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NONCONVEX VALUED IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH VARIABLE TIMES
ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНІ НЕОПУКЛІ ВКЛЮЧЕННЯ З ІМПУЛЬСНОЮ ДІЄЮ

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In this paper, a fixed point theorem due to Schaefer combined with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued operators with nonempty closed and decomposable values is used to investigate the existence of solutions for first and second order impulsive functional differential inclusions with variable times.

З допомогою теореми Шефера про нерухому точку, а також теореми Брессана і Коломбо про вибір для нижньо напівнеперервних багатозначних операторів із непорожніми замкненими розкладними значеннями вивчено питання існування розв'язків функціонально-диференціальних включень першого та другого порядків зі змінним часом.

1. Introduction. In this paper, we are concerned with the existence of solutions to some classes of initial value problems for first and second order impulsive functional and neutral functional differential inclusions. Initially, in Section 3, we will consider the first order impulsive functional differential inclusion

$$y'(t) \in F(t, y_t), \quad \text{a.e.} \quad t \in J := [0, T], \quad t \neq \tau_k(y(t)),$$

$$y(t^+) = I_k(y(t^-)), \quad t = \tau_k(y(t)), \quad k = 1, \ldots, m, \tag{1.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0],$$

where $F : J \times \mathcal{D} \to \mathcal{P}(\mathrm{I\!R})$ is a multivalued map with nonempty compact values, $\mathcal{D} = \{\psi : [-r, 0] \to \mathrm{I\!R}; \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t}^-)$ and $\psi(\bar{t}^+)$ exist and $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in \mathcal{D}$, $0 < r < \infty$, $\tau_k : \mathrm{I\!R} \to \mathrm{I\!R}$, $I_k : \mathrm{I\!R} \to \mathrm{I\!R}$, $k = 1, 2, \ldots, m$, are given functions satisfying some assumptions that will be specified later.

For any function y defined on $[-r, T]$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t. Later, in

Section 4, we study the second order impulsive functional differential inclusion of the form

$$\begin{aligned} y'' &\in F(t, y_t), \text{ a.e. } t \in J := [0,T], \;\; t \neq \tau_k(y(t)), \\ y(t^+) &= I_k(y(t^-)), \;\; t = \tau_k(y(t)), \\ y'(t^+) &= \overline{I}_k(y(t^-)), \;\; t = \tau_k(y(t)), \;\; k = 1, \ldots, m, \\ y(t) &= \phi(t), \;\; t \in [-r, 0], \; y'(0) = \eta, \end{aligned} \tag{1.2}$$

where F, I_k, and ϕ are as in problem (1.1), $\overline{I}_k \in C(\mathbb{R}, \mathbb{R})$ and $\eta \in \mathbb{R}$. Sections 5 and 6 are devoted to the existence of solutions for initial value problems for first and second order impulsive neutral functional differential inclusions. More precisely, in these last sections, we consider the IVPs

$$\begin{aligned} \frac{d}{dt}[y(t) - g(t, y_t)] &\in F(t, y_t), \text{ a.e. } t \in J = [0,T], \; t \neq \tau_k(y(t)), \\ y(t^+) &= I_k(y(t^-)), \;\; t = \tau_k(y(t)), \; k = 1, \ldots, m, \\ y(t) &= \phi(t), \;\; t \in [-r, 0], \end{aligned} \tag{1.3}$$

where F, I_k are as in problem (1.1), $g : J \times \mathcal{D} \to \mathbb{R}$ is a given function, and

$$\begin{aligned} \frac{d}{dt}[y'(t) - g(t, y_t)] &\in F(t, y_t), \quad \text{a.e.} \quad t \in J = [0,T], \; t \neq \tau_k(y(t)), \\ y(t^+) &= I_k(y(t^-)), \quad t = \tau_k(y(t)), \\ y'(t^+) &= \overline{I}_k(y(t^-)), \quad t = \tau_k(y(t)), \;\; k = 1, \ldots, m, \\ y(t) &= \phi(t), \quad t \in [-r, 0], \; y'(0) = \eta, \end{aligned} \tag{1.4}$$

where F, I_k, ϕ, g, η and $\overline{I}_k$ are as in the above cited problems.

