline{w})| \le \sigma(t, \|w - \overline{w}\|_D)$$

are satisfied on $E \times F(D,R)$.

Theorem 3. *Suppose that Assumptions $H[T_h]$ and $H[f,g]$are satisfied and*

1) $h \in H$ and the function $u_h : \Omega_h \to R$ is a solution of (4), (22) and there is $\alpha_0 H \to R_+$ such that

$$\left|\varphi^{(r,m)} - \varphi_h^{(r,m)}\right| \le \alpha_0(h) \quad \text{on} \quad E_{h,0} \cup \partial_0 E_h \quad \text{and} \quad \lim_{h\to 0} \alpha_0(h) = 0; \tag{24}$$

2) $v : \Omega \to R$ is a solution of (1), (2) and v is of class C^1 on Ω.

Then there is $\alpha : H \to R_+$ such that

$$\left|u_h^{(r,m)} - v_h^{(r,m)}\right| \le \alpha(h) \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{25}$$

Proof. We prove that the functions

$$f_h(t,x,w) = f(t,x,T_h[w]), \quad g_h(t,x,w) = g(t,x,T_h[w])$$

satisfy all the assumptions of Theorem 2. We first show that problem (17), (18) is stable in the sense of Assumption $H[f_h, g_h]$. Let the functions $\alpha_0, \gamma : H \to R_+$be such that

$$\lim_{h\to 0} \alpha_0(h) = 0, \quad \lim_{h\to 0} \gamma(h) = 0.$$

Let us consider the difference problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \bar{c} \sigma_h \left(t^{(r)}, \eta^{(r)} \right) + h_0 \gamma(h), \quad 0 \le r \le K-1, \tag{26}$$

$$\eta^{(0)} = \alpha_0(h), \tag{27}$$

and its solution $\eta_h : I_h \to R_+$. Denote by $\omega_h : [0, a) \to R_+$ the maximal solution of the problem

$$w'(t) = \bar{c}\sigma(t, w) + \gamma(h), \tag{28}$$

$$w(t) = \alpha_0(h). \tag{29}$$

It is easily seen that $\eta_h^{(i)} \le \omega_h^{(i)}$ for $0 \le i \le N_0$ and

$$\lim_{h \to 0} \omega(t) = 0 \quad \text{uniformly on} \quad [0, a].$$

Then we have $\eta_h^{(i)} \le \omega_h(a)$ for $0 \le i \le N_0$ and problem (28), (29) with $\varepsilon = 0$ is stable. Moreover we have

$$\|f_h(t, x, w) - f_h(t, x, \overline{w})\| = \|f(t, x, T_h[w]) - f(t, x, T_h[\overline{w}])\| \le$$
$$\le \sigma\left(t, \|T_h[w - \overline{w}]\|_D\right) \le \sigma_h\left(t, x, \|w - \overline{w}\|_{D_h}\right).$$

In the same way we get the inequality

$$\|g_h(t, x, w) - g_h(t, x, \overline{w})\| \le \sigma_h\left(t, \|w - \overline{w}\|_{D_h}\right).$$

Then the assertion of Theorem 3 follows from Theorem 2.

Remark 1. There are the following consequences of Theorem 3. In classical theorems concerning difference methods for quasilinear functional differential problems it is assumed that

$$1 - h_0 \sum_{i=1}^{n} \frac{1}{h_i} |f_i(t, x, w)| \ge 0 \quad \text{on} \quad E \times C(D, R), \tag{30}$$

see [3]. It is important in our considerations that we have omitted the above assumption.

Now we give an example of the operator T_h satisfying Assumption $H[T_h]$. Put

$$S_* = \{(j, s) : j \in \{0, 1\},\, s = (s_1, \ldots, s_n),\, s_i \in \{0, 1\} \quad \text{for} \quad 1 \le i \le n\}.$$

Let $w \in F(D_h, R)$ and $(t, x) \in D$. There exists $(t^{(r)}, x^{(m)}) \in D_h$ such that

$$t^{(r)} \le t \le t^{(r+1)}, \quad x^{(m)} \le x \le x^{(m+1)}, \quad \left(t^{(r+1)}, x^{(m+1)}\right) \in D_h.$$

