UDC 517.9

ON HIGHER ORDER GENERALIZED EMDEN – FOWLER DIFFERENTIAL EQUATIONS WITH DELAY ARGUMENT*

ПРО УЗАГАЛЬНЕНІ ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ ЕМДЕНА – ФАУЛЕРА ВИЩИХ ПОРЯДКІВ ІЗ ЗАГАЮВАННЯМ В АРГУМЕНТІ

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In the paper the differential equation

$$u^{(n)}(t) + p(t)|u(\tau(t))|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0, \qquad (*)$$

is considered. Here, we assume that $n \geq 3$, $p \in L_{\rm loc}(R_+; R_-)$, $\mu \in C(R_+; (0, +\infty))$, $\tau \in C(R_+; R_+)$, $\tau(t) \leq t$ *for* $t \in R_+$ *and* $\lim_{t \to +\infty} \tau(t) = +\infty$. *In case* $\mu(t) \equiv \text{const} > 0$, *oscillatory properties of equation* $(*)$ *have been extensively studied, where as if* $\mu(t) \not\equiv \text{const}$, *to the extent of authors' knowledge, the analogous questions have not been examined. In this paper, new sufficient conditions for the equation* $(*)$ *to have Property* **B** *are established.*

Розглянуто диференціальне рівняння

$$u^{(n)}(t) + p(t)|u(\tau(t))|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0, \qquad (*)$$

де $n \geq 3$, $p \in L_{\rm loc}(R_+; R_-)$, $\mu \in C(R_+; (0, +\infty))$, $\tau \in C(R_+; R_+)$, $\tau(t) \leq t$ *для* $t \in R_+$ *та* $\lim_{t \to +\infty} \tau(t) = +\infty$. *У випадку* $\mu(t) \equiv \text{const} > 0$ *осциляційні властивості рівняння* $(*)$ *було детально вивчено, тоді як у випадку* $\mu(t) \not\equiv \text{const}$, *наскільки відомо авторам, подібні питання не було розглянуто. У статті наведено нові достатні умови для того, щоб рівняння* $(*)$ *мало властивість* **B**.

1. Introduction. This work deals with oscillatory properties of solutions of a functional differential equations of the form

$$u^{(n)}(t) + p(t)|u(\tau(t))|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0, \qquad (1.1)$$

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where

$$n \geq 3, \quad p \in L_{\mathrm{loc}}(R_+; R_-), \quad \mu \in C(R_+; (0, +\infty)), \quad \tau \in C(R_+; R_+),$$
$$\tau(t) \leq t \quad \text{for} \quad t \in R_+ \quad \text{and} \quad \lim_{t \to +\infty} \tau(t) = +\infty. \tag{1.2}$$

It wiell always be assumed that the condition

$$p(t) \leq 0 \quad \text{for} \quad t \in R_+ \tag{1.3}$$

is fulfilled.

Let $t_0 \in R_+$. A function $u : [t_0; +\infty) \to R$ is said to be a proper solution of equation (1.1) if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive, $\sup\{|u(s)| : s \in [t, +\infty)\} > 0$ for $t \geq t_0$ and there exists a function $\overline{u} \in C(R_+; R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality $\overline{u}^{(n)}(t) + p(t)|\overline{u}(\tau(t))|^{\mu(t)} \mathrm{sign}\, \overline{u}(\tau(t)) = 0$ holds almost everywhere for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1. *We say that equation (1.1) has Property* **A** *if any proper solution* u *is oscillatory if* n *is even and is either oscillatory or satisfies*

$$|u^{(i)}(t)| \downarrow 0 \quad as \quad t \uparrow +\infty, \quad i = 0, \ldots, n-1, \tag{1.4}$$

if n *is odd.*

Definition 1.2. *We say that equation (1.1) has Property* **B** *if any proper solution* u *is either oscillatory, satisfies (1.4), or satisfies*

$$|u^{(i)}(t)| \uparrow +\infty \quad as \quad t \uparrow +\infty, \quad i = 0, \ldots, n-1, \tag{1.5}$$

if n *is even, and is either oscillatory or satisfies (1.5) if* n *is odd.*

Definition 1.3. *We say that equation (1.1) is almost linear if the condition* $\lim_{t \to +\infty} \mu(t) = 1$ *holds, while if* $\limsup_{t \to +\infty} \mu(t) \neq 1$ *or* $\liminf_{t \to +\infty} \mu(t) \neq 1$, *then we say that the equation is an essentially nonlinear differential equation.*

Oscillatory properties of almost linear and essentially nonlinear differential equation with advanced argument are studied well enough in [1–6]. For Emden–Fowler differential equations with deviating arguments, an essential contribution was made in [7–13]. In the present paper sufficient conditions are established for the equation (1.1) to have Property **B**. Analogous results for Property **A** see in [14].

2. Some auxiliary lemmas. The following notation will be used throughout the work: $\widetilde{C}^{n-1}_{\mathrm{loc}}([t_0, +\infty))$ will denote the set of all function $u : [t_0, +\infty) \to R$, absolutely continuous on any finite subinterval of $[t_0, +\infty)$ along with their derivatives of order up to including $n-1$;

$$\alpha = \inf\{\mu(t), t \in R_+\}, \quad \beta = \sup\{\mu(t), t \in R_+\}, \tag{2.1}$$

$$\tau_{(-1)}(t) = \sup\{s \geq 0; \tau(s) \leq t\}, \quad \tau_{(-k)} = \tau_{(-1)} \circ \tau_{(-(k-1))}, \quad k = 2, 3, \ldots. \tag{2.2}$$

Clearly, $\tau_{(-1)}(t) \geq t$ and $\tau_{(-1)}$ is nondecreasing and coincidence with the inverse of σ when the latter exists.

Lemma 2.1 [12]. *Let* $u \in \widetilde{C}^{n-1}_{\rm loc}([t_0,+\infty))$, $u(t) > 0$, $u^{(n)}(t) \ge 0$ *for* $t \ge t_0$ *and* $u^{(n)}(t) \not\equiv 0$ *in any neighborhood of* $+\infty$. *Then there exists* $t_1 \ge t_0$ *and* $\ell \in \{0,\dots,n\}$ *such that* $\ell+n$ *is even and*

$$\begin{gathered} u^{(i)}(t) > 0 \quad \text{for} \quad t \ge t_1, \quad i = 0,\dots,\ell-1, \\ (-1)^{i+\ell}u^{(i)}(t) \ge 0 \quad \text{for} \quad t \ge t_1, \quad i = \ell,\dots,n. \end{gathered} \tag{2.3$_\ell$}$$

In the case $\ell = 0$ we mean that only the second inequality in (2.3_ℓ) holds, while if $\ell = n$ only the first inequality holds and $u^{(n)}(t) \ge 0$.

Lemma 2.2 [15]. *Let* $u \in \widetilde{C}^{n-1}_{\rm loc}([t_0,+\infty))$, $u^{(n)}(t) \ge 0$ *and* (2.3_ℓ) *be satisfied for some* $\ell \in \{1,\dots,n-2\}$, *where* $\ell+n$ *is even. Then*

$$\int\limits_{t_0}^{+\infty} t^{n-\ell-1}u^{(n)}(t)\,dt < +\infty. \tag{2.4}$$

Moreover, if

$$\int\limits_{t_0}^{+\infty} t^{n-\ell}u^{(n)}(t)\,dt = +\infty, \tag{2.5$_\ell$}$$

then there exists $t_1 \ge t_0$ *such that*

$$u(t) \ge \frac{t^{\ell-1}}{\ell!}\,u^{(\ell-1)}(t) \quad \text{for} \quad t \ge t_1, \tag{2.6}$$

$$\frac{u^{(i)}(t)}{t^{\ell-i}}\downarrow, \quad \frac{u^{(i)}(t)}{t^{\ell-i-1}}\uparrow, \quad i = 0,\dots,\ell-1, \tag{2.7$_i$}$$

and

$$u^{(\ell-1)}(t) \ge \frac{t}{(n-\ell)!}\int\limits_{t}^{+\infty} s^{n-\ell-1}u^{(n)}(s)\,ds + \frac{1}{(n-\ell)!}\int\limits_{t_1}^{t} s^{n-\ell}u^{(n)}(s)\,ds. \tag{2.8}$$

Definition 2.1. *Let* $t_0 \in R_+$. *By* $\mathbf{U}_{\ell,t_0}$ *we denote the set of all solutions of equation (1.1) satisfying the condition* (2.3_ℓ).

