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BARGMANN TYPE FINITE-DIMENSIONAL REDUCTIONS OF THE LAX INTEGRABLE SUPERSYMMETRIC BOUSSINESQ HIERARCHY AND THEIR INTEGRABILITY

СКІНЧЕННОВИМІРНІ РЕДУКЦІЇ ТИПУ БАРГМАНА ІНТЕГРОВНОЇ ЗА ЛАКСОМ СУПЕРСИМЕТРИЧНОЇ ІЄРАРХІЇ БУССІНЕСКА ТА ЇХ ІНТЕГРОВНІСТЬ

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For the supersymmetric Boussinesq hierarchy, related with the Lax type flows on the space dual to the Lie algebra of superintegro-differential operators of one anticommuting variable for some non-self-adjoint superdifferential operator, the method of the Bargmann type finite-dimensional reductions is developed. We prove existence of an even exact supersymplectic structure on the corresponding invariant finite-dimensional supersubspace of the supersymmetric Boussinesq hierarchy as well as the Lax – Liouville integrability of commuting vector fields, generated by the hierarchy and reduced to this supersubspace.

Для суперсиметричної ієрархії Буссінеска, пов'язаної з потоками типу Лакса на спряженому просторі до алгебри Лі суперінтегро-диференціальних операторів однієї антикомутативної змінної для несамоспряженого супердиференціального оператора, розвинено метод скінченновимірних редукцій типу Баргмана. Доведено існування парної точної суперсимплектичної структури на відповідному інваріантному скінченновимірному суперпідпросторі суперсиметричної ієрархії Буссінеска та інтегровність за Лаксом – Ліувіллем редукованих на цей суперпідпростір комутуючих векторних полів, породжених ієрархією.

1. Introduction. In the framework of the different Lie-algebraic approaches, a wide class of supersymmetric nonlinear dynamical systems, possessing matrix Lax type representations [1 – 4] and infinite sequences of local conservation laws, has been constructed in [5 – 10] and many others. For such nonlinear dynamical systems, defined on suitable functional manifolds, a method of reducing the system to the invariant subspaces, generated by critical points of the related conservation laws has been developed in [4, 11 – 14]. In particular, in [4, 12, 13] it has been shown that the exact symplectic structure on the corresponding invariant space can be obtained by means of the Gelfand – Dikii relationship [15, 16] for the differential of the Lagrangian functional on a suitably extended phase space [11], and the corresponding Hamiltonian functions of the reduced vector fields generated by the systems have been constructed.

In [13, 17, 18] the reduction method has been further developed for superconformal nonlinear dynamical systems as well as for supersymmetric ones defined on supermanifolds of one commuting and one anticommuting independent variables. In particular, in the paper [18] this method has been used for investigating Neumann type invariant reductions [19] of the Laberge – Mathieu supersymmetric hierarchy, related to the Lax type flows on the space dual to the Lie algebra of superintegro-differential operators of two anticommuting variables for some self-adjoint superdifferential operator.

In this article, the reduction method is applied to the supersymmetric Boussinesq hierarchy [6] associated with a non-self-adjoint superdifferential operator depending on one anticommuting variable.

The second section contains a preliminary description of Lie-algebraic and differential-geometric properties, being important for a better understanding of the used techniques and the obtained results.

In the third section, we establish existence of an even exact supersymplectic structure on the invariant supersubspace determined by the Bargmann type constraints [14] by means of the superanalog of the Gelfand – Dikii relationship [18, 20] and the Hamiltonian functions for the reduced commuting vector fields, generated by the hierarchy.

In the fourth section making use of the differential-geometric properties of the supertrace gradient for the monodromy supermatrix of the related periodic matrix linear spectral problem, we obtain the Lax representations for these reduced vector fields. The algorithm for reducing of monodromy supermatrix upon the invariant supersubspace is described. A complete set of functionally independent conservation laws, being involutive with respect to the Poisson bracket related with the obtained even supersymplectic structure, is also found. It ensures the complete Liouville integrability [21] of the reduced vector fields.

2. The Lax integrability of the supersymmetric Boussinesq hierarchy. The supersymmetric Boussinesq hierarchy [6] can be represented in the form of the Lax type flows

$$\frac{dl}{dt_j} = \left[(l^{(3j+1)/3})_+, l\right], \qquad \frac{dl}{d\tilde{t}_j} = \left[(l^{(3j+2)/3})_+, l\right], \tag{1}$$

where $l = \partial^3 + \phi D_\theta \partial + a\partial - \chi D_\theta - b \in \mathcal{G}^*$, $w = (a, b, \phi, \chi)^\top \in M^{2|2} \subset C^\infty(\mathbb{S}^{1|1}; \mathbb{R}^{2|2})$, $(x, \theta) \in \mathbb{S}^{1|1}$, $\mathbb{S}^{1|1} \simeq \mathbb{S} \times \Lambda_1$, $\mathbb{S} \simeq \mathbb{R}/2\pi\mathbb{Z}$, Λ_1 is a subalgebra of anticommuting elements of the Grassmann algebra $\Lambda := \Lambda_0 \oplus \Lambda_1$ over the field $\mathbb{R} \subset \Lambda_0$, $\mathbb{R}^{2|2} := \Lambda_0^2 \times \Lambda_1^2$, $\partial := \dfrac{\partial}{\partial x}$, $D_\theta := \dfrac{\partial}{\partial \theta} + \theta \dfrac{\partial}{\partial x}$ is a superderivative and $t_j, \tilde{t}_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$ are evolution parameters. Here the lower index "+" denotes the pure differential part of a superintegro-differential operator from the space $\mathcal{G}^* \simeq \mathcal{G}$ being the dual space to the Lie algebra $\mathcal{G}$ of superintegro-differential operators

$$\mathcal{A} := \partial^q + \sum_{p<2q-1} a_p D_\theta^p \in \mathcal{G}, \quad p \in \mathbb{Z}, \quad q \in \mathbb{N},$$

with coefficients $a_p := a_p(x, \theta) = a_p^0(x) + \theta a_p^1(x)$, $a_p \in C^\infty(\mathbb{S}^{1|1}; \Lambda_0)$ if $p = 2r$ and $a_p := a_p(x, \theta) = a_p^1(x) + \theta a_p^0(x)$, $a_p \in C^\infty(\mathbb{S}^{1|1}; \Lambda_1)$ if $p = 2r - 1$, $r < q$, $q \in \mathbb{N}$, subject to the scalar product

$$(\mathcal{A}, \mathcal{B}) := \int_0^{2\pi} dx \int d\theta \operatorname{res} \mathcal{A}\mathcal{B}, \quad \mathcal{A}, \mathcal{B} \in \mathcal{G},$$

where the symbol "res" designates the coefficient at the operator D_θ^{-1}. The evolution with respect to the parameter $\tilde{t}_1$ is given by the supersymmetric nonlinear dynamical system [6] such as

$$\frac{da}{d\tilde{t}_1} = \frac{1}{3}\,(4\phi\chi + 2\phi\phi_x - 6b_x - 3a_{xx}),$$

$$\frac{db}{d\tilde{t}_1} = \frac{1}{3}\,(2(D_\theta a)\chi + 2\phi(D_\theta b) + 2aa_x - 2(D_\theta a_x)\phi + 3b_{xx} + 2a_{xxx}),$$

$$\frac{d\phi}{d\tilde{t}_1} = -2\chi_x - \phi_{xx},$$

$$\frac{d\chi}{d\tilde{t}_1} = \frac{1}{3}\,(2\phi(D_\theta\chi) - 2(D_\theta\phi)\chi + 2(a\phi)_x + 2\phi(D_\theta\phi_x) + 3\chi_{xx} + 2\phi_{xxx}),$$

which entails the Boussinesq system [22] at $a = b = 0$, $\phi = \theta u$, $\chi = \theta v$ and $(u, v)^\top \in C^\infty(\mathbb{S}; \mathbb{R}^2)$.