We define

$$(T_h w)(t,x) = \sum_{(j,s)\in S_*} w^{(r+j,m+s)} \left(\frac{Y - Y^{(r,m)}}{h}\right)^{(j,s)} \left(1 - \frac{Y - Y^{(r,m)}}{h}\right)^{1-(j,s)}$$

where

$$\left(\frac{Y - Y^{(r,m)}}{h}\right)^{(j,s)} = \left(\frac{t - t^{(r)}}{h_0}\right)^{j} \prod_{k=1}^{n} \left(\frac{x_k - x_k^{(m_k)}}{h_k}\right)^{s_k}$$

and

$$\left(1 - \frac{Y - Y^{(r,m)}}{h}\right)^{1-(j,s)} = \left(1 - \frac{t - t^{(r)}}{h_0}\right)^{1-j} \prod_{k=1}^{n} \left(1 - \frac{x_k - x_k^{(m_k)}}{h_k}\right)^{1-s_k}$$

and we take $0^0 = 1$ in the above formulas.

Lemma 2. *Suppose that the function $w : D \to R$ is of class C^2 and denote by w_h the restriction of w to the set D_h. Let $\widetilde{C}$ be a constant such that*

$$|\partial_{tt}(t,x)| \le \widetilde{C}, \quad \left|\partial_{tx_j}(t,x)\right| \le \widetilde{C}, \quad \left|\partial_{x_i x_j}(t,x)\right| \le \widetilde{C} \quad \text{on} \quad D,$$

where $1 \le i, j \le n$. Then

$$\|T_h[w] - w\|_D \le \widetilde{C} \left[h_0^2 + 2h_0 \sum_{i=1}^{n} h_i + \sum_{j,i=1}^{n} h_j h_i\right].$$

The proof of the above lemma can found in [7], Chapter 5. We omit details.

Now we give an error estimate for method (4), (22).

Theorem 4. *Suppose that*

1) all the assumptions of Theorem 3 are satisfied with $\sigma(t,p) = Lp$ and the solution $v : \Omega \to R$ of differential problem (1), (2) is of class C^2;

2) the constant $\widetilde{C} \in R_+$ is such that

$$|\partial_{tt} v(t,x)| \le \widetilde{C}, \quad \left|\partial_{tx_j} v(t,x)\right| \le \widetilde{C}, \quad \left|\partial_{x_i x_j} v(t,x)\right| \le \widetilde{C} \quad \text{on} \quad \Omega,$$

where $1 \le i, j \le n$, and there exists $d \in R_+$ such that

$$|f_i(t,x,p)| \le d, \quad 1 \le i \le n.$$

Then

$$\left|u_h^{(r,m)} - v_h^{(r,m)}\right| \le \widetilde{\eta}_h^{(r)} \tag{31}$$

where

$$\widetilde{\eta}_h^{(r)} = \alpha_0(h)(1 + h_0\bar{c}L)^r + \gamma(h)\frac{(1 + h_0\bar{c}L)^r - 1}{\bar{c}L},$$

$$\gamma(h) = \frac{1}{2}\widetilde{C}h_0 + (1+c_0)L\widetilde{\gamma}(h) + \frac{1}{2}d\widetilde{C}\sum_{i=1}^{n} h_i,$$

$$\widetilde{\gamma}(h) = \widetilde{C}\left[h_0^2 + 2h_0\sum_{i=1}^{n} h_i + \sum_{j,i=1}^{n} h_j h_i\right].$$

Proof. The difference operators δ_0 and δ satisfy the conditions

$$\left|\delta_0 v^{(r,m)} - \partial_t v^{(r,m)}\right| \le \frac{1}{2}\widetilde{C}h_0,$$

$$\left|\delta_i v^{(r,m)} - \partial_{x_i} v^{(r,m)}\right| \le \frac{1}{2}\widetilde{C}h_i, \quad 1 \le i \le n.$$

It follows from above estimates and from Assumption $H[T_h]$ that

$$\begin{aligned}\left|\Gamma_h^{(r,m)}\right| \le {} & \left|\delta_0 v^{(r,m)} - \partial_t v^{(r,m)}\right| + \\ & + \left|\sum_{i=1}^{n} f_i\left(t^{(r)}, x^{(m)}, (T_h[v])_{(r,m)}\right)\delta_i v^{(r+1,m)} - \sum_{i=1}^{n} f_i\left(t^{(r)}, x^{(m)}, v_{(r,m)}\right)\partial_{x_i} v^{(r+1,m)}\right| + \\ & + \left|g\left(t^{(r)}, x^{(m)}, (T_h[v])_{(r,m)}\right) - g\left(t^{(r)}, x^{(m)}, v_{(r,m)}\right)\right| \le \\ & \le \frac{1}{2}\widetilde{C}h_0 + (1+c_0)L\,\widetilde{\gamma}(h) + \frac{1}{2}d\widetilde{C}\sum_{i=1}^{n} h_i.\end{aligned}$$