Lemma 2.3. *Let the conditions (1.2), (1.3) be fulfilled,* $\ell \in \{1,\dots,n-2\}$*with* $\ell+n$ *even and equation (1.1) have positive proper solution* $u : [t_0,+\infty) \to (0,+\infty)$ *such that* $u \in \mathbf{U}_{\ell,t_0}$. *Moreover, let* $\alpha \ge 1$ *and*

$$\int\limits_{t_0}^{+\infty} t^{n-\ell}(c\,\tau^{\ell-1}(t))^{\mu(t)}|p(t)|dt = +\infty \quad \text{for} \quad c \in (0,1], \tag{2.9$_{\ell,c}$}$$

then for any $\gamma \in (1, +\infty)$ *there exists* $t_* > t_0$ *such that for any* $k \in N$

$$u^{(\ell-1)}(t) \geq \rho^{(\alpha)}_{k,\ell,t_*}(t) \quad \textit{for} \quad t \geq \tau_{(-k)}(t_*), \tag{2.10}$$

where

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) = \ell\,!\exp\left\{\gamma_\ell(\alpha) \int\limits_{\tau_{(-1)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}|p(\xi)|d\xi\, ds\right\}, \tag{2.11$_\ell$}$$

$$\rho^{(\alpha)}_{i,\ell,t_*}(t) = \ell\,! + \frac{1}{(n-\ell)!} \int\limits_{\tau_{(-1)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)} \times$$

$$\times \left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{i-1,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\, ds, \quad i = 2, \ldots, k, \tag{2.12$_\ell$}$$

$$\gamma_\ell(\alpha) = \begin{cases} \dfrac{1}{\ell\,!\,(n-\ell)!} & \textit{if} \quad \alpha = 1, \\ \gamma & \textit{if} \quad \alpha > 1, \end{cases} \tag{2.13$_\ell$}$$

α *is given by first equality of (2.1).*

Proof. Let $t_0 \in R_+$, $\ell \in \{1, \ldots, n-2\}$ with $\ell + n$ even and $u \in \mathbf{U}_{\ell,t_0}$. According to (1.1), (2.3$_\ell$) and (2.9$_{\ell,c}$) it is clear that condition (2.5$_\ell$) is fulfilled. Indeed, by (2.3$_\ell$) there exists $t_1 > t_0$ and $c \in (0, 1]$ such that

$$u(\tau(t)) \geq c(\tau(t))^{\ell-1} \quad \text{for} \quad t \geq t_1.$$

Thus from (1.1) we have

$$\int\limits_{t_1}^{t} s^{n-\ell} u^{(n)}(s)ds \geq \int\limits_{t_1}^{t} s^{n-\ell} \left(c\,\tau^{\ell-1}(s)\right)^{\mu(s)} |p(s)|ds \quad \text{for} \quad t \geq t_1.$$

Passing to the limit in the latter inequality, by (2.9$_{\ell,c}$) we get (2.5$_\ell$).

According to Lemma 2.2 there exists $t_2 > t_1$ such that conditions (2.6) – (2.8) are fulfilled for $t \geq t_2$ and

$$u^{(\ell-1)}(t) \geq \frac{t}{(n-\ell)!} \int\limits_{t}^{+\infty} s^{n-\ell-1}(u(\tau(s)))^{\mu(s)}|p(s)|ds +$$

$$+ \frac{1}{(n-\ell)!} \int\limits_{t_{(-1)}(t_2)}^{t} s^{n-\ell}(u(\tau(s)))^{\mu(s)}|p(s)|ds \quad \text{for} \quad t \geq \tau_{(-1)}(t_2).$$

Therefore, by (2.6) we get

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int\limits_{\tau_{(-1)}(t_2)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{(\ell-1)\mu(\xi)} \times$$

$$\times \left(\frac{1}{\ell\,!}\, u^{(\ell-1)}(\tau(\xi)) \right)^{\mu(\xi)} |p(\xi)| d\xi\, ds \quad \text{for} \quad t \geq \tau_{(-1)}(t_2). \tag{2.14}$$

According to $(2.7_{\ell-1})$ and $(2.9_{\ell,c})$ choose $t_* > \tau_{(-1)}(t_2)$ such that

$$\frac{1}{(n-\ell)!} \int\limits_{\tau_{(-1)}(t_2)}^{t_*} \int\limits_{s}^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\, u^{(\ell-1)}(\tau(\xi)) \right)^{\mu(\xi)} |p(\xi)| d\xi\, ds > \ell\,!. \tag{2.15}$$

By (2.14) and (2.15) we have

$$u^{(\ell-1)}(t) \geq \ell\,! + \frac{1}{(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{(\ell-1)\mu(\xi)} \times$$

$$\times \left(\frac{1}{\ell\,!}\, u^{(\ell-1)}(\tau(\xi)) \right)^{\mu(\xi)} |p(\xi)| d\xi\, ds \quad \text{for} \quad t \geq t_*. \tag{2.16}$$

Let $\alpha = 1$. Since $u^{(\ell-1)}(t)/t$ is a nonincreasing function, from (2.16) we obtain

$$u^{(\ell-1)}(t) \geq \ell\,! + \frac{1}{\ell\,!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} \times$$

$$\times\, u^{(\ell-1)}(\xi) |p(\xi)| d\xi\, ds \quad \text{for} \quad t \geq t_*. \tag{2.17}$$

By the second condition of $(2.7_{\ell-1})$, it is obviously that

$$x'(t) \geq \frac{u^{(\ell-1)}(t)}{\ell\,!(n-\ell)!} \int\limits_{t}^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi, \tag{2.18}$$

where

$$x(t) = \ell\,! + \frac{1}{\ell\,!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} u^{(\ell-1)}(\xi) |p(\xi)| d\xi\, ds. \tag{2.19}$$

Thus, according to (2.17), (2.18) and (2.19) we get

$$x'(t) \geq \frac{x(t)}{\ell\,!(n-\ell)!} \int\limits_{t}^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi \quad \text{for} \quad t \geq t_*.$$

Therefore, since $x(t_*) = \ell\,!$, we have

$$x(t) \geq \ell\,! \exp\left\{\frac{1}{\ell\,!(n-\ell)!}\int\limits_{t_*}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}|p(\xi)|d\xi\,ds\right\} \quad \text{for} \quad t \geq t_*.$$

Hence by (2.16) and (2.19)

$$u^{(\ell-1)}(t) \geq \rho^{(1)}_{1,\ell,t_*}(t) \quad \text{for} \quad t \geq t_*, \tag{2.20}$$

where

$$\rho^{(1)}_{1,\ell,t_*}(t) = \ell\,! \exp\left\{\frac{1}{\ell\,!(n-\ell)!}\int\limits_{t_*}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}|p(\xi)|d\xi\,ds\right\}. \tag{2.21}$$

Thus, by (2.14) and (2.20),

$$u^{(\ell-1)}(t) \geq \rho^{(1)}_{i,\ell,t_*}(t) \quad \text{for} \quad t \geq \tau_{(-i)}(t_*), \quad i = 1,\ldots,k, \tag{2.22}$$

where

$$\rho^{(1)}_{i,\ell,t_*}(t) = \ell\,! + \frac{1}{(n-\ell)!}\int\limits_{\tau_{(-i)}(t_*)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}\times$$

$$\times\left(\frac{1}{\ell\,!}\,\rho^{(1)}_{i-1,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)}|p(\xi)|d\xi\,ds \quad i = 2,\ldots,k. \tag{2.23}$$

Now assume that $\alpha > 1$ and $\gamma \in (1,+\infty)$. Since $u^{(\ell-1)}(t) \uparrow +\infty$ for $t \uparrow +\infty$, without loss of generality we can assume that $\left(\frac{1}{\ell\,!}\,u^{(\ell-1)}(\tau(t))\right)^{\alpha-1} \geq \ell\,!(n-\ell)!\,\gamma$ for $t \geq t_*$. From (2.16) we obtain

$$u^{(\ell-1)}(t) \geq \ell\,! + \gamma\int\limits_{t_*}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}u^{(\ell-1)}(\xi)|p(\xi)|d\xi\,ds \quad \text{for} \quad t \geq t_*. \tag{2.24}$$