The evolution equations (1) can be considered as a compatibility condition for the spectral relationship

$$ly = \lambda y, \tag{2}$$

where $\lambda \in \Lambda_0 \supset \mathbb{C}$ is a spectral parameter, being invariant with respect to the evolution flows (1), $y \in L_2(\mathbb{S}^{1|1}; \mathbb{C}^{1|0})$, and the evolution equations

$$\frac{dy}{dt_j} = (l^{(3j+1)/3})_+ y,$$

and

$$\frac{dy}{d\tilde{t}_j} = (l^{(3j+2)/3})_+ y.$$

The corresponding adjoint spectral relationship and the adjoint evolutions take the form

$$\begin{aligned} l^* z &= \lambda z, \\ \frac{dz}{dt_j} &= -(l^{(3j+1)/3})_+^* z, \\ \frac{dz}{d\tilde{t}_j} &= -(l^{(3j+2)/3})_+^* z, \end{aligned} \tag{3}$$

where $l^* = -\partial^3 - D_\theta \partial \phi - \partial a - D_\theta \chi - b$ is the operator adjoint to l and defined by means of the integral relationship

$$\int_0^{2\pi} dx \int d\theta z(ly) = \int_0^{2\pi} dx \int d\theta (l^* z) y,$$

for all $y \in L_2(\mathbb{S}^{1|1}; \mathbb{C}^{1|0})$ and $z \in L_2(\mathbb{S}^{1|1}; \mathbb{C}^{0|1})$.

The mentioned above spectral problems can be suitably rewritten in the equivalent matrix forms [6],

$$D_\theta Y = AY, \tag{4}$$

$$D_\theta Z = -A^{\top_s} Z, \tag{5}$$

where $A \in C^\infty(\mathbb{S}^{1|1}; \mathrm{gl}\,(3|3))$, $A := A[w;\lambda]$, $Y := Y(x,\theta;\lambda) \in \mathcal{W} := L_2(\mathbb{S}^{1|1}; \mathbb{C}^{3|3})$, $Z := Z(x,\theta;\lambda) \in \mathcal{W}$, $Y = (y_0, y_2, y_4, y_1, y_3, y_5)^\top$, $y_0 := y$, $Z = (z_0, z_2, z_4, z_1, z_3, z_5)^\top$, $z_5 := z$ and

$$A := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ b+\lambda & a & 0 & \chi & \phi & 0 \end{pmatrix}.$$

Here the upper index "$\top_s$" denotes the supermatrix supertransposition acting by the rule

$$W^{\top_s} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}^{\top_s} = \begin{pmatrix} W_{11}^\top & W_{21}^\top \\ -W_{12}^\top & W_{22}^\top \end{pmatrix}$$

for any supermatrix $W \in \mathrm{gl}(m|n)$.

The associated evolutions are written as

$$\frac{dY}{dt_j} = B_j Y = (\lambda^j S)_+ Y, \tag{6}$$

$$\frac{dY}{d\tilde{t}_j} = \tilde{B}_j Y = (\lambda^j S^2)_+ Y, \tag{7}$$

and

$$\frac{dZ}{dt_j} = -I B_j^{\top_s} I Z, \tag{8}$$

$$\frac{dZ}{d\tilde{t}_j} = -I \tilde{B}_j^{\top_s} I Z, \tag{9}$$

where $I := \mathrm{diag}\,(1,1,1,-1,-1,-1)$, $B_j := B_j[w;\lambda]Y = (\lambda^j S)_+ Y$, $\tilde{B}_j := \tilde{B}_j[w;\lambda]Y = (\lambda^j S^2)_+ Y$, $S \simeq \sum_{j \in \mathbb{Z}_+} \check{S}_{j-1} \lambda^{-j+1}$ is an asymptotical expansion of the monodromy supermatrix $S(x,\theta;\lambda) := Y(x, x+2\pi, \theta; \lambda)$ for the periodic matrix spectral problem (4) as $|\lambda| \to \infty$, $Y(\check{x}, x, \theta; \lambda)$ is a fundamental solution of the linear equation (4), that is, $Y(x,x,\theta;\lambda) = \mathbf{1}$, $\check{x} \in \mathbb{S}$, $\mathbf{1} \in \mathrm{gl}\,(6)$ is the unit (6×6)-matrix (see [18]),

$$\check{S}_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

the lower index "+" denotes the polynomial part of the corresponding Laurent series. The hierarchy (1) possesses two sequences of the Casimir type conservation laws

$$\begin{aligned} \gamma_j &:= \int_0^{2\pi} dx \int d\theta \gamma_j[w] = \frac{3}{3j+1} \int_0^{2\pi} dx \int d\theta \operatorname{res} l^{(3j+1)/3}, \\ \tilde{\gamma}_j &:= \int_0^{2\pi} dx \int d\theta\, \tilde{\gamma}_j[w] = \frac{3}{3j+2} \int_0^{2\pi} dx \int d\theta \operatorname{res} l^{(3j+2)/3}, \end{aligned} \tag{10}$$

where some four of them are

$$\gamma_0 = -\int_0^{2\pi} dx \int d\theta \phi, \quad \tilde{\gamma}_0 = \int_0^{2\pi} dx \int d\theta \chi,$$

$$\gamma_1 = \frac{1}{3} \int_0^{2\pi} dx \int d\theta (a\chi + b\phi + \phi(D_\theta \chi)),$$

$$\begin{aligned} \tilde{\gamma}_1 = \frac{1}{27} \int_0^{2\pi} dx \int d\theta (&-18b\chi + 9b_x\phi + 3a^2\phi + 9a\chi_x + 6a\phi_{xx} + \\ &+ 3a\phi(D_\theta\phi) + \phi(D_\theta\phi)^2 + 9\phi(D_\theta\chi_x) - 9\chi(D_\theta\chi) + 3\phi(D_\theta\phi_{xx}), \quad \text{etc.} \end{aligned}$$

These conservation laws are connected to each other by means of the Magri [23] relationships,

$$\mathcal{M}\, \varphi(x, \theta; \bar{\lambda}) = \bar{\lambda}\, \mathcal{L}\, \varphi(x, \theta; \bar{\lambda}), \tag{11}$$

$$\mathcal{M}\, \tilde{\varphi}(x, \theta; \bar{\lambda}) = \bar{\lambda}\, \mathcal{L}\, \tilde{\varphi}(x, \theta; \bar{\lambda}), \tag{12}$$

where $\varphi(x, \theta; \bar{\lambda}) = \operatorname{grad} \operatorname{str} S(x, \theta; \lambda)$, $\tilde{\varphi}(x, \theta; \bar{\lambda}) = \operatorname{grad} \operatorname{str} S^2(x, \theta; \lambda)$,

$$\varphi(x, \theta; \bar{\lambda}) \simeq \sum_{j \in \mathbb{Z}_+} \varphi_j \bar{\lambda}^{-j}, \quad \varphi_j = \operatorname{grad} \gamma_j[w],$$

$$\tilde{\varphi}(x, \theta; \bar{\lambda}) \simeq \sum_{j \in \mathbb{Z}_+} \tilde{\varphi}_j \bar{\lambda}^{-j}, \quad \tilde{\varphi}_j = \operatorname{grad} \tilde{\gamma}_j[w],$$

and $\mathcal{L} : T(M^{2|2}) \to T^*(M^{2|2})$ and $\mathcal{M} : T(M^{2|2}) \to T^*(M^{2|2})$, are a pair of compatible linear Poisson operators [23], constructed before in [6]. Here the symbol "grad" denotes, as usually, the left gradient of the corresponding functional. The operators $\mathcal{L}$ and $\mathcal{M}$ generate a bi-Hamiltonian representation for the hierarchy (1) in the form

$$\begin{aligned} \frac{dw}{dt_j} &= -\mathcal{L} \operatorname{grad} \gamma_{j+1} = -\mathcal{M} \operatorname{grad} \gamma_j, \\ \frac{dw}{d\tilde{t}_j} &= -\mathcal{L} \operatorname{grad} \tilde{\gamma}_{j+1} = -\mathcal{M} \operatorname{grad} \tilde{\gamma}_j, \end{aligned} \tag{13}$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & 2\phi + 3D_\theta\partial & 0 & -3\partial \\ 2\phi + 3D_\theta\partial & 2\chi + (D_\theta a) + \phi_x & -3\partial & \phi D_\theta \\ 0 & -3\partial & 0 & 0 \\ -3\partial & -(D_\theta\phi) + \phi D_\theta & 0 & 0 \end{pmatrix}$$

and $\mathcal{M}$ has a more cumbersome form (see [6]). The vector fields $\dfrac{d}{dt_{j_1}}$ and $\dfrac{d}{dt_{j_2}}$, $j_1, j_2 \in \mathbb{Z}_+$, commute with each other, i.e.,

$$\left[\frac{d}{dt_{j_1}}, \frac{d}{dt_{j_2}}\right] = 0, \quad \left[\frac{d}{dt_{j_1}}, \frac{d}{d\tilde{t}_{j_2}}\right] = 0, \quad \left[\frac{d}{d\tilde{t}_{j_1}}, \frac{d}{d\tilde{t}_{j_2}}\right] = 0. \tag{14}$$

The existence of conservation laws (10) and matrix Lax type linearizations (4), (6), (7) and (5), (8), (9) allow us to reduce the hierarchy (1) to its invariant supersubspaces,

$$M_N^{2|2} = \{w \in M^{2|2} : \operatorname{grad} L_N[w] = 0\},$$

generated by the Lagrangian functionals

$$L_N := \int_0^{2\pi} dx \int d\theta L_N[w] = \sum_{k_1=0}^{P} a_{k_1}\gamma_{k_1} + \sum_{k_2=0}^{Q} b_{k_2}\tilde{\gamma}_{k_2} + \sum_{i=1}^{N} c_i\lambda_i,$$

where $a_{k_1}, b_{k_2}, c_i \in \Lambda_0 \supset \mathbb{C}$ are some coefficients and $\lambda_i \in \Lambda_0 \supset \mathbb{C}$, $i = \overline{1, N}$, are different eigenvalues of the periodic spectral problem (4) for arbitrarily chosen orders $P, Q, N \in \mathbb{Z}_+$.