The function $\widetilde{\eta}_h$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)}(1 + h_0\bar{c}L) + h_0\gamma(h), \quad 0 \le i \le N_0 - 1 \quad \eta^{(0)} = \alpha_0(h),$$

which is equivalent to (26), (27) for $\sigma(t,p) = Lp$. Then from Theorem 3 we get the assertion (31).

5. Numerical examples. For $n = 2$ we put

$$E = [0,1]\times[-1,1]\times[-1,1], \quad E_0 = \{0\}\times[-1,1]\times[-1,1].$$

Consider the quasilinear differential equation

$$\begin{aligned}\partial_t z(t,x,y) = {} & xy^2\partial_x z(t,x,y) + yx^2\partial y z(t,x,y) + f(t,x,y)z(t,x,y) + \\ & + z(t, 0,5(y-x), 0,5(x+y)) + z(t, 0,5(x+y), 0,5(x-y)) - e^{-tx} - e^{ty},\end{aligned} \tag{32}$$

where

$$f(t,x,y) = (x-y)(1+txy),$$

with the initial boundary conditions

$$\begin{aligned} z(0,x,y) &= 1, \quad (x,y) \in [-1,1]\times[-1,1], \\ z(t,-1,y) &= e^{t(-1-y)}, \quad t\in[0,1],\ y\in[-1,1], \\ z(t,1,y) &= e^{t(1-y)}, \quad t\in[0,1],\ y\in[-1,1], \\ z(t,x,-1) &= e^{t(x+1)}, \quad t\in[0,1],\ x\in[-1,1], \\ z(t,x,1) &= e^{t(x+1)}, \quad t\in[0,1],\ x\in[-1,1]. \end{aligned} \tag{33}$$

The solution of the above problem is given by $v(t,x) = e^{t(x-y)}$. Put $h = (h_0, h_1, h_2)$ and assume that $h_1 = h_2$.

Write

$$E_h = \{(t^r, x^{m_1}, y^{m_2}) : 0 \le r \le N_0,\ -K \le m_1, m_2 \le K\}$$

where $N_0 h_0 \le 1 \le (N_0+1)h_0$, $Kh_1 = Kh_2 = 1$. Let us denote by $z_h : E_h \to R$ the solution of the implicit difference problem corresponding to (32), (33). We consider also the function $\widetilde{z}_h : E_h \to R$ which is a solution of the classical difference equation corresponding to (32), (33). Write

$$\eta_h^{(r)} = \frac{1}{(2K+1)^2} \sum_{m_1=-K}^{K} \sum_{m_2=-K}^{K} \left| z_h^{(r,m_1,m_2)} - v^{(r,m_1,m_2)} \right|,$$

$$\widetilde{\eta}_h^{(r)} = \frac{1}{(2K+1)^2} \sum_{m_1=-K}^{K} \sum_{m_2=-K}^{K} \left| \widetilde{z}_h^{(r,m_1,m_2)} - v^{(r,m_1,m_2)} \right|.$$

The numbers $\eta_h^{(r)}$ and $\widetilde{\eta}_h^{(r)}$ are the arithmetical mean of the errors with fixed $t^{(r)}$. The values of the functions η_h and $\widetilde{\eta}_h$ are listed in the table.

Table of errors $(\widetilde{\eta}_h, \eta_h)$ for $h_0 = 0,01,\ h_1 = h_2 = 0,01$:

$t = 0,60$	$1,41414e+003$	$1,65029e-003$
$t = 0,70$	$4,06659e+004$	$2,03410e-003$
$t = 0,80$	$1,10407e+006$	$2,50142e-003$
$t = 0,90$	$2,91547e+007$	$3,07663e-003$
$t = 1,00$	$7,59963e+008$	$3,78804e-003$

It follows that the results obtained by the implicit difference method are better than those obtained by the classical scheme. This is due to the fact that we need the relation (30) for steps of the mesh in the classical case. We do not need this assumption in our implicit difference method.

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