By (2.24), as above we can find that if $\alpha > 1$, then

$$u^{(\ell-1)}(t) \geq \rho^{(\alpha)}_{k,\ell,t_*}(t) \quad \text{for} \quad t \geq \tau_{(-k)}(t_*), \tag{2.25}$$

where

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) = \ell\,! \exp\left\{\gamma\int\limits_{\tau_{(-1)}(t_*)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}\times\right.$$

$$\left.\times\,|p(\xi)|d\xi\,ds\right\} \quad \text{for} \quad t \geq \tau_{(-1)}(t_*), \tag{2.26}$$

$$\rho^{(\alpha)}_{i,\ell,t_*}(t) = \ell\,! + \frac{1}{(n-\ell)!}\int\limits_{\tau_{(-i)}(t_*)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}\times$$

$$\times\left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{i-1,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)}|p(\xi)|d\xi\,ds \quad \text{for} \quad t \ge \tau_{(-i)}(t_*), \quad i = 2,\ldots,k. \qquad (2.27)$$

According to (2.20) – (2.23) and (2.25) – (2.27) it is clear that for any $\alpha \ge 1$, $k \in N$, and $\gamma \in \in (1,+\infty)$, there exists $t_* \in R_+$ such that (2.10) holds, where $\gamma_\ell(\alpha)$ is given by (2.13_ℓ), which proves the validity of the lemma.

Remark 2.1. It is obvious that, if $\beta < +\infty$ and ($2.9_{\ell,1}$) holds, then for any $c \in (0,1]$ the condition ($2.9_{\ell,c}$) is fulfilled.

Remark 2.2. Condition ($2.9_{\ell,1}$) is not suffices for condition (2.5) to be fulfilled. Therefore, in this case, it can happen that Lemma 2.3 is not correct. Indeed, let $\delta \in (0,1)$. Consider equation (1.1), where n is odd and

$$\tau(t) \equiv t, \quad p(t) = -\frac{n!\,t^{\log_{1/\delta} t}}{t^{n+1}(\delta t - 1)^{\log_{1/\delta} t}}, \quad \mu(t) = \log_{1/\delta} t, \quad t \ge \frac{2}{\delta}.$$

It is clear that the function $u(t) = \delta - \dfrac{1}{t}$ is solution of the equation (1.1) and satisfies condition (2.3_1) for $t \ge \dfrac{2}{\delta}$. On the other hand condition ($2.9_{1,1}$) holds, but condition (2.5_1) is not fulfilled.

3. Necessary conditions for the existence of solutions of type (2.3_ℓ).

Theorem 3.1. *Let $\ell \in \{1,\ldots,n-2\}$ with $\ell + n$ be even, conditions (1.2), (1.3), ($2.9_{\ell,c}$) and*

$$\int\limits_0^{+\infty} t^{n-\ell-1}(\tau(t))^{\ell\mu(t)}|p(t)|dt = +\infty \qquad (3.1_\ell)$$

be fulfilled and for some $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} \ne \varnothing$. Then there exists $t_* > t_0$ such that, if $\alpha = 1$, then, for any $k \in N$,

$$\lim_{t\to+\infty}\frac{1}{t}\int\limits_{\tau_{(-k)}(t_*)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}\left(\frac{1}{\ell\,!}\,\rho^{(1)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)}|p(\xi)|d\xi\,ds = 0 \qquad (3.2)$$

and if $\alpha > 1$, then, for any $k \in N$, $\gamma \in (1,+\infty)$ and $\delta \in (1,\alpha]$,

$$\int\limits_{\tau_{(-i)}(t_*)}^{+\infty}\int\limits_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}\left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta}|p(\xi)|d\xi\,ds < +\infty, \qquad (3.3)$$

where α is defined by first equality of (2.1) and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by (2.11) – (2.13).

Proof. Let $t_0 \in R_+$, $\ell \in \{1,\ldots,n-2\}$, $\mathbf{U}_{\ell,t_0} \ne \varnothing$ and $\gamma \in (1,+\infty)$. By definition (see Definition 2.1) equation (1.1) has a proper solution $u \in \mathbf{U}_{\ell,t_0}$ satisfying condition (2.3_ℓ) with some $t_1 \geq t_0$. Due to (1.1), (2.3_ℓ) and $(2.9_{\ell,c})$, it is obvious that condition (2.5_ℓ) holds. Thus, by Lemma 2.2 there exists $t_1 > t_0$ such that conditions (2.6), (2.7_i) are fulfilled. On the other hand, according to Lemma 2.3 (and its proof), there exist $t_2 > t_1$ and $t_* > t_2$ such that

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int\limits_{t_2}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(u(\tau(\xi)))^{\mu(\xi)}|p(\xi)|d\xi\, ds \quad \text{for} \quad t \geq t_2 \tag{3.4}$$

and relation (2.10) is fulfilled. Without loss of generality we can assume that $\tau(t) \geq t_2$ for $t \geq t_*$. Therefore, by (2.10), from (3.4) we get

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int\limits_{\tau_{(-k)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\, u^{(\ell-1)}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\, ds. \tag{3.5}$$

Assume that $\alpha = 1$. Then by (2.10) and (3.5) we have

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int\limits_{\tau_{(-k)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)} \times$$

$$\times \left(\frac{1}{\ell\,!}\, \rho^{(1)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\, ds \quad \text{for} \quad t \geq \tau_{(-k)}(t_*). \tag{3.6}$$

On the other hand, according to $(2.7_{\ell-1})$ and (3.1_ℓ) it is obvious that

$$u^{(\ell-1)}(t)/t \ \downarrow 0 \quad \text{for} \quad t \uparrow +\infty. \tag{3.7}$$

Therefore by (3.7), from (3.6) we get

$$\lim_{t\to+\infty} \frac{1}{t} \int\limits_{\tau_{(-k)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\, \rho^{(1)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\, ds = 0. \tag{3.8}$$

Now assume that $\alpha > 1$ and $\delta \in (1, \alpha]$. Then by $(2.7_{\ell-1})$, (2.10) and (3.7) we obtain

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int\limits_{\tau_{(-k)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \times$$

$$\times \left(\frac{1}{\ell\,!}\, \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta} \left(\frac{1}{\ell\,!}\, u^{(\ell-1)}(\xi)\right)^{\delta} |p(\xi)|d\xi\, ds \geq$$

$$\geq \frac{1}{(n-\ell)!} \int\limits_{\tau_{(-k)}(t_*)}^{t} \left(\frac{1}{\ell\,!}\, u^{(\ell-1)}(\xi)\right)^{\delta} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \times$$

$$\times \left(\frac{1}{\ell\,!}\, \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta} |p(\xi)|d\xi\, ds.$$

Thus we have

$$(v(t))^{\delta} \geq \frac{1}{(\ell\,!(n-\ell)!)^{\delta}} \Bigg(\int\limits_{\tau_{(-k)}(t_*)}^{t} v^{\delta}(s) \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \times$$

$$\times \left(\frac{1}{\ell\,!} \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi)) \right)^{\mu(\xi)-\delta} |p(\xi)| d\xi\, ds \Bigg)^{\delta}, \tag{3.9}$$

where $v(t) = \dfrac{1}{\ell\,!} u^{(\ell-1)}(t)$.

By (3.1_ℓ), it is obvious that there exists $t_1 > \tau_{(-k)}(t_*)$ such that

$$\int\limits_{\tau_{(-k)}(t_*)}^{t} v^{\delta}(s) \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!} \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi)) \right)^{\mu(\xi)-\delta} \times$$

$$\times |p(\xi)| d\xi\, ds > 0 \quad \text{for} \quad t \geq t_1.$$

Therefore, from (3.9) we get

$$\int\limits_{t_1}^{t} \frac{\varphi'(s) ds}{(\varphi(s))^{\delta}} \geq \frac{1}{(\ell\,!(n-\ell)!)^{\delta}} \int\limits_{t_1}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \times$$

$$\times \left(\frac{1}{\ell\,!} \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi)) \right)^{\mu(\xi)-\delta} |p(\xi)| d\xi\, ds \quad \text{for} \quad t \geq t_1, \tag{3.10}$$

where

$$\varphi(t) = \int\limits_{\tau_{(-k)}(t_*)}^{t} (v(s))^{\delta} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!} \rho^{(\alpha)}_{k,\ell,t_*}(\tau(s)) \right)^{\mu(\xi)-\delta} |p(\xi)| d\xi\, ds.$$

From (3.10) we obtain

$$\int\limits_{t_1}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!} \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi)) \right)^{\mu(\xi)-\delta} |p(\xi)| d\xi\, ds \leq$$

$$\leq \frac{(\ell\,!(n-\ell)!)^{\delta}}{\delta-1} \left(\varphi^{1-\delta}(t_1) - \varphi^{1-\delta}(t) \right) \leq \frac{(\ell\,!(n-\ell)!)^{\delta}}{\delta-1} \varphi^{1-\delta}(t_1) \quad \text{for} \quad t \geq t_1.$$

Hence,

$$\int\limits_{t_1}^{+\infty} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!} \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi)) \right)^{\mu(\xi)-\delta} |p(\xi)| d\xi\, ds \leq +\infty. \tag{3.11}$$

According to (3.8) and (3.11) conditions (3.2) and (3.3) hold, with proves the validity of the theorem.