3. The supersymplectic structure on some invariant supersubspace. Below we shall study the reductions of the vector fields $\dfrac{d}{dt_j}$ and $\dfrac{d}{d\tilde{t}_j}$, $j \in \mathbb{Z}_+$, on the invariant supersubspace determined by the Bargmann type constraints [14]

$$\begin{gathered} M_N^{2|2} = \left\{w \in M^{2|2} : \operatorname{grad} L_N[w] = 0\right\}, \\ L_N := \int_0^{2\pi} dx \int d\theta L_N[w] = -3\gamma_1 + \sum_{i=1}^{N} c_i\lambda_i, \end{gathered} \tag{15}$$

where $\lambda_i \in \Lambda_0 \supset \mathbb{C}$, are some different eigenvalues of the periodic spectral problem (2), being considered as smooth by Frechet functionals on $M^{2|2}$, i.e., $\lambda_i \in \mathcal{D}(M^{2|2})$, with the corresponding eigenvectors $Y_i = (y_{0i}, y_{2i}, y_{4i}, y_{1i}, y_{3i}, y_{5i})^\top \in \mathcal{W}$ and adjoint eigenvectors $Z_i = (z_{0i}, z_{2i}, z_{4i}, z_{1i}, z_{3i}, z_{5i})^\top \in \mathcal{W}$, $c_i \in \Lambda_0 \supset \mathbb{C}$, $i = \overline{1, N}$.

First, we shall analyze the differential-geometric structure of the invariant supersubspace $M_N^{2|2} \subset M^{2|2}$. To describe this supersubspace, evidently we shall construct the gradients of the eigenvalues $\lambda_i \in \mathcal{D}(M^{2|2})$, $i = \overline{1, N}$, making use the relationships

$$\int_0^{2\pi} dx \int d\theta \langle D_\theta Y_i, \bar{Z}_i\rangle = \int_0^{2\pi} dx \int d\theta \langle A[w, \lambda_i] Y_i, \bar{Z}_i\rangle, \quad i = \overline{1, N}, \tag{16}$$

where the brackets $\langle\,,\,\rangle$ denotes the standard scalar product on $\mathbb{C}^{6N|6N}$ and $\bar{Z}_i = (\bar{z}_{0i}, \bar{z}_{2i}, \bar{z}_{4i}, \bar{z}_{1i}, \bar{z}_{3i}, \bar{z}_{5i})^\top$ is complex conjugate to the vector Z_i, which follows from the spectral problem (4). These gradients are written as

$$\operatorname{grad}\lambda_i = -\frac{1}{\mu_i}\left(\bar{y}_{2i}\bar{z}_{5i}, \bar{y}_{0i}\bar{z}_{5i}, \bar{y}_{3i}\bar{z}_{5i}, \bar{y}_{1i}\bar{z}_{5i}\right)^\top,$$

where $\bar{Y}_i = (\bar{y}_{0i}, \bar{y}_{2i}, \bar{y}_{4i}, \bar{y}_{1i}, \bar{y}_{3i}, \bar{y}_{5i})^\top$ is complex conjugate to the vector Y_i,

$$\mu_i := \int_0^{2\pi}\int d\theta\, y_{0i}z_{5i}, \quad i = \overline{1,N},$$

are normalizing multipliers, being invariant with respect to the vector fields $\dfrac{d}{dt_j}$ and $\dfrac{d}{d\tilde{t}_j}$ for all $j \in \mathbb{Z}_+$.

In case of $\mu_i = -c_i$, $i = \overline{1,N}$, the condition (15) takes the form of the Bargmann type constraints,

$$M_N^{2|2}\bigcap H_c = \Bigg\{ w \in M^{2|2} : a = -\sum_{i=1}^{N} y_{0i}z_{4i}, b = -\sum_{i=1}^{N} y_{2i}z_{4i},$$
$$\phi = \sum_{i=1}^{N} y_{0i}z_{5i}, \chi = \sum_{i=1}^{N} y_{2i}z_{5i} \Bigg\}, \tag{17}$$

where $H_c := \{(w, \mathcal{Y}, \mathcal{Z})^\top \in \hat{M}^{2|2} : \mu_i = -c_i, c_i \in \Lambda_0, i = \overline{1,N}\}$ are common level surfaces of the invariant functionals μ_i, $i = \overline{1,N}$, in the phase space $\hat{M}^{2|2} := M^{2|2} \times W^{2N}$ of the hierarchies of coupled dynamical systems (13), (6), (7) and (8), (9) with the parameters λ_i, $i = \overline{1,N}$, and $\mathcal{Y} := (Y_1, Y_2, \ldots, Y_N)^\top$, $\mathcal{Z} := (Z_1, Z_2, \ldots, Z_N)^\top$.

From the relationships (17) it follows that the solutions to the supersymmetric Boussinesq hierarchy on the invariant supersubspace (17) are expressed by means of the coordinates of the eigenvectors Y_i and Z_i, $i = \overline{1,N}$.

The exact supersymplectic structure on the invariant supersubspace $M_N^{2|2} \subset M^{2|2}$ can be obtained by means of the analog [18, 20] of the Gelfand – Dikii relationship on the functional supermanifold $M^{2|2}$ similarly as it was done in the paper [16] for subspaces of critical points of local conservation laws. To make use this relationship we need the evident forms of the Frechet smooth functionals λ_i, $i = \overline{1,N}$, on H_c. From the equalities (16) we have

$$\lambda_i' = \int_0^{2\pi} dx \int d\theta \Bigg(-\sum_{s=0}^{5} (D_\theta y_{si}) z_{si} + y_{1i}z_{0i} + y_{3i}z_{2i} + y_{5i}z_{4i} -$$
$$- y_{2i}z_{1i} - y_{4i}z_{3i} + b y_{0i}z_{5i} + a y_{2i}z_{5i} + \chi y_{1i}z_{5i} + \phi y_{3i}z_{5i} \Bigg), \tag{18}$$

where $\lambda_i' := \lambda_i|_{\hat{M}^{2|2}\bigcap H_c}$, $i = \overline{1,N}$, on the level surfaces H_c, $c := (c_1, c_2, \ldots, c_N)^\top \in \Lambda_0^N$, in the phase space $\hat{M}^{2|2}$.

Therefore, taking into account the evident dependence (18) of $\lambda_i' \in \mathcal{D}(M^{2|2})$, $i = \overline{1,N}$, on the functions $(w, \mathcal{Y}, \mathcal{Z})^\top \in \hat{M}^{2|2}$ on the level surfaces H_c, $c \in \Lambda_0^N$, we can apply an analog of the Gelfand–Dikii relationship to the Lagrangian functional $\hat{L}_N := \int_0^{2\pi} dx \int d\theta \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \in \in \mathcal{D}(\hat{M}^{2|2})$ such as

$$\hat{L}_N = -3\gamma_1 + \sum_{i=1}^{N} \lambda_i' + \sum_{i=1}^{N} s_i \mu_i,$$

where $s_i \in \Lambda_0 \supset \mathbb{C}$, $i = \overline{1,N}$, are Lagrangian multipliers.

Owing to the well known Lax theorem [1, 4], the condition $\operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] = 0$ determines an invariant supersubspace $\tilde{M}_N^{2|2} \subset \hat{M}^{2|2}$ of the hierarchy of the coupled dynamical systems (13), (6), (7) and (8), (9) with the parameters λ_i, $i = \overline{1,N}$, such as

$$\tilde{M}_N^{2|2} = \Bigg\{ (w, \mathcal{Y}, \mathcal{Z})^\top \in M^{2|2} : a = -\sum_{i=1}^{N} y_{0i} z_{4i}, b = -\sum_{i=1}^{N} y_{2i} z_{4i},$$

$$\phi = \sum_{i=1}^{N} y_{0i} z_{5i}, \chi = \sum_{i=1}^{N} y_{2i} z_{5i},$$

$$D_\theta Y_i = A[w; s_i] Y_i, D_\theta Z_i = -A^{\top_s}[w; s_i] Z_i, i = \overline{1,N} \Bigg\}.$$

Thus, the supersubspace $M_N^{2|2} \bigcap H_c \subset M^{2|2}$ is diffeomorphic to the supersubspace $\tilde{M}_N^{2|2} \subset \subset \hat{M}^{2|2}$ if $s_i = \lambda_i$, $i = \overline{1,N}$, for every $c \in \Lambda_0^N$.