The theory of impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [1], Lakshmikantham et al. [2], and Samoilenko and Perestyuk [3] and the references therein. The theory of impulsive differential equations with variable times is relatively less developed due to the difficulties created by the state-dependent impulses. Recently, some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [4], Frigon and O'Regan [5 – 7], Kaul et al. [8], Kaul and Liu [9, 10] Lakshmikantham et al. [11, 12], Liu and by Ballinger [13] and Vatsala and Vasundara Devi [14, 15]. Very recently, by mean of Schaefer's theorem and the concept of upper and lower solutions, Benchohra et al. [16–18] have considered different classes of impulsive functional differential equations. The same tools have been applied to a variety of impulsive functional differential inclusions with convex valued right-hand side by the same authors [19, 20]. The main theorems of this paper extend those considered by Benchohra et al. [19, 20]. Our approach here is based on the Schaefer's fixed point theorem [21, p. 29] combined with a selection theorem due to Bressan and Colombo [22] for lower semicontinuous multivalued operators with nonempty closed and decomposable values.

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

For $\phi \in \mathcal{D}$ the norm of ϕ is defined by

$$\|\phi\|_{\mathcal{D}} = \sup\{|\phi(\theta)| : \theta \in [-r, 0]\}.$$

$L^1([0,T], \mathbb{R})$ denotes the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} := \int_0^T |y(t)| dt \quad \text{for all} \quad y \in L^1(J, \mathbb{R}).$$

$AC^i([0,T], \mathbb{R})$ is the space of i-times differentiable functions $y : [0,T] \to \mathbb{R}$, whose ith derivative, $y^{(i)}$, is absolutely continuous.

Let A be a subset of $[0,T] \times \mathcal{D}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$, where $\mathcal{J}$ is Lebesgue measurable in J and D is Borel measurable in $\mathcal{D}$. A subset A of $L^1([0,T], \mathbb{R})$ is decomposable if for all $u, v \in A$ and $\mathcal{J} \subset [0,T]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in A$, where χ_J stands for the characteristic function of J. Let E be a Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \neq \varnothing\}$ is open for any open set B in E. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. For more details on multivalued maps we refer to the books of Deimling [23], Górniewicz [24] and Hu and Papageorgiou [25].

Definition 2.1. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T], \mathbb{R}))$ be a multivalued operator. We say N has property (BC) if*

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F : J \times \mathcal{D} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([-r,T], \mathbb{R}) \to \mathcal{P}(L^1([0,T], \mathbb{R}))$$

by letting

$$\mathcal{F}(y)=\{w\in L^1([0,T],\mathbb{R}):w(t)\in F(t,y_t)\quad\text{for a.e.}\quad t\in[0,T]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2.2. *Let $F:\ J\times\mathcal{D}\to\mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Next, we state a selection theorem due to Bressan and Colombo in [22].

Theorem 2.1. *Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1([0,T],\ \mathbb{R}))$ be a multivalued operator which has property (BC). Then N has a continuous selection; i.e., there exists a continuous function (single-valued) $g:Y\to L^1(J,\mathbb{R})$ such that $g(y)\in N(y)$ for every $y\in Y$.*

Let us introduce the following hypotheses which are assumed hereafter:

(H_1) $F:[0,T]\times\mathcal{D}\longrightarrow\mathcal{P}(\mathbb{R})$ is a nonempty, compact-valued, multivalued map such that:

a) $(t,u)\mapsto F(t,u)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable;

b) $u\mapsto F(t,u)$ is lower semi-continuous for a.e. $t\in[0,T]$;

(H_2) for each $r>0$, there exists a function $h_r\in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}}:=\sup\{|v|:\ v\in F(t,u)\}\le h_r(t)\quad\text{for a.e.}\quad t\in[0,T];$$

and for $u\in\mathcal{D}$ with $\|u\|_{\mathcal{D}}\le r$.

The following lemma is crucial in the proof of our main theorem:

Lemma 2.1 [26]. *Let $F:[0,T]\times\mathcal{D}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty, compact values. Assume (H_1) and (H_2) hold. Then F is of l.s.c. type.*

3. First order impulsive FDIs. The main result of this section concerns the IVP (1.1). Before stating and proving this one, we give the definition of a solution of the IVP (1.1). We shall consider the space

$$\Omega=\{y:[0,T]\longrightarrow\mathbb{R}:\text{there exist }0<t_1<\ldots<t_m<T\text{ such that }t_k=\tau_k(y(t_k)),$$
$$y_k\in C(J_k,\mathbb{R}),k=0,\ldots,m,\ \text{and there exist }y(t_k^-)\text{ and }y(t_k^+),k=1,\ldots,m,$$
$$\text{with }y(t_k^-)=y(t_k)\}.$$

Here $y_k:=y_{/J_k}$, $k=0,\ldots,m$, $J_k=(t_k,t_{k+1}]$, $t_0=0$ and $t_{m+1}=T$.

$$PC=\{y:[-r,T]\to\mathbb{R}:y\in\mathcal{D}\cap\Omega\}.$$

Definition 3.1. *A function $y\in PC\cap\cup_{k=0}^m AC((t_k,t_{k+1}),\mathbb{R})$ is said to be a solution of (1.1) if there exists $v(t)\in F(t,y_t)$ a.e. $t\in[0,T]$ such that $y'(t)=v(t)$ a.e. $t\in[0,T]$, $t\neq\tau_k(y(t))$, $y(t^+)=I_k(y(t))$, $t=\tau_k(y(t))$, $k=1,\ldots,m$ and $y(t)=\phi(t)$, $t\in[-r,0]$.*

We are now in a position to state and prove our existence result for the problem (1.1). We first list the following additional hypotheses.