Corollary 3.1. *Let* $\ell \in \{1,\dots,n-1\}$ *with* $\ell+n$ *be even,* $\beta < +\infty$, *conditions (1.2), (1.3),* $(2.9_{\ell,1})$, (3.1_ℓ) *be fulfilled and for some* $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} \neq \varnothing$. *Then for any* $\gamma > 1$ *there exists* $t_* > t_0$ *such that if* $\alpha = 1$, *for any* $k \in N$, *(3.2) holds and if* $\alpha > 1$, *then, for any* $k \in N$ *and* $\delta \in (1,\alpha]$, *(3.3) holds, where* α *and* β *are defined by (2.1) and* $\rho^{(\alpha)}_{k,\ell,t_*}$ *is given by (2.11) – (2.13).*

Proof. According to Remark 2.1, it suffices to note that, since $\beta < +\infty$, by $(2.9_{\ell,1})$, for any $c \in (0,1]$ conditions $(2.9_{\ell,c})$ is fulfilled.

4. Sufficient conditions for nonexistence of solutions of the type (2.3_ℓ).

Theorem 4.1. *Let* $\ell \in \{1,\dots,n-2\}$ *with* $\ell+n$ *be even, conditions (1.2), (1.3),* $(2.9_{\ell,c})$ *and* (3.1_ℓ) *be fulfilled and if* $\alpha = 1$ *for large* $t_* \in R_+$ *and for some* $k \in N$,

$$\limsup_{t\to+\infty} \frac{1}{t} \int\limits_{\tau_{(-k)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\,ds > 0, \tag{4.1_ℓ}$$

or if $\alpha > 1$, *for some* $k \in N$ *and* $\delta \in (1,\alpha]$,

$$\int\limits_{\tau_{(-k)}(t_*)}^{+\infty} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta} |p(\xi)|d\xi\,ds = +\infty. \tag{4.2_ℓ}$$

Then for any $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$, *where* α *is defined by the first equality of (2.1), and* $\rho^{(\alpha)}_{k,\ell,t_*}$ *is given by (2.11) – (2.13).*

Proof. Assume the contrary. Let there exist $t_0 \in R_+$ such that $\mathbf{U}_{\ell,t_0} \neq \varnothing$ (see Definition 2.1). Then equation (1.1) has a proper solution $u : [t_0,+\infty) \to R$ satisfying condition (2.3_ℓ). Since the conditions of Theorem 3.1 are fulfilled, there exists $t_* > t_0$ such that if $\alpha = 1$ ($\alpha > 1$) condition (3.2) (condition (3.3)) holds, which contradicts (4.1_ℓ) ((4.2_ℓ)). The obtained contradiction proves the validity of the theorem.

Theorem 4.1′. *Let* $\ell \in \{1,\dots,n-2\}$ *with* $\ell+n$ *be even, conditions (1.2), (1.3),* $(2.9_{\ell,1})$ *and* (3.1_ℓ) *be fulfilled and* $\beta < +\infty$. *Moreover, if* $\alpha = 1$, $\alpha > 1$, *for any large* $t_* \in R_+$ *and for some* $k \in N$ (*for some* $k \in N$ *and* $\delta \in (1,\alpha]$), (4.1_ℓ) *holds* ((4.2_ℓ) *holds*), *then* $\mathbf{U}_{\ell,t_0} = \varnothing$, *where* α *and* β *are given by (2.1).*

Proof. If suffices to note that, since $\beta < +\infty$, by $(2.9_{\ell,1})$ for any $c \in (0,1]$ the condition $(2.9_{\ell,c})$ is fulfilled. Therefore all the conditions of Theorem 4.1 hold, which proves the validity of the theorem.

Corollary 4.1. *Let* $\ell \in \{1,\dots,n-2\}$ *with* $\ell+n$ *be even,* $\alpha = 1$, *conditions (1.2), (1.3),* $(2.9_{\ell,c})$ *and* (3.1_ℓ) *be fulfilled and*

$$\limsup_{t\to+\infty} \frac{1}{t} \int\limits_{0}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}|p(\xi)|d\xi\,ds > 0. \tag{4.3_ℓ}$$

Then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, *where* α *is defined by the first equality of (2.1).*

Proof. Since

$$\rho^{(1)}_{1,\ell,t_*}(\tau(t)) \geq \ell \quad \text{for large} \quad t,$$

it is suffices to note that by (4.3_ℓ) for $\alpha = 1$ and $k = 1$ condition (4.1_ℓ) ise fulfilled.

Corollary 4.1′. *Let $\ell \in \{1, \dots, n-2\}$ with $\ell + n$ be even, conditions (1.2), (1.3), (4.3_ℓ) and (3.1_ℓ) be fulfilled. Moreover, if $\alpha = 1$ and $\beta < +\infty$, then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, where α and β are given by (2.1).*

Proof. To prove the corollary it is suffices to note that, since $\beta < +\infty$, by (4.3_ℓ) condition ($2.9_{\ell,c}$) holds.

Corollary 4.2. *Let $\ell \in \{1, \dots, n-2\}$ with $\ell + n$ even, conditions (1.2), (1.3) and ($2.9_{\ell,c}$) be fulfilled, $\alpha = 1$ and*

$$\liminf_{t\to+\infty} t \int\limits_t^{+\infty} s^{n-\ell-2}(\tau(s))^{1+(\ell-1)\mu(s)}|p(\xi)|ds = \gamma > 0. \qquad (4.4_\ell)$$

Moreover, if for some $\varepsilon \in (0, \gamma)$

$$\limsup_{t\to+\infty} \frac{1}{t} \int\limits_0^t \int\limits_s^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{\mu(\xi)\left(\ell-1+\frac{\gamma-\varepsilon}{\ell\,!(n-\ell)!}\right)}|p(\xi)|d\xi\, ds > 0, \qquad (4.5_\ell)$$

then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, where α is given by the first equality of (2.1).

Proof. Let $\varepsilon \in (0, \gamma)$. According to ($4.4_\ell$), (2.11) and (2.13) it is clear that $\rho^{(1)}_{1,\ell,t_*}(\tau(t)) \geq$ $\geq \ell\,!(\tau(t))^{\frac{\gamma-\varepsilon}{\ell\,!(n-\ell)!}}$ for large t. Therefore, by (4.5_ℓ), for $k = 1$, (4.1_ℓ) holds, which proves the validity of the corollary.

Corollary 4.2′. *Let $\ell \in \{1, \dots, n-2\}$ with $\ell + n$ even, conditions (1.2), (1.3), (3.1_ℓ), (4.4_ℓ) and (4.5_ℓ) be fulfilled. Moreover, if $\alpha = 1$ and $\beta < +\infty$, then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, where α and β are given by (2.1).*

Proof. To prove the corollary, it is suffices to note that, since $\beta < +\infty$ by (4.4_ℓ) the condition ($2.9_{\ell,c}$) holds.

Corollary 4.3. *Let $\ell \in \{1, \dots, n-2\}$ with $\ell+n$ even, conditions (1.2), (1.3), ($2.9_{\ell,c}$) and (3.1_ℓ) be fulfilled. Moreover, if $\alpha > 1$ and, for some $\delta \in (1, \alpha]$,*

$$\int\limits_0^{+\infty} \int\limits_s^{+\infty} \xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}|p(\xi)|d\xi\, ds = +\infty. \qquad (4.6_\ell)$$

Then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, where α is defined by the first condition of (2.1).

Proof. By (4.6_ℓ), for $k = 1$ condition (4.2_ℓ) holds, which proves the validity of the corollary.