By means of the analog of Gelfand–Dikii differential relationship [18, 20] for the Lagrangian functional $\hat{L}_N \in \mathcal{D}(\hat{M}^{2|2})$,

$$d\hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] = \Big\langle (dw, d\mathcal{Y}, d\mathcal{Z})^\top, \operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \Big\rangle + D_\theta \alpha^{(1)}, \tag{19}$$

where $(\phi, \chi, \mathcal{Y}, \mathcal{Z})^\top$ are coordinates on a suitably truncated functional supermanifold $\hat{M}_N^{2|2} \subset \subset \hat{M}^{2|2}$, "$d$" is a symbol of the exterior differentiation in the Grassmann algebra of differential forms on $\mathbb{C}^{(6N+2)|(6N+2)}$ and the brackets $\langle , \rangle$ denotes the standard scalar product on $\mathbb{C}^{(6N+2)|(6N+2)}$, we can construct the even exact two-form $\hat{\omega}^{(2)} = d\alpha^{(1)}$,

$$\hat{\omega}^{(2)} = \sum_{i=1}^{N} \sum_{s=0}^{5} y_{si} \wedge z_{si} + d\phi \wedge d\chi, \tag{20}$$

where "$\wedge$" is a symbol of the exterior product on the Grassmann algebra of differential forms on $\mathbb{C}^{(6N+2)|(6N+2)}$. The reduced two-form $\omega^{(2)} := \hat{\omega}^{(2)}|_{\tilde{M}_N^{2|2}}$ defines a supersymplectic structure on the supersubspace $M_N^{2|2} \bigcap H_c \simeq \tilde{M}_N^{2|2} \subset \hat{M}_N^{2|2}$, which is smoothly embedded in the superspace $\hat{M}_N^{2|2}$ owing to the relationships (17).

The expression (19) ensures the invariance of the reduced two-form $\omega^{(2)}$ with respect to the superdifferentiation D_θ, that is,

$$D_\theta \omega^{(2)} = 0.$$

Since $D_\theta^2 = \dfrac{d}{dx}$, the two-form $\omega^{(2)}$ is also invariant with respect to the vector field $\dfrac{d}{dx}$ on $\hat{M}_N^{2|2}$.

Taking into account that the supersubspace $M_N^{2|2} \bigcap H_c \subset M^{2|2}$ is diffeomorphic to the finite-dimensional supersubmanifold $M_{\mathcal{F}} \subset \mathbb{R}^{6N|(6N+2)}$, determined by the constraints

$$F_1 := \phi - \sum_{i=1}^{N} y_{0i} z_{5i} = 0, \quad F_2 := \chi - \sum_{i=1}^{N} y_{2i} z_{5i}, \tag{21}$$

in the superspace $\mathbb{R}^{6N|(6N+2)}$, we can obtain a supersymplectic structure on $M_N^{2|2} \bigcap H_c$ as a natural Dirac type reduction of the two-form $\hat{\omega}^{(2)}$ on $M_{\mathcal{F}}$.

The two-form $\hat{\omega}^{(2)}$ generates the Poisson bracket on the superspace $\mathbb{R}^{6N|(6N+2)}$,

$$\{F, G\}_{\hat{\omega}^{(2)}} = \sum_{i=1}^{N} \sum_{s=0,2,4} \left(\frac{\partial F}{\partial z_{si}} \frac{\partial G}{\partial y_{si}} - \frac{\partial F}{\partial y_{si}} \frac{\partial G}{\partial z_{si}} \right) -$$

$$- \sum_{i=1}^{N} \sum_{s=1,3,5} \left(\frac{\partial_r F}{\partial y_{si}} \frac{\partial_l G}{\partial z_{si}} + \frac{\partial_r F}{\partial z_{si}} \frac{\partial_l G}{\partial y_{si}} \right) + \frac{\partial_r F}{\partial \phi} \frac{\partial_l G}{\partial \chi} + \frac{\partial_r F}{\partial \chi} \frac{\partial_l G}{\partial \phi}, \tag{22}$$

where $\dfrac{\partial_l}{\partial \zeta}$ and $\dfrac{\partial_r}{\partial \zeta}$ denote operators of the left and the right derivatives with respect to the anticommuting variable $\zeta \in \Lambda_1$, for arbitrary smooth functions $F \in C^\infty(\mathbb{R}^{6N|(6N+2)}; \mathbb{R}^{1|0})$ or $C^\infty(\mathbb{R}^{6N|(6N+2)}; \mathbb{R}^{0|1})$ and $G \in C^\infty(\mathbb{R}^{6N|(6N+2)}; \mathbb{R}^{1|0})$ or $C^\infty(\mathbb{R}^{6N|(6N+2)}; \mathbb{R}^{0|1})$.

Since the matrix of constraints $\{F_{\kappa_1}, F_{\kappa_2}\}_{\hat{\omega}^{(2)}}$, $\kappa_1, \kappa_2 = 1, 2$, is nondegenerate, the standard Dirac type reduction procedure [4, 25] entails the following Poisson bracket:

$$\{F, G\}_{\omega_{\mathcal{F}}^{(2)}} = \{F, G\}_{\hat{\omega}^{(2)}} - \{F, F_1\}_{\hat{\omega}^{(2)}} \{F_2, G\}_{\hat{\omega}^{(2)}} - \{F, F_2\}_{\hat{\omega}^{(2)}} \{F_1, G\}_{\hat{\omega}^{(2)}} = \{F, G\}_{\hat{\omega}^{(2)}} -$$

$$- \left(\sum_{i_1=1}^{N} \left(-\frac{\partial F}{\partial z_{0i_1}} z_{5i_1} - \frac{\partial_r F}{\partial y_{5i_1}} y_{0i_1} \right) + \frac{\partial_r F}{\partial \chi} \right) \times$$

$$\times \left(\sum_{i_2=1}^{N} \left(z_{5i_2} \frac{\partial G}{\partial z_{2i_2}} - y_{2i_2} \frac{\partial_l G}{\partial y_{5i_2}} \right) + \frac{\partial_l G}{\partial \phi} \right) -$$

$$- \left(\sum_{i_1=1}^{N} \left(-\frac{\partial F}{\partial z_{2i_1}} z_{5i_1} - \frac{\partial_r F}{\partial y_{5i_1}} y_{2i_1} \right) + \frac{\partial_r F}{\partial \phi} \right) \times$$

$$\times \left(\sum_{i_2=1}^{N} \left(z_{5i_2} \frac{\partial G}{\partial z_{0i_2}} - y_{0i_2} \frac{\partial_l G}{\partial y_{5i_2}} \right) + \frac{\partial_l G}{\partial \chi} \right), \tag{23}$$

related with the supersymplectic structure $\omega^{(2)}_{\mathcal{F}} := \omega^{(2)}$ on $M_{\mathcal{F}} \simeq M_N^{2|2}$.

From the equalities

$$\frac{dL_N}{dt_j} = 0, \quad \frac{dL_N}{d\tilde{t}_j} = 0, \quad j \in \mathbb{Z}_+,$$

it follows that there exist functions $\hat{h}^{(t_j)}, \hat{h}^{(\tilde{t}_j)} \in D(\hat{M}^{2|2})$, obeying the relationships

$$\begin{gathered} \left\langle \left(\frac{dw}{dt_j}, \frac{d\mathcal{Y}}{dx}, \frac{d\mathcal{Z}}{dx}\right)^\top, \operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \right\rangle = D_\theta \hat{h}^{(t_j)}, \\ \left\langle \left(\frac{dw}{d\tilde{t}_j}, \frac{d\mathcal{Y}}{dx}, \frac{d\mathcal{Z}}{dx}\right)^\top, \operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \right\rangle = D_\theta \hat{h}^{(\tilde{t}_j)}, \end{gathered} \tag{24}$$

where the brackets $\langle\, , \rangle$ denotes the standard scalar product on $C^{(6N+2)|(6N+2)}$. The functions $\hat{h}^{(t_j)}$ and $\hat{h}^{(\tilde{t}_j)}$ on $\tilde{M}_N^{2|2}$ satisfy the following equalities:

$$i_{\frac{d}{dt_j}} \hat{\omega}^{(2)} = -d\hat{h}^{(t_j)}, \quad i_{\frac{d}{d\tilde{t}_j}} \hat{\omega}^{(2)} = -d\hat{h}^{(\tilde{t}_j)}, \quad j \in \mathbb{Z}_+, \tag{25}$$

where $i_{\frac{d}{dt_j}}, i_{\frac{d}{d\tilde{t}_j}}$ are inner differentiations with respect to the vector fields $\frac{d}{dt_j} : \tilde{M}_N^{2|2} \to \to T(\tilde{M}_N^{2|2})$ and $\frac{d}{d\tilde{t}_j} : \tilde{M}_N^{2|2} \to T(\tilde{M}_N^{2|2})$, $j \in \mathbb{Z}_+$, in the Grassmann algebra of differential forms on $\mathbb{C}^{(6N+2)|(6N+2)}$.