Corollary 4.3′. *Let $\ell \in \{1, \dots, n-2\}$ with $\ell + n$ even, conditions (1.2), (1.3), (3.1_ℓ), ($2.9_{\ell,1}$) and (4.6_ℓ) be fulfilled. Moreover, if $\alpha > 1$ and $\beta < +\infty$, then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, where α and β are given by (2.1).*

Proof. According to Corollary 4.3, it is suffices to note that, since $\beta < +\infty$ by ($2.9_{\ell,1}$) for any $c \in (0, 1]$, condition ($2.9_{\ell,c}$) hold.

Corollary 4.4. *Let $\ell \in \{1, \dots, n-2\}$ with $\ell + n$ even, conditions (1.2), (1.3), ($2.9_{\ell,c}$), (3.1_ℓ), (4.4_ℓ) and (4.6_ℓ) be fulfilled. Moreover, if $\alpha > 1$ and there exists $m \in N$ such that*

$$\liminf_{t\to+\infty} \frac{\tau^m(t)}{t} > 0, \qquad (4.7)$$

then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, where α is given by the first condition of (2.1).

Proof. By (4.4_ℓ) there exists $c > 0$ and $t_1 \in R_+$ such that

$$t \int\limits_t^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi \geq c \quad \text{for} \quad t \geq t_1. \tag{4.8}$$

Let $\delta = \dfrac{1+\alpha}{2}$ and $m_0 = \dfrac{\delta(m-1)}{c(\alpha-\delta)}$. Then by (4.8) and (2.26), there exists $t_* > t_1$ such that

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) \geq t^{m_0 c} \quad \text{for} \quad t \geq t_*.$$

Therefore, for large t we have

$$\left(\frac{\tau(t)}{t}\right)^{\delta} \left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{1,\ell,t_*}(\tau(t))\right)^{\mu(t)-\delta} \geq \left(\frac{\tau(t)}{t}\right)^{\delta} \left(\frac{1}{\ell\,!}\,\tau^{m_0 c}(t)\right)^{\alpha-\delta} =$$

$$= \frac{1}{(\ell\,!)^{\alpha-\delta}} \left(\frac{(\tau(t))^{1+\frac{m_0 c(\alpha-\delta)}{\delta}}}{t}\right)^{\delta} = (\ell\,!)^{\delta-\alpha} \left(\frac{\tau^m(t)}{t}\right)^{\delta}.$$

Thus, by (4.7) and (4.6_ℓ) it is obvious that (4.2_ℓ) holds, which proves the corollary.

Corollary 4.4′. *Let $\ell \in \{1,\ldots,n-2\}$ with $\ell + n$ be even and conditions (1.2), (1.3), (3.1_ℓ), (4.6_ℓ) and (4.7) be fulfilled. Moreover, if $\alpha > 1$ and $\beta < +\infty$, then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, where α and β are given by (2.1).*

Proof. Since $\beta < +\infty$, it suffices to note that all conditions of Corollary 4.4 are satisfied.

Quite similarly one can prove the following corollary.

Corollary 4.5. *Let $\ell \in \{1,\ldots,n-2\}$ with $\ell+n$ even, conditions (1.2), (1.3), (3.1_ℓ) and ($2.9_{\ell,c}$) be fulfilled and $\alpha > 1$. Moreover, if*

$$\liminf_{t\to+\infty} t\,\ln t \int\limits_t^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi > 0 \tag{4.9_ℓ}$$

and for some $\delta \in (1,\alpha]$ and $m \in N$

$$\int\limits_0^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} (\ln \tau(\xi))^m |p(\xi)| d\xi\, ds = +\infty, \tag{4.10_ℓ}$$

then for any $t_0 \in R_+$ we have $\mathbf{U}_{\ell,t_0} = \varnothing$, where α is defined by the first equality of (2.1).

Corollary 4.5′. *Let $\ell \in \{1,\ldots,n-2\}$ with $\ell + n$ even, conditions (1.2), (1.3), ($2.9_{\ell,1}$), (4.9_ℓ) and (4.10_ℓ) be fulfilled. Moreover, if $\alpha > 1$ and $\beta < +\infty$, then for any $t_0 \in R_+$ we have $\mathbf{U}_{\ell,t_0} = \varnothing$, where α and β are given by (2.1).*

Corollary 4.6. *Let $\alpha > 1$, $\ell \in \{1,\dots,n-2\}$ with $\ell+n$ even, conditions (1.2), (3.1$_\ell$) and (2.9$_{\ell,c}$) be fulfilled. Moreover, assume there exist $\gamma \in (0,1)$ and $r \in (0,1)$ such that*

$$\liminf_{t\to+\infty} t^{\gamma} \int_t^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}|p(\xi)|d\xi > 0, \tag{4.11$_\ell$}$$

$$\liminf_{t\to+\infty} \frac{\tau(t)}{t^r} > 0 \tag{4.12}$$

and at last one of the conditions

$$r\,\alpha \ge 1 \tag{4.13}$$

or $r\,\alpha < 1$ holds, and for some $\varepsilon > 0$ and $\delta \in (1,\alpha)$,

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-\ell-1-\delta-\varepsilon+\frac{r(1-\gamma)(\alpha-\delta)}{1-\alpha r}}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}|p(\xi)|d\xi\, ds = +\infty, \tag{4.14$_\ell$}$$

is fulfilled. Then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \varnothing$, where α is defined by first equality of (2.1).

Proof. It suffices to show that condition (4.2$_\ell$) is satisfied for some $k \in N$. Indeed, according to (4.11$_\ell$) and (4.12) there exist $\gamma \in (0,1)$, $r \in (0,1)$, $c > 0$ and $t_1 \in R_+$ such that

$$t^{\gamma}\int_t^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}|p(\xi)|d\xi \ge c \quad \text{for} \quad t \ge t_1 \tag{4.15}$$

and

$$\tau(t) \ge c\,t^r \quad \text{for} \quad t \ge t_1. \tag{4.16}$$

By (2.12$_\ell$), (2.11$_\ell$) and (4.15) we have

$$\rho^{(\alpha)}_{2,\ell,t_*}(t) \ge \frac{c}{(n-\ell)!}\int_{\tau_{(-1)}(t_*)}^{t} s^{-\gamma}ds = \frac{c\left(t^{1-\gamma}-\tau^{1-\gamma}_{(-1)}(t_*)\right)}{(n-\ell)!(1-\gamma)} \quad \text{for} \quad t \ge \tau_{(-1)}(t_*).$$

Choose $t_2 > \tau_{(-1)}(t_*)$ and $c_1 \in (0,c)$ such that

$$\rho^{(\alpha)}_{2,\ell,t_*}(t) \ge c_1\,t^{1-\gamma} \quad \text{for} \quad t \ge t_2.$$

Therefore, by (4.15) and (4.16) we can find $t_3 > t_2$ and $c_2 \in (0,c_1)$ such that from (2.12) we get

$$\rho^{(\alpha)}_{3,\ell,t_*}(t) \ge c_2\,t^{(1-\gamma)(1+\alpha\,r)} \quad \text{for} \quad t \ge t_3.$$

Hence for any $k_0 \in N$, there exist t_{k_0} and $c_{k_0-1} > 0$ such that

$$\rho^{(\alpha)}_{k_0,\ell,t_*}(t) \geq c_{k_0-1}\, t^{(1-\gamma)(1+\alpha r+\ldots+(\alpha r)^{k_0-2})} \quad \text{for} \quad t \geq t_{k_0}. \tag{4.17}$$

Assume that (4.13) is fulfilled. Choose $k_0 \in N$ such that $k_0 - 1 \geq \dfrac{\delta}{r(\alpha-\delta)(1-\gamma)}$. Then by (4.16), (4.17) and ($2.9_{\ell,1}$) condition (4.2_ℓ) holds for $k = k_0$.

In this case, the validity of the corollary has already been proved.

Assume now that $\alpha r < 1$ and for some $\varepsilon \in (0, (1-\gamma(\alpha-\delta)r)$, ($4.14_\ell$) is fulfilled. Choose $k_0 \in N$ such that $1 + \alpha r + \ldots + (\alpha r)^{k_0-2} \geq \dfrac{1}{1-\alpha r} - \dfrac{\varepsilon}{(1-\gamma)(\alpha-\delta)r}$. Then by ($4.14_\ell$), (4.16) and (4.17) it is obvious that (4.2_ℓ) holds for $k = k_0$. The proof the corollary is complete.