To state the first equality in (25) we need to calculate the expressions

$$i_{\frac{d}{dt_j}} \left\langle (dw, d\mathcal{Y}, d\mathcal{Z})^\top, \operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \right\rangle = D_\theta \hat{h}^{(t_j)}.$$

Then we have

$$d i_{\frac{d}{dt_j}} \left\langle (dw, d\mathcal{Y}, d\mathcal{Z})^\top, \operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \right\rangle = -D_\theta d\hat{h}^{(t_j)}.$$

From (19) we easily obtain that the identities

$$d \left\langle (dw, d\mathcal{Y}, d\mathcal{Z})^\top, \operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \right\rangle = D_\theta d\alpha^{(1)}$$

and

$$i_{\frac{d}{dt_j}} d \left\langle (dw, d\mathcal{Y}, d\mathcal{Z})^\top, \operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \right\rangle = i_{\frac{d}{dt_j}} D_\theta \hat{\omega}^{(2)} \tag{26}$$

hold owing to the relations

$$dD_\theta = -D_\theta d, \quad i_{\frac{d}{dt_j}} D_\theta = -D_\theta i_{\frac{d}{dt_j}}.$$

Since the Lie derivative with respect to the vector field $\frac{d}{dt_j} : \tilde{M}_N^{2|2} \to T(\tilde{M}_N^{2|2})$ can be represented as

$$\frac{d}{dt_j} = di_{\frac{d}{dt_j}} + i_{\frac{d}{dt_j}} d$$

the resulting relationship

$$\frac{d}{dt_j} \left\langle (dw, d\mathcal{Y}, d\mathcal{Z})^\top , \operatorname{grad} \hat{L}_N[w, \mathcal{Y}, \mathcal{Z}] \right\rangle = -D_\theta \left(d\hat{h}^{(t_j)} + i_{\frac{d}{dt_j}} \hat{\omega}^{(2)} \right)$$

holds on $\tilde{M}^{2|2}$. The latter proves the first equality in (25), which takes place on $\tilde{M}_N^{2|2}$. The second equality in (25) can be proved analogously.

The obtained relationships (25) lead to important equalities,

$$\frac{d}{dt_j}\, \omega^{(2)} = 0, \quad \frac{d}{d\tilde{t}_j}\, \omega^{(2)} = 0,$$

on $M_N^{2|2} \bigcap H_c \simeq \tilde{M}_N^{2|2}$. Therefore, the functions $h^{(t_j)} := \hat{h}^{(t_j)}\Big|_{M_N^{2|2} \bigcap H_c}$, $h^{(\tilde{t}_j)} := \hat{h}^{(\tilde{t}_j)}\Big|_{M_N^{2|2} \bigcap H_c}$ are Hamiltonians subject to the vector fields $\frac{d}{dt_j}$ and $\frac{d}{d\tilde{t}_j}$ on $M_N^{2|2} \bigcap H_c \simeq M_{\mathcal{F}}$ for all $j \in \mathbb{Z}_+$ if $s_i = \lambda_i$, $i = \overline{1, N}$, that is,

$$i_{\frac{d}{dt_j}} \omega_{\mathcal{F}}^{(2)} = -dh^{(t_j)}, \quad i_{\frac{d}{d\tilde{t}_j}} \omega_{\mathcal{F}}^{(2)} = -dh^{(\tilde{t}_j)}, \quad j \in \mathbb{Z}_+.$$

For example, the Hamiltonian function of the vector field $\frac{d}{dx} := \frac{d}{dt_0}$ on the supersubspace $M_N^{2|2} \bigcap H_c \subset M^{2|2}$ equals, by definition, $h^{(x)} := \hat{h}^{(x)}\Big|_{M_N^{2|2} \bigcap H_c}$, where

$$\begin{aligned}
\hat{h}^{(x)} = \sum_{i=1}^{N} \Bigg(& \lambda_i (y_{0i} z_{4i} + y_{1i} z_{5i}) - y_{2i} z_{0i} - y_{4i} z_{2i} - y_{3i} z_{1i} - y_{5i} z_{3i} + \\
& + \phi (y_{4i} z_{5i} - y_{2i} z_{3i}) - \chi (y_{2i} z_{5i} - y_{0i} z_{3i}) + \\
& + \left(\sum_{k=1}^{N} y_{0k} z_{5k} \right) (y_{3i} z_{4i} + y_{2i} z_{3i}) + \left(\sum_{k=1}^{N} y_{2k} z_{5k} \right) (y_{1i} z_{4i} + y_{0i} z_{3i}) + \\
& + \left(\sum_{k=1}^{N} y_{1k} z_{5k} \right) y_{3i} z_{5i} - \left(\sum_{k=1}^{N} y_{2k} z_{4k} \right) y_{0i} z_{4i} \Bigg),
\end{aligned}$$

$$\begin{aligned}
h^{(x)} = \sum_{i=1}^{N} \Bigg(& \lambda_i (y_{0i} z_{4i} + y_{1i} z_{5i}) - y_{2i} z_{0i} - y_{4i} z_{2i} - y_{3i} z_{1i} - y_{5i} z_{3i} + \\
& + \left(\sum_{k=1}^{N} y_{0k} z_{5k} \right) (y_{3i} z_{4i} + y_{4i} z_{5i}) + \left(\sum_{k=1}^{N} y_{2k} z_{5k} \right) y_{1i} z_{4i} +
\end{aligned}$$

$$+\left(\sum_{k=1}^{N} y_{1k}z_{5k}\right) y_{3i}z_{5i} - \left(\sum_{k=1}^{N} y_{2k}z_{4k}\right) y_{0i}z_{4i}\Bigg).$$

Analogously we can find Hamiltonians of the other vector fields $\frac{d}{dt_j}$ and $\frac{d}{d\tilde{t}_j}$, $j \in \mathbb{Z}_+$, on the supersubspace $M_N^{2|2} \bigcap H_c \subset M^{2|2}$. Therefore, the following theorem holds.

Theorem 1. *The vector fields* $\frac{d}{dt_j}$ *and* $\frac{d}{d\tilde{t}_j}$, $j \in \mathbb{Z}_+$, *generated by the Boussinesq hierarchy (1), allow invariant reductions on the finite-dimensional supersubspaces* $M_N^{2|2} \bigcap H_c \subset M^{2|2}$ *for each* $N \in \mathbb{N}$, *which are diffeomorphic to the finite-dimensional supermanifold* $M_{\mathcal{F}}$, *smoothly embedded into the superspace* $\mathbb{R}^{6N|(6N+2)}$ *and endowed with the even, reduced via the Dirac scheme, Poisson bracket (23). On these supersubspaces the vector fields* $\frac{d}{dt_j}$ *and* $\frac{d}{d\tilde{t}_j}$, $j \in \mathbb{Z}_+$, *generated by the equations (6), (7) and (8), (9) under* $\lambda = \lambda_i$, $i = \overline{1,N}$, *are Hamiltonian with respect to the Poisson bracket (23). The corresponding Hamiltonians* $h^{(t_j)}, h^{(\tilde{t}_j)} \in C^\infty(\mathbb{R}^{6N|(6N+2)}; \mathbb{R}^{1|0})$ *are reductions on* $M_N^{2|2} \bigcap H_c \subset M^{2|2}$ *of suitably constructed functions* $\hat{h}^{(t_j)}, \hat{h}^{(\tilde{t}_j)} \in D(\hat{M}^{2|2})$ *satisfying the equalities (24). The relationships (17) describe all periodic and quasiperiodic solutions to the Boussinesq hierarchy (1) on the supersubspace* $M_N^{2|2} \bigcap H_c$.