5. Differential equations with property B.

Theorem 5.1. *Let conditions (1.2) and (1.3) be fulfilled and for any $\ell \in \{1,\ldots,n\}$ with $\ell+n$ even, conditions ($2.9_{\ell,c}$), (3.1_ℓ) and, for even n, ($2.9_{1,c}$) hold. Moreover, let for any large $t_* \in R_+$ and $\ell \in \{1,\ldots,n-2\}$ with $\ell+n$ even for some $k \in N$, condition (4.1_ℓ) hold, when $\alpha = 1$, or for some $k \in N$, $\gamma \in (1,+\infty)$ and $\delta \in (1,\alpha]$ (4.2_ℓ) hold when $\alpha > 1$. Then equation (1.1) has Property* **B***, where α is defined by the first condition of (2.1) and $\rho^{\alpha)}_{k,\ell,t_*}$ is given by (2.11) – (2.13).*

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : [t_0,+\infty) \to (0,+\infty)$ (the case $u(t) < 0$ is similar). Then by (1.2), (1.3) and Lemma 2.1 there exists $\ell \in \{1,\ldots,n\}$ such that $\ell+n$ is even and condition (2.3_ℓ) holds. Since for any $\ell \in \{1,\ldots,n-2\}$, with $\ell+n$ even, the conditions of Theorem 4.1 are fulfilled, we have $\ell \notin \{1,\ldots,n-2\}$. Let $\ell = n$. Then by (2.3_n) it is clear that there exists $c \in (0,1]$ such that for large t, $u(\tau(t)) \geq c\tau^{n-1}(t)$. Thus from (1.1) by ($2.9_{n,c}$) we have

$$u^{(n-1)}(t) \geq \int\limits_{t_1}^{t} (c\tau^{n-1}(s))^{\mu(s)}|p(s)|ds \to +\infty \quad \text{for} \quad t \to +\infty,$$

where t_1 is a sufficiently large number. That is, condition (1.4) is fulfilled. Now assume that $\ell = 0$ and n is even and there exists $c \in (0,1]$ such that $u(t) \geq c$ for $t \geq t_2$, where t_2 is a sufficiently large number. According to (2.3_0) from (1.1) we get

$$\sum_{i=0}^{n-1}(n-i-1)!\, t_1|u^{(i)}(t_1)| \geq \int\limits_{t_1}^{t} s^{n-1}c^{\mu(s)}|p(s)|ds \quad \text{for} \quad t \geq t_2.$$

The last inequality contradicts conditions ($2.9_{1,c}$). The obtained contradiction proves that condition (1.5) holds, that is equation (1.1) has Property **B**.

Theorem 5.1′. *Let conditions (1.2), (1.3) be fulfilled and for any $\ell \in \{1,\ldots,n\}$ with $\ell+n$ even, conditions ($2.9_{\ell,1}$), (3.1_ℓ) and, for even n, ($2.9_{1,1}$) hold. Moreover, let $\beta < +\infty$ and for any large $t_* \in R_+$ and $\ell \in \{1,\ldots,n-2\}$ with $\ell+n$ even for some $k \in N$, condition (4.1_ℓ) be fulfilled, when $\alpha = 1$ or for some $k \in N$, $\gamma \in (1,+\infty)$ and $\delta \in (1,\alpha]$ (4.2_ℓ) hold, when $\alpha > 1$. Then the equation (1.1) has Property* **B***, where α and β are defined by the first condition of (2.1) and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by (2.11_ℓ) – (2.13_ℓ).*

Proof. Since $\beta < +\infty$, by $(2.9_{\ell,1})$ for any $\ell \in \{1,\ldots,n\}$ with $\ell+n$ even, condition $(2.9_{\ell,c})$ holds. That is conditions of Theorem 5.1 are fulfilled, which proves the validity of the theorem.

Theorem 5.2. *Let $\alpha > 1$, conditions (1.2), (1.3), $(2.9_{1,c})$, (3.1_1) be fulfilled and*

$$\liminf_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t} > 0. \tag{5.1}$$

Moreover, if for some $\delta \in (1,\alpha)$

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-2-\delta}(\tau(\xi))^{\delta}|p(\xi)|d\xi\, ds = +\infty, \tag{5.2}$$

when n is odd and

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-3-\delta}(\tau(\xi))^{\delta+\mu(\xi)}|p(\xi)|d\xi\, ds = +\infty, \tag{5.3}$$

when n is even, then equation (1.1) has Property **B**, *where α is given by the first condition of (2.1).*

Proof. According to $(2.9_{1,c})$, (3.1_1) and (5.1) it is obvious that for any $\ell \in \{1,\ldots,n\}$ conditions $(2.9_{\ell,c})$ and (3.1_ℓ) hold. On the other hand by (5.1), (5.2) and (5.3), for any $\ell \in \{1,\ldots,n-2\}$ with $\ell+n$ even condition (4.2_ℓ) holds. That is if $\alpha > 1$, then all conditions of Theorem 5.1 hold, which proves the validity of the theorem.

Theorem 5.2′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions (1.2), (1.3), $(2.9_{1,1})$, (3.1_1) and (5.1) be fulfilled. Moreover, let for some $\delta \in (1,\alpha]$, the condition (5.2) hold when n is odd and the condition (5.3) hold, when n is even. Then equation (1.1) has Property* **B**, *where α and β are given by (2.1).*

Proof. Since $\beta < +\infty$, by $(2.9_{1,1})$ it is obvious that for any $c \in (0,1]$ condition $(2.9_{1,c})$ holds. That is all conditions of Theorem 5.2 are fulfilled, which proves the validity of the theorem.

Corollary 5.1. *Let $\alpha > 1$, conditions (1.2), (1.3), $(2.9_{1,c})$, (3.1_1) and (5.1) be fulfilled and*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-3}\tau(s)|p(s)|ds > 0. \tag{5.4}$$

Moreover, if for some $\delta \in (1,\alpha]$ and $\gamma > 0$

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-2-\delta}(\tau(\xi))^{\delta+\gamma(\mu(\xi)-\delta)}|p(\xi)|d\xi\, ds = +\infty, \tag{5.5}$$

then equation (1.1) has Property **B**, *where α is defined by the first condition of (2.1).*

Proof. Since $\alpha > 1$. By (5.4), (2.11_1) and (2.13_1) for any $\gamma > 0$, there exists $t_\gamma \in R_+$ such that $\rho^{(\alpha)}_{1,1,t_*}(t) \geq \ell!\, t^{\gamma}$ for $t \geq t_\gamma$. Therefore, by (5.4), (5.5) and (5.1) for any $\ell \in \{1,\ldots,n-2\}$ condition (4.2_ℓ) holds. That is for $\alpha > 1$ all conditions of Theorem 5.1′ hold. Therefore according to the same theorem, equation (1.1) has Property **B**.

By Corollary 5.1, Theorem 5.2$'$ can be proved similarly.

Corollary 5.1$'$. *Let $\alpha > 1$ and $\beta < +\infty$, conditions (1.2), (1.3), (2.9$_{1,1}$), (3.1$_1$), (5.1) and (5.4) be fulfilled. Moreover, if for some $\delta \in (1, \alpha]$ and $\gamma > 0$ condition (5.5) holds, then equation (1.1) has Property* **B**, *where α and β are given by (2.1).*

Corollary 5.2. *Let $\alpha > 1$, conditions (1.2), (1.3), (2.9$_{1,c}$), (3.1$_1$), (5.1) and (5.4) be fulfilled and there exist $m \in N$ such that condition (4.7) holds. Then equation (1.1) has Property* **B**, *where α is defined by the first condition of (2.1).*

Proof. By (5.1), (2.9$_{1,c}$), (3.1$_1$) and (5.4) it is obvious that for any $\ell \in \{1, \ldots, n\}$ conditions (2.9$_{\ell,c}$), (3.1$_\ell$) and (4.6$_\ell$) hold.

Let equation (1.1) have a nonoscillatory proper solution $u : (t_0, +\infty) \to (0, +\infty)$. Then by (1.2), (1.3) and Lemma 2.1, there exists $\ell \in \{1, \ldots, n\}$ such that $\ell + n$ is even and the condition (2.3$_\ell$) holds. By Corollary 4.4, $\ell \notin \{1, \ldots, n-2\}$. If $\ell = n$ (if n is even and $\ell = 0$) by (2.9$_{n,c}$) ((2.9$_{1,c}$)) analogously to Theorem 5.1, we show that condition (1.4) (condition (1.5)) holds, that is equation (1.1) has Property **B**.

Corollary 5.2$'$. *Let $\alpha > 1$ and $\beta < +\infty$, conditions (1.2), (1.3), (2.9$_{1,1}$), (3.1$_1$), (5.1) and (5.4) be fulfilled. Moreover, if there exists $m \in N$ such that condition* (4.10) *holds, then equation (1.1) has Property* **B**, *where α and β are given by (2.1).*

Corollary 5.3. *Let $\alpha > 1$, conditions (1.2), (1.3), (2.9$_{1,c}$), (3.1$_1$) and (5.1) be fulfilled. Assume, moreover, that there exist $\gamma \in (0, 1)$ and $r \in (0, 1)$ such that conditions (4.14$_1$) and (4.15) hold and at least one of the conditions (4.16) or $r\,\alpha < 1$ and for some $\varepsilon > 0$ and $\delta \in (1, \alpha)$ (4.17$_1$) are fulfilled. Then equation (1.1) has Property* **B**, *where α is defined by the first condition of (2.1).*

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : (t_0, +\infty) \to (0, +\infty)$. Then by (1.2), (1.3) and Lemma 2.1, there exists $\ell \in \{1, \ldots, n\}$ such that $\ell + n$ is even and condition (2.3$_\ell$) holds. Since by (2.9$_{1,c}$), (3.1$_1$), (4.14$_1$) and (5.1) for any $\ell \in \{1, \ldots, n-2\}$, conditions (2.9$_{\ell,c}$), (3.1$_\ell$) and (4.14$_\ell$) are fulfilled, then according to Corollary 4.6, we have $\ell \notin$ $\notin \{1, \ldots, n-2\}$. On the other hand analogously to Theorem 5.1, we show that if $\ell = 0$ ($\ell = n$) the condition (1.4) ((1.5)) is fulfilled, that is the equation (1.1) has Property **B**.

Corollary 5.3$'$. *Let $\alpha > 1$ and $\beta < +\infty$, conditions (1.2), (1.3), (2.9$_{1,1}$), (3.1$_1$) and (5.1) be fulfilled and for some $\gamma \in (0, 1)$ and $r \in (0, 1)$, conditions (4.14$_1$) and (4.15) hold. Then equation (1.1) has Property* **B**, *where α and β are given by (2.1).*

Theorem 5.3. *Let $\alpha > 1$, conditions (1.2), (1.3), (2.9$_{n,c}$) and (3.1$_{n-1}$) be fulfilled and*

$$\limsup_{t \to +\infty} \frac{(\tau(t))^{\mu(t)}}{t} < +\infty. \tag{5.6}$$

Moreover, if for some $\delta \in (1, \alpha]$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{1-\delta} (\tau(\xi))^{\delta + (n-3)\mu(\xi)} |p(\xi)| d\xi\, ds = +\infty, \tag{5.7}$$

then equation (1.1) has Property **B**, *where α is given by the first condition of (2.1).*

Proof. According to (2.9$_{n,c}$), (3.1$_{n-1}$) and (5.6) it is obvious that for any $\ell \in$ $\in \{1, \ldots, n-1\}$ the conditions (2.9$_{\ell,c}$) and (3.1$_\ell$) hold. On the other hand by (5.6) and (5.7) for any $\ell \in \{1, \ldots, n-2\}$, with $\ell + n$ even the condition (4.2$_\ell$) holds. That is, if $\alpha > 1$, then all conditions of Theorem 5.1 hold, which proves the validity of the theorem.

Theorem 5.3′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions (1.2), (1.3), (2.9$_{n,1}$), (3.1$_{n-1}$), (5.6) and for some $\delta \in (1, \alpha)$ condition (5.7) be fulfilled. Then equation (1.1) has Property* **B***, where α and β are given by (2.1).*

Proof. Since $\beta < +\infty$, by (2.9$_{n,1}$) it is obvious that for any $c \in (0, 1]$ conditions (2.9$_{n,c}$) hold. That is all conditions of Theorem 5.3 are fulfilled, which proves the validity of the theorem.

Corollary 5.4. *Let $\alpha > 1$, conditions (1.2), (1.3), (2.9$_{n,c}$), (3.1$_{n-1}$) and (5.6) be fulfilled and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} (\tau(s))^{1+(n-3)\mu(s)} |p(s)| ds > 0. \tag{5.8}$$

Moreover, if for some $\delta \in (1, \alpha]$ and $\gamma > 0$

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{-1-\delta} (\tau(\xi))^{\delta+(n-3)\mu(\xi)+\gamma(\mu(\xi)-\delta)} |p(\xi)| d\xi\, ds = +\infty, \tag{5.9}$$

then equation (1.1) has Property **B***, where α is given by the first condition of (2.1).*

Proof. Since $\alpha > 1$, by (5.8), (2.11$_{n-2}$) and (2.13$_{n-2}$), for any $\gamma > 0$ there exists $t_* \in R_+$ such that $\rho^{(\alpha)}_{1,n-2,t_*}(t) \geq \ell!\, t^{\gamma}$ for $t \geq t_{\gamma}$. Therefore by (5.6), (5.8) and (5.9) for any $\ell \in \{1, \ldots, n-2\}$ the conditions (4.2$_\ell$) hold. Therefore, according to the same theorem, equation (1.1) has Property **B**.

Corollary 5.4′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions (1.2), (1.3), (2.9$_{n,1}$), (3.1$_{n-1}$), (5.8) and (5.9) be fulfilled. Then equation (1.1) has Property* **B***, where α and β are given by (2.1).*

By (5.6), repeating the arguments given in Corollary 5.3, we easily ascertain that the corollary below is true.

Corollary 5.5. *Let $\alpha > 1$, conditions (1.2), (1.3), (2.9$_{n,c}$), (3.1$_{n-1}$) and (5.6) be fulfilled. Moreover, let there exist $\gamma \in (0, 1)$ and $r \in (0, 1)$ such that conditions (4.14$_{n-2}$), (4.15) and at last one of the conditions (4.16) or $r\,\alpha < 1$ and for some $\varepsilon > 0$ and $\delta \in (1, \alpha]$, (4.17$_{n-2}$) be fulfilled. Then equation (1.1) has Property* **B***, where α is defined by the first condition of (2.1).*

Corollary 5.5′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions (1.2), (1.3), (2.9$_{n,1}$), (3.1$_{n-1}$) and (5.6) be fulfilled. Moreover, let there exist $\gamma \in (0, 1)$ and $r \in (0, 1)$ such that conditions (4.14$_{n-2}$), (4.15) and at last one of conditions (4.16) or $r\,\alpha < 1$ and for some $\varepsilon > 0$ and $\delta \in (1, \alpha]$, (4.17$_{n-2}$) be fulfilled. Then equation (1.1) has Property* **B***, where α and β are given by (2.1).*

Theorem 5.4. *Let $\alpha = 1$, conditions (1.2), (1.3), (2.9$_{1,c}$), (3.1$_1$) and (5.1) be fulfilled. Moreover, if*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-2} |p(\xi)| d\xi\, ds > 0, \tag{5.10}$$

when n is odd and

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-3} (\tau(\xi))^{\mu(\xi)} |p(\xi)| d\xi\, ds > 0, \tag{5.11}$$

when n is even, then equation (1.1) has Property **B**, *where α is given by the first condition of (2.1).*

Proof. According to $(2.9_{1,c})$, (3.1_1) and (5.1) for any $\ell \in \{1,\ldots,n\}$ conditions $(2.9_{\ell,c})$ and (3.1_ℓ) holds. On the other hand by (5.1), (5.10) and (5.11) for any $\ell \in \{1,\ldots,n-2\}$, with $\ell+n$ even condition (4.1_ℓ) holds. That is, if $\alpha = 1$, then all conditions of Theorem 5.1 hold, which proves the validity of the theorem.

Theorem 5.4′. *Let $\alpha = 1$ and $\beta < +\infty$, conditions (1.2), (1.3), $(2.9_{1,1})$, (3.1_1), (5.1), (5.10) and (5.11) be fulfilled. Then equation (1.1) has Property* **B**, *where α and β are defined by (2.1).*

Proof. Since $\beta < +\infty$, by $(2.9_{1,1})$, for any $c \in (0,1]$ condition $(2.9_{1,c})$ holds. That is all conditions of Theorem 5.4 are fulfilled, which proves the validity of the theorem.