4. The Lax – Liouville integrability of the reduced commuting vector fields. To state the Liouville integrability of the Hamiltonian vector fields $\frac{d}{dt_j}$ and $\frac{d}{d\tilde{t}_j}$, $j \in \mathbb{Z}_+$, on $\tilde{M}_N^{2|2}$ for all $N \in \mathbb{N}$ we need to construct for these flows the related matrix Lax type representations, depending on the spectral parameter $\lambda \in \mathbb{C}$, making use of the reduction procedure for the monodromy supermatrix of the periodic spectral problem (4). We can formulate the following theorem.

Theorem 2. *On the finite-dimensional supersymplectic superspace* $M_N^{2|2} \bigcap H_c$, $c \in \Lambda_0^N$, *there exist matrix Lax representations*

$$\frac{dS_N}{dt_j} = [B_{j,N}, S_N], \tag{27}$$

$$\frac{dS_N}{d\tilde{t}_j} = [\tilde{B}_{j,N}, S_N], \tag{28}$$

for the Hamiltonian vector fields $\frac{d}{dt_j}$ *and* $\frac{d}{d\tilde{t}_j}$, $j \in \mathbb{Z}_+$, *where*

$$B_{j,N} := B_{j,N}(\mathcal{Y}, \mathcal{Z}; \lambda) = B[w;\lambda]\big|_{M_N^{2|2} \bigcap H_c},$$

and

$$\tilde{B}_{j,N} := \tilde{B}_{j,N}(\mathcal{Y}, \mathcal{Z}; \lambda) = \tilde{B}[w;\lambda]\Big|_{M_N^{2|2} \bigcap H_c}$$

are projections of the corresponding supermatrices on $M_N^{2|2} \bigcap H_c$ *and the reduced monodromy* *supermatrix* $S_N := S_N(\mathcal{Y}, \mathcal{Z}; \lambda) = S(x, \theta; \lambda)\|_{M_N^{2|2} \bigcap H_c}$ *equals*

$$S_N := \sum_{i=1}^{N} \frac{S_i}{\lambda - \lambda_i} + S_0 + S_{-1}\lambda =$$

$$= \sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i} \begin{pmatrix} y_{0i}z_{0i} & y_{0i}z_{2i} & y_{0i}z_{4i} & -y_{0i}z_{1i} & -y_{0i}z_{3i} & -y_{0i}z_{5i} \\ y_{2i}z_{0i} & y_{2i}z_{2i} & y_{2i}z_{4i} & -y_{2i}z_{1i} & -y_{2i}z_{3i} & -y_{2i}z_{5i} \\ y_{4i}z_{0i} & y_{4i}z_{2i} & y_{4i}z_{4i} & -y_{4i}z_{1i} & -y_{4i}z_{3i} & -y_{4i}z_{5i} \\ y_{1i}z_{0i} & y_{1i}z_{2i} & y_{1i}z_{4i} & -y_{1i}z_{1i} & -y_{1i}z_{3i} & -y_{1i}z_{5i} \\ y_{3i}z_{0i} & y_{3i}z_{2i} & y_{3i}z_{4i} & -y_{3i}z_{1i} & -y_{3i}z_{3i} & -y_{3i}z_{5i} \\ y_{0i}z_{0i} & y_{5i}z_{2i} & y_{5i}z_{4i} & -y_{5i}z_{1i} & -y_{5i}z_{3i} & -y_{5i}z_{5i} \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ a & 0 & 3 & \phi & 0 & 0 \\ -b + \sum_{\ell=1}^{N} y_{0\ell}z_{2\ell} & -2a & 0 & -\chi - \sum_{\ell=1}^{N} y_{0\ell}z_{3\ell} & -2\phi & 0 \\ \phi & 0 & 0 & 0 & 3 & 0 \\ \chi + (D_\theta a) & \phi & 0 & a + (D_\theta \phi) & 0 & 3 \\ \begin{matrix} -2(D_\theta b) - (D_\theta a_x) + \\ + \sum_{\ell=1}^{N} y_{4\ell}z_{5\ell} \end{matrix} & \begin{matrix} -\chi - (D_\theta a) + \\ + \sum_{\ell=1}^{N} y_{1\ell}z_{4\ell} \end{matrix} & -2\phi & \begin{matrix} -b - (D_\theta \chi) - \\ - \sum_{\ell=1}^{N} y_{1\ell}z_{3\ell} \end{matrix} & -2a - 2(D_\theta \phi) & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{pmatrix} \lambda, \tag{29}$$

with the functions a, b, ϕ, χ *given by the expressions (17) and the differential relationships*

$$D_\theta a = -\sum_{i=1}^{N} y_{1i}z_{4i} - \sum_{i=1}^{N} y_{0i}z_{3i}, \quad D_\theta b = -\sum_{i=1}^{N} y_{3i}z_{4\ell} - \sum_{i=1}^{N} y_{2i}z_{3i},$$

$$D_\theta \phi = \sum_{i=1}^{N} y_{1i}z_{5i} - a, \quad D_\theta \chi = \sum_{i=1}^{N} y_{3i}z_{5i} - b,$$

$$D_\theta a_x = \sum_{i=1}^{N} y_{3i}z_{4\ell} + \sum_{i=1}^{N} y_{2i}z_{3i} - \sum_{i=1}^{N} y_{1i}z_{2i} - \sum_{i=1}^{N} y_{0i}z_{1i} + a\phi,$$

Proof. Making use of the spectral problem (4) we can express the gradient $\varphi(x, \theta; \bar{\lambda})$ of the supertrace of the corresponding monodromy supermatrix

$$S := \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \tag{30}$$

by means of its elements

$$\varphi(x,\theta;\bar\lambda) = \begin{pmatrix} \mathrm{str}\,(I\bar S I A_a) \\ \mathrm{str}\,(I\bar S I A_b) \\ \mathrm{str}\,(\bar S A_\phi) \\ \mathrm{str}\,(\bar S A_\chi) \end{pmatrix} = -\begin{pmatrix} \bar S_{26} \\ \bar S_{16} \\ \bar S_{56} \\ \bar S_{46} \end{pmatrix},$$

where $\bar S$ is a supermatrix with the elements being complex conjugate to the corresponding ones of S.

Using the equality

$$\varphi(x,\theta;\bar\lambda_i) = \overline{\left(\frac{d}{d\lambda}\,\mathrm{str}\,S(x,\theta;\lambda)\Big|_{\lambda=\lambda_i}\right)}\,\mathrm{grad}\,\lambda_i,$$

we can obtain, in particular, the Magri type relationship [23]

$$\mathcal{M}\,\mathrm{grad}\,\lambda_i = \bar\lambda_i\,\mathcal{L}\,\mathrm{grad}\,\lambda_i,$$

for all $i = \overline{1,N}$ as well as expansions of the elements S_{16}, S_{26}, S_{46} and S_{56} of the reduced on $M_N^{2|2}$ monodromy supermatrix S_N at their poles.

Another elements of the supermatrix S_N can be extracted from the supergeneralization of Novikov – Lax [2, 3, 18] monodromy supermatrix equation

$$D_\theta S = AS - (ISI)A. \tag{31}$$

From this equation we directly obtain the elements S_{13}, S_{23}, S_{43}, S_{15}, S_{45} and S_{12}. To find the other monodromy supermatrix elements we can use the relationship

$$\mathrm{str}\,S := S_{11} + S_{22} + S_{33} - (S_{44} + S_{55} + S_{66}) \equiv C(\lambda), \quad D_\theta\,C(\lambda) = 0,$$

where $C(\lambda)$ is some Laurent series with constant even coefficients, which follows from the equation (31). Since

$$D_\theta(S_{14} + S_{25} + S_{36}) = (S_{44} + S_{55} + S_{66} - S_{11} + S_{22} + S_{33}) - \chi S_{16} - \phi S_{26},$$

this property of the monodromy supermatrix supertrace allows us to obtain S_{25} from the equality

$$D_\theta(3S_{25} + aS_{16} + \phi S_{46} + D_\theta(S_{45} - S_{12}) + D_\theta(S_{23} - S_{56})) = -\chi S_{16} - \phi S_{26} - C(\lambda).$$

It is evident that the function $C(\lambda)$ can be chosen arbitrarily since the supermatrix $Y\mathbf{C}(\lambda)Y^{-1}$ with $Y := Y(x,\theta;\lambda)$ is some even fundamental supermatrix for the spectral problem (4) and $\mathbf{C}(\lambda)$ is some even supermatrix with elements in the forms of Laurent series with constant coeffecients and satisfies the equation (31). If

$$C(\lambda) = \sum_{i=1}^{N} \frac{\sigma_i}{\lambda - \lambda_i}, \qquad \sigma_i = \sum_{s=0}^{5} y_{si} z_{si}, \tag{32}$$

the elements S_{25}, S_{14} and S_{36} look like

$$S_{25} = \sum_{i=1}^{N} \frac{y_{2i} z_{3i}}{\lambda - \lambda_i} + C_1(\lambda), \quad S_{14} = \sum_{i=1}^{N} \frac{y_{0i} z_{1i}}{\lambda - \lambda_i} + C_1(\lambda),$$

$$S_{36} = \sum_{i=1}^{N} \frac{y_{4i} z_{5i}}{\lambda - \lambda_i} + C_1(\lambda), \quad D_\theta C_1(\lambda) = 0,$$

where $C_1(\lambda)$ is some Laurent series with odd coefficients.