Theorem 5.5. *Let $\alpha = 1$, conditions (1.2), (1.3), $(2.9_{1,c})$, (3.1_1) and (5.1) be fulfilled. Moreover, if*

$$\liminf_{t\to+\infty} t \int_0^t \int_s^{+\infty} \xi^{n-3}\tau(\xi)|p(\xi)|d\xi\, ds > \max\left(\frac{\ell\,!(n-\ell)!}{\omega^{\ell-1}}, \ell \in \{1,2,\ldots,n-2\}\right) \tag{5.12}$$

and

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-2}(\tau(\xi))^{\mu(\xi)}|p(\xi)|d\xi\, ds > 0, \tag{5.13}$$

then equation (1.1) has Property **B**, *where*

$$\omega = \liminf_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t}. \tag{5.14}$$

Proof. By (5.12), (5.14) and (2.11_ℓ) it is obvious that for large t we have

$$\rho^{(1)}_{1,\ell,t_*}(t) \geq \ell\,!\,t, \quad \ell \in \{1,\ldots,n-2\}. \tag{5.15}$$

On the other hand according to $(2.9_{1,c})$, (3.1_1), (5.1), (5.14), (5.15) and (5.13) for any $\ell \in$ $\in \{1,\ldots,n-1\}$, conditions $(2.9_{\ell,c})$, (3.1_ℓ) and (4.1_ℓ) hold. That is if $\alpha = 1$, then all conditions of Theorem 5.1 hold, which proves the validity of the theorem.

The proof of Theorem 5.4 has been a guide for us in proving Theorem 5.4′. In the same way we will be guided by the proof of Theorem 5.5 to show that the next theorem is valid.

Theorem 5.5′. *Let $\alpha = 1$ and $\beta < +\infty$, conditions (1.2),(1.3), $(2.9_{1,1})$, (3.1_1), (5.1), (5.12) and (5.13) be fulfilled. Then equation (1.1) has Property* **B**, *where ω is defined by condition (5.14).*

Theorem 5.6. *Let $\alpha = 1$, conditions (1.2), (1.3), $(2.9_{n,c})$, (3.1_{n-1}) and (5.6) be fulfilled. Moreover, if*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi(\tau(\xi))^{(n-3)r(\xi)}|p(\xi)|d\xi\, ds > 0, \tag{5.16}$$

then equation (1.1) has Property **B**, *where α is given by the first condition of (2.1).*

Proof. According to ($2.9_{n,c}$), (3.1_{n-1}) and (5.6) for any $\ell \in \{1, \ldots, n-1\}$ conditions ($2.9_{\ell,c}$) and (3.1_{ℓ}) hold. On the other hand by (5.6) and (5.16) for any $\ell \in \{1, \ldots, n-2\}$, with $\ell + n$ even condition (4.1_{ℓ}) holds. That is, if $\alpha = 1$, then all conditions of Theorem 5.1 hold, which proves the validity of the theorem.

Theorem 5.6′. *Let $\alpha = 1$ and $\beta < +\infty$, conditions (1.2), (1.3), ($2.9_{n,1}$), (3.1_{n-1}), (5.6) and (5.16) be fulfilled. Then equation (1.1) has Property* **B***, where α and β are given by (2.1).*

Proof. Since $\beta < +\infty$, it suffices to show that by ($2.9_{n,1}$), for any $c \in (0, 1]$ condition ($2.9_{n,c}$) hold.

Theorem 5.7. *Let $\alpha = 1$, conditions (1.2), (1.3), ($2.9_{n,c}$), (3.1_{n-1}) and (5.6) be fulfilled. Moreover, if*

$$\liminf_{t \to +\infty} t \int_0^t \int_s^{+\infty} (\tau(\xi)^{1+(n-3)\mu(\xi)} |p(\xi)| d\xi \, ds > \max\left(\frac{\ell\,!(n-\ell)!}{\omega^{n-\ell-2}}, \ell \in \{1, 2, \ldots, n-2\} \right), \quad (5.17)$$

then the condition

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi(\tau(\xi))^{(n-2)\mu(\xi)} |p(\xi)| d\xi \, ds > 0, \quad (5.18)$$

is sufficient for equation (1.1) to have Property **B***, where*

$$\omega = \liminf_{t \to +\infty} \frac{t}{(\tau(t))^{\mu(t)}} \,. \quad (5.19)$$

Proof. By (5.17), (5.19) and (2.11) it is obvious that for large t condition (5.15) holds.

On the other hand according to ($2.9_{n,c}$), (3.1_{n-1}), (5.6), (5.15), (5.18) and (5.19) for any $\ell \in \{1, \ldots, n-1\}$, conditions ($2.9_{\ell,c}$), ($3.1_{\ell}$) and ($4.1_{\ell}$) hold. That is, if $\alpha = 1$, then all conditions of Theorem 5.1 hold, which proves the validity of the theorem.

Theorem 5.7′. *Let $\alpha = 1$ and $\beta < +\infty$, conditions (1.2), (1.3), ($2.9_{n,1}$), (3.1_{n-1}) and (5.6) be fulfilled. Moreover, if conditions (5.17) and (5.18) hold, then equation (1.1) has Property* **B***, where α, β and ω are given by (2.1) and (5.19).*

Proof. Since $\beta < +\infty$, it suffices to show that by ($2.9_{n,1}$), for any $c \in (0, 1]$ conditions ($2.9_{n,c}$) hold.

1. *Graef I., Koplatadze R., Kvinikadze G.* Nonlinear functional differential equations with Properties A and B // J. Math. Anal. and Appl. — 2005. — **306**, № 1. — P. 136–160.
2. *Koplatadze R.* On oscillatory properties of solutions of generalized Emden–Fowler type differential equations // Proc. A. Razmadze Math. Inst. — 2007. — **145**. — P. 117–121.
3. *Koplatadze R.* Quasi-linear functional differential equations with property A // J. Math. Anal. and Appl. — 2007. — **330**, № 1. — P. 483–510.
4. *Koplatadze R.* On asymptotic behavior of solutions of "almost linear" and essentially nonlinear differential equations // Nonlinear Anal.: Theory, Methods and Appl. — 2009. — **71**, № 12. — P. 396–400.
5. *Koplatadze R., Litsyn E.* Oscillation criteria for higher order "almost linear" functional differential equation // Funct. Different. Equat. — 2009. — **16**, № 3. — P. 387–434.
6. *Koplatadze R.* On higher order functional differential equations with Property A // Georg. Math. J. — 2004. — **11**, № 2. — P. 307–336.

7. *Koplatadze R., Chanturia T.* On oscillatory properties of differential equations with a deviating argument (in Russian). — Tbilisi: Tbilis. State Univ. Press, 1977.

8. *Ladd G. S., Lakshmikantham V., Zhang B. G.* Oscillation theory of differential equations with deviating arguments. — New York: Dekker, 1987.

9. *Koplatadze R.* On oscillatory properties of solutions of functional differential equations // Mem. Different. Equat. Math. Phys. — 1994. — **3**. — P. 3 – 179.

10. *Erbe Y. H., Kong Q., Zhang B. G.* Oscillation theory for functional differential equations. — New York: Dekker, 1995.

11. *Agarval R. P., Grace S. R., O'Regan D.* Oscillation theory for second order linear, half-linear, singular and sublinear dynamik equations. — Dordrecht: Kluwer Acad. Publ., 2002.

12. *Kiguradze I., Stavroulakis I.* On the solutions of higher order Emden – Fowler advanced differential equations // Appl. Anal. — 1998. — **70**. — P. 97 – 112.

13. *Koplatadze R.* On asymptotic behavior of solutions of n-th order Emden – Fowler differential equations with advanced argument // Czechoslovak Math. J. — 2010. — **60(135)**, № 3. — P. 817 – 833.

14. *Domoshnitski A., Koplatadze R.* On asymptotic behavior of solutions of generalized Enmden – Fowler differential equations with delay argument // Abstrs and Appl. Anal. — 2014. —Art. ID 168425.

15. *Gramatikopoulos M. K., Koplatadze R., Kvinikadze G.* Linear functional differential equations with Property A // J. Math. Anal. and Appl. — 2003. — **284**, № 1. — P. 294 – 314.

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