Then the elements S_{42}, S_{53}, $S_{44} - S_{11}$, $S_{55} - S_{22}$, $S_{66} - S_{33}$, $S_{44} - S_{22}$, $S_{55} - S_{33}$, S_{24} and S_{35} can be found successfully.

From the evident expressions for the differences $S_{44} - S_{11}$, $S_{55} - S_{22}$, $S_{66} - S_{33}$, $S_{44} - S_{22}$ and $S_{55} - S_{33}$ we further obtain the diagonal elements of the reduced monodromy supermatrix S_N in the forms

$$S_{11} = \sum_{i=1}^{N} \frac{y_{0i} z_{0i}}{\lambda - \lambda_i} + \mathcal{P}, \quad S_{22} = \sum_{i=1}^{N} \frac{y_{2i} z_{2i}}{\lambda - \lambda_i} + \mathcal{P},$$

$$S_{33} = \sum_{i=1}^{N} \frac{y_{4i} z_{4i}}{\lambda - \lambda_i} + \mathcal{P}, \quad S_{44} = -\sum_{i=1}^{N} \frac{y_{1i} z_{1i}}{\lambda - \lambda_i} + \mathcal{P},$$

$$S_{55} = -\sum_{i=1}^{N} \frac{y_{3i} z_{3i}}{\lambda - \lambda_i} + \mathcal{P}, \quad S_{66} = -\sum_{i=1}^{N} \frac{y_{5i} z_{5i}}{\lambda - \lambda_i} + \mathcal{P},$$

where $\mathcal{P} = \mathcal{P}(\mathcal{Y}, \mathcal{Z}; \lambda)$ is some still undefined function on $M_N^{2|2}$.

From the relationships for the superderivatives of the diagonal elements of S_N we look for the elements S_{41}, S_{52} and S_{63}, depending on $(D_\theta \mathcal{P})$. Moreover, from the similar relationships for the elements in the first column of S_N we obtain expressions for the elements S_{21}, S_{51}, S_{31}, S_{61} and the differential relationship

$$-D_\theta \mathcal{P}_{xx} = -a\mathcal{P}_x + \chi(D_\theta \mathcal{P}) - \phi(D_\theta \mathcal{P}_x). \tag{33}$$

Thereby, the elements S_{54}, S_{32}, S_{65}, S_{34}, S_{62} and S_{64} can also be found.

Having calculated now the elements S_{65}, S_{62} and S_{64} successfully in other way we obtain the expressions

$$\mathcal{P}_x = \frac{2}{3}\,\phi C_1(\lambda), \quad (D_\theta\phi)C_1(\lambda) = 0,$$

$$3\mathcal{P}_{xx} = -(2\chi + (D_\theta a))C_1(\lambda) + \phi_x C_1(\lambda) - \phi(D_\theta\,\mathcal{P}).$$

From these equations it follows that

$$D_\theta\mathcal{P}_x = 0, \quad -\frac{2}{3}\,a\phi C_1(\lambda) + \chi(D_\theta\,\mathcal{P}) = 0,$$

$$-(2\chi + (D_\theta a))C_1(\lambda) - \phi(D_\theta\,\mathcal{P}) = 0.$$

allowing for the vanishing solution for $\mathcal{P}$ and $C_1(\lambda)$. The latter entails the reduced monodromy supermatrix S_N expression (29).

In addition, the relationships (27), (28) follow from the compatibility conditions of the equations (4) and (6),

$$\frac{dA}{dt_j} - (D_\theta B_j) = (IB_jI)A - AB_j, \quad j \in \mathbb{Z}_+,$$

as well as from those for the equations (5) and (7)

$$\frac{dA}{d\tilde{t}_j} - (D_\theta \tilde{B}_j) = (I\tilde{B}_jI)A - A\tilde{B}_j, \quad j \in \mathbb{Z}_+.$$

This finishes the proof.

Owing to the equations (27), (28) the functionals $\frac{1}{\alpha}\operatorname{str} S_N^\alpha$, $\alpha \in \mathbb{N}$, are invariant with respect to the vector fields $\frac{d}{dt_j}$ and $\frac{d}{d\tilde{t}_j}$, $j \in \mathbb{Z}_+$. Then the coefficients in the expansions of these functionals at their poles appear to be conservation laws of the reduced on $M_N^{2|2} \bigcap H_c$ vector fields from the hierarchy (1). Among them the coefficients σ_i, $\hat{\sigma}_i$, $\breve{\sigma}_i \in C^\infty(\mathbb{R}^{6N|(6N+2)};\mathbb{R}^{1|0})$, $i = \overline{1,N}$, are given by the expansions of the invariant functionals $\operatorname{str} S_N$, $\frac{1}{2}\operatorname{str} S_N^2$ and $\frac{1}{3}\operatorname{str} S_N^3$ as follows:

$$\operatorname{str} S_N = \sum_{i=1}^{N} \frac{\sigma_i}{\lambda - \lambda_i},$$

$$\frac{1}{2}\operatorname{str} S_N^2 = \frac{1}{2}\sum_{i=1}^{N} \frac{\sigma_i^2}{(\lambda - \lambda_i)^2} + \sum_{i=1}^{N} \frac{\hat{\sigma}_i}{\lambda - \lambda_i}, \tag{34}$$

$$\hat{\sigma}_i = \sum_{k=1,k\neq i}^{N} \frac{\operatorname{str}(S_i S_k)}{\lambda_i - \lambda_k} + \lambda_i \operatorname{str}(S_{-1}S_i) + \operatorname{str}(S_0 S_i) =$$

$$= \sum_{k=1,k\neq i}^{N} \frac{\left(\sum_{s=0}^{5} y_{si} z_{sk}\right)\left(\sum_{r=0}^{5} y_{rk} z_{ri}\right)}{\lambda_i - \lambda_k} - \lambda_i g_{ii} + f_{ii},$$

and

$$\frac{1}{3}\operatorname{str} S_N^3 = \frac{1}{3}\sum_{i=1}^{N} \frac{\sigma_i^3}{(\lambda-\lambda_i)^3} + \sum_{i=1}^{N} \frac{\sigma_i \hat{\sigma}_i}{(\lambda-\lambda_i)^2} + \sum_{i=1}^{N} \frac{\breve{\sigma}_i}{\lambda-\lambda_i}, \tag{35}$$

$$\breve{\sigma}_i = \sum_{\substack{k,\ell=1,\\ k,\ell\neq i,\ k\neq \ell}}^{N} \frac{\operatorname{str}(S_i S_k S_\ell)}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_\ell)} + \sum_{k=1,\ k\neq i}^{N} \frac{\operatorname{str}(S_i S_k)(\sigma_k - \sigma_i)}{(\lambda_i-\lambda_k)^2} +$$

$$+ \sum_{k=1,\ k\neq i}^{N} \frac{\lambda_i \operatorname{str}(S_{-1}(S_i S_k + S_k S_i))}{\lambda_i - \lambda_k} +$$

$$+ \sum_{k=1,\ k\neq i}^{N} \frac{\operatorname{str}(S_0(S_i S_k + S_k S_i))}{\lambda_i - \lambda_k} +$$

$$+ \lambda_i \operatorname{str}((S_{-1}S_0 + S_0 S_{-1})S_i) + \operatorname{str}(S_0^2 S_i) =$$

$$= \sum_{\substack{k,\ell=1,\\ k,\ell\neq i,\ k\neq \ell}}^{N} \frac{\left(\sum_{s=0}^{5} y_{si} z_{s\ell}\right)\left(\sum_{\kappa=0}^{5} y_{\kappa\ell} z_{\kappa k}\right)\left(\sum_{r=0}^{5} y_{rk} z_{ri}\right)}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_\ell)} +$$

$$+ \sum_{k=1,\ k\neq i}^{N} \frac{\left(\sum_{s=0}^{5} y_{si} z_{sk}\right)\left(\sum_{r=0}^{5} y_{rk} z_{ri}\right)\sum_{\kappa=0}^{5}(y_{\kappa i} z_{\kappa i} - y_{\kappa k} z_{\kappa k})}{(\lambda_i-\lambda_k)^2} +$$

$$+ \sum_{k=1,\ k\neq i}^{N} \frac{\lambda_i g_{ik}\left(\sum_{s=0}^{5} y_{sk} z_{si}\right) + \lambda_i g_{ki}\left(\sum_{s=0}^{5} y_{si} z_{sk}\right)}{\lambda_i - \lambda_k} +$$

$$+ \sum_{k=1,\ k\neq i}^{N} \frac{f_{ik}\left(\sum_{s=0}^{5} y_{sk} z_{si}\right) + f_{ki}\left(\sum_{s=0}^{5} y_{si} z_{sk}\right)}{\lambda_i - \lambda_k} - \lambda_i p_{ii} + q_{ii},$$

where

$$
\begin{aligned}
g_{ik} &= 3(y_{0i}z_{4k} + y_{1i}z_{5k}),\\
f_{ik} &= 3y_{2i}z_{0k} + ay_{0i}z_{2k} + 3y_{4i}z_{2k} + \phi y_{1i}z_{2k} + \left(-b + \sum_{\ell=1}^{N} y_{0\ell}z_{2\ell}\right) y_{0i}z_{4k} - 2ay_{2i}z_{4k} -\\
&\quad - \left(\chi + \sum_{\ell=1}^{N} y_{0\ell}z_{3\ell}\right) y_{1i}z_{4k} - 2\phi y_{3i}z_{4k} + \phi y_{0i}z_{1k} + 3y_{3i}z_{1k} +\\
&\quad + (\chi + (D_\theta a))y_{0i}z_{3k} + \phi y_{2i}z_{3k} + (a + (D_\theta \phi))y_{1i}z_{3k} +\\
&\quad + 3y_{5i}z_{3k} - \left(2(D_\theta b) + (D_\theta a_x) - \sum_{\ell=1}^{N} y_{4\ell}z_{5\ell}\right) y_{5i}z_{0k} -\\
&\quad - \left(\chi + (D_\theta a) - \sum_{\ell=1}^{N} y_{1\ell}z_{4\ell}\right) y_{2i}z_{5k} - 2\phi y_{4i}z_{5k} -\\
&\quad - \left(b + (D_\theta \chi) + \sum_{\ell=1}^{N} y_{1\ell}z_{3\ell}\right) y_{1i}z_{5k} - 2(a + (D_\theta \phi))y_{3i}z_{5k},\\
p_{ik} &= 9(y_{0i}z_{2k} + y_{2i}z_{4k} + y_{1i}z_{3k} + y_{3i}z_{5k}) - 3\phi y_{0i}z_{5k},\\
q_{ik} &= 3ay_{0i}z_{0i} + 9y_{4i}z_{0i} + 3\phi y_{1i}z_{0i} + 3\left(-b + \sum_{\ell=1}^{N} y_{0\ell}z_{2\ell}\right) y_{0i}z_{2k} -\\
&\quad - 3ay_{2i}z_{2k} - 3\left(\chi + \sum_{\ell=1}^{N} y_{0\ell}z_{3\ell}\right) y_{1i}z_{2k} - 3\phi y_{3i}z_{2k} +\\
&\quad + \left(-2a^2 + \phi\left(-\chi - 2(D_\theta a) + \sum_{\ell=1}^{N} y_{0\ell}z_{3\ell}\right)\right) y_{0i}z_{4k} +\\
&\quad + 3\left(-b + \sum_{\ell=1}^{N} y_{0\ell}z_{2\ell}\right) y_{2i}z_{4k} - 6ay_{4i}z_{4k} - 2\phi(2a + (D_\theta \phi))y_{1i}z_{4k} -\\
&\quad - 3\left(\chi + \sum_{\ell=1}^{N} y_{0\ell}z_{3\ell}\right) y_{3i}z_{4k} - 6\phi y_{5i}z_{4k} + 3(\chi + (D_\theta a))y_{0i}z_{1k} +\\
&\quad + 6\phi y_{2i}z_{1k} + 3(a + (D_\theta \phi))y_{1i}z_{1k} + 9y_{5i}z_{1k} +\\
&\quad + \left(\phi(2a + (D_\theta \phi)) - 6(D_\theta b) - 3(D_\theta a_x) + 3\sum_{\ell=1}^{N} y_{4\ell}z_{5\ell}\right) y_{0i}z_{3k} +
\end{aligned}
$$

$$+3\left(\sum_{\ell=1}^{N} y_{1\ell}z_{4\ell}\right)y_{2i}z_{3k}-3\phi y_{4i}z_{3k}-$$

$$-3\left(b+(D_\theta\chi)+\sum_{\ell=1}^{N} y_{1\ell}z_{3\ell}\right)y_{1i}z_{3k}-3(a+(D_\theta\phi))y_{3i}z_{3k}+$$

$$+\left(-3a(\chi+(D_\theta a))+a\sum_{\ell=1}^{N} y_{1\ell}z_{4\ell}-2(D_\theta\phi)(\chi+(D_\theta a))+\right.$$

$$\left.+\phi\left(b-(D_\theta\chi)-\sum_{\ell=1}^{N}(y_{1\ell}z_{3\ell}+2y_{0\ell}z_{2\ell})\right)\right)y_{0i}z_{5k}+$$

$$+\left(2\phi(a-(D_\theta\phi))-6(D_\theta b)-3(D_\theta a_x)+3\sum_{\ell=1}^{N} y_{4\ell}z_{5\ell}\right)y_{2i}z_{5k}-$$

$$-3\left(\chi+(D_\theta a)-\sum_{\ell=1}^{N} y_{1\ell}z_{4\ell}\right)y_{4i}z_{5k}+$$

$$+\left(-2(a+(D_\theta\phi))^2+\phi\left(3\chi+\sum_{\ell=1}^{N}(y_{0\ell}z_{3\ell}-2y_{1\ell}z_{4\ell})\right)\right)y_{1i}z_{5k}-$$

$$-3\left(b+(D_\theta\chi)+\sum_{\ell=1}^{N} y_{1\ell}z_{3\ell}\right)y_{3i}z_{5k}-6(a+(D_\theta\phi))y_{5i}z_{5k},$$

and are functionally independent on $M_N^{2|2}\bigcap H_c$ and involutive with respect to the Poisson bracket $\{.,.\}_{\omega^{(2)}}$ on $M_N^{2|2}\bigcap H_c$. Thus, owing to the superanalog [29] of Liouville integrability theorem the vector fields $\dfrac{d}{dt_j}$ and $\dfrac{d}{d\tilde{t}_j}$, $j\in\mathbb{Z}_+$, are superintegrable flows on the finite-dimensional supersubspace $M_N^{2|2}\bigcap H_c\subset M^{2|2}$.

5. Conclusion. In the present paper the generalized invariant reduction technique, devised before in [18], for investigating Lax type integrable supersymmetric nonlinear dynamical systems, has been used to study the Bargmann type reductions of the vector fields generated by the supersymmetric Boussinesq hierarhy related with a non-self-adjoint superdifferential operator of one anticommuting variable. It has been established that the corresponding invariant finite-dimensional supersubspace is diffeomorphic to some supersymplectic supermanifold, smoothly embedded into superspace $R^{6N|(6N+2)}$, $N\in\mathbb{N}$, with an even supersymplectic structure.

The invariant reduction procedure can be applied to a wide class of other Lax integrable supersymmetric nonlinear dynamical systems on the functional supermanifolds of one commuting and one anticommuting independent variables, associated with the linear matrix spectral relationships. The devised technique can also be effectively used for investigating reductions of $(2|1+1)$- and $(2|2+1)$-dimensional supersymmetric nonlinear dynamical systems with triple matrix Lax linearizations, described before in the papers [26, 27, 28], upon suitably determined invariant finite-dimensional supersubspaces. The latter is planned to be a subject of future studies.

It is worth to mention here that the reduction method contributes to solving Lax integrable supersymmetric nonlinear dynamical systems on functional supermanifolds (of one commuting and one anticommuting independent variables) by means of the integration of the Liouville integrable systems on suitably determined finite-dimensional supermanifolds with even super-symplectic structures. Thus, there is a need of developing the devised in [30] integration method, based on specially constructed Picard – Fuchs type differential-functional equations generating Hamiltonian – Jacobi transformations, and applying it to the Liouville – Lax integrable dynamical systems on supersymplectic finite-dimensional supermanifods.

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