(H_3) The functions $\tau_k \in C^1(\mathbb{R}, \mathbb{R})$ for $k = 1, \ldots, m$. Moreover,

$$0 < \tau_1(x) < \ldots < \tau_m(x) < T \quad \text{for all} \quad x \in \mathbb{R}.$$

(H_4) There exist constants $c_k > 0$ such that

$$|I_k(x)| \leq c_k \quad \text{for each} \quad x \in \mathbb{R}, \quad k = 1, \ldots, m.$$

(H_5) There exists a continuous nondecreasing function $\psi \ : [0, +\infty) \longrightarrow (0, +\infty)$, and $p \in L^1([0,T], \mathbb{R}_+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(\|u\|_{\mathcal{D}}) \quad \text{for} \quad \text{a.e.} \quad u \in [0,T] \quad \text{and each} \quad u \in \mathcal{D}$$

with

$$\int_1^{\infty} \frac{d\gamma}{\psi(\gamma)} = +\infty.$$

(H_6) For all $(t,x) \in [0,T] \times \mathbb{R}$ and for all $y_t \in \mathcal{D}$ we have

$$\tau_k'(x)v(t) \neq 1 \quad \text{for} \quad k = 1, \ldots, m, \quad \text{for all} \quad v \in \mathcal{F}(y).$$

(H_7) For all $x \in \mathbb{R}$

$$\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x)) \quad \text{for} \quad k = 1, \ldots, m-1.$$

Theorem 3.1. *Suppose that hypotheses (H_1) – (H_7), are satisfied. Then the impulsive initial value problem (1.1) has at least one solution.*

Proof. (H_1) and (H_2) imply, by Lemma 2.1, that F is of lower semi-continuous type. Then from Theorem 2.1 there exists a continuous function $f : C([-r,T], \mathbb{R}) \to L^1([0,T], \mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in PC$.

Step 1. Consider the problem,

$$\begin{aligned} y'(t) &= f(y_t), \quad t \in [0,T], \\ y(t) &= \phi(t), \quad t \in [-r,0]. \end{aligned} \tag{3.1}$$

It is obvious that if $y \in C([-r,T], \mathbb{R})$ is a solution of the problem (3.1), then y is a solution to the problem (1.1). Transform the problem into a fixed point problem. Consider the operator $N : C([-r,T], \mathbb{R}) \to C([-r,T], \mathbb{R})$ defined by:

$$N(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + \int_0^t f(y_s)ds & \text{if } t \in [0,T]. \end{cases}$$

We shall show that N is a continuous and completely continuous operator.

Claim 1. *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $C([-r,T],\mathbb{R})$. Then

$$|N(y_n(t)) - N(y(t))| \le \int_0^t |f(y_{n_s}) - f(y_s)|ds \le \int_0^T |f(y_{n_s}) - f(y_s)|ds.$$

Since the function f is continuous, then

$$\|N(y_n) - N(y)\|_\infty \le \|f(y_n) - f(y)\|_{L^1} \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 2. *N maps bounded sets into bounded sets in $C([-r,T],\mathbb{R})$.*

Indeed, it is enough to show that for any $q > 0$ there exists a positive constant ℓ such that, for each $y \in B_q := \{y \in C([-r,T],\mathbb{R}) : \|y\|_\infty \le q\}$, we have $\|N(y)\|_\infty \le \ell$. From (H_2) and (H_3) we have

$$|N(y)(t)| \le \|\phi\|_{\mathcal{D}} + \int_0^t |f(y_s)|ds \le \|\phi\|_{\mathcal{D}} + \|h_q\|_{L^1} = \ell.$$

Claim 3. *N maps bounded sets into equicontinuous sets of $C([-r,T],\mathbb{R})$.*

Let $\tau_1, \tau_2 \in [0,T]$, $\tau_1 < \tau_2$, and B_q be a bounded set of $C([-r,T],\mathbb{R})$. Let $y \in B_q$. Then

$$|N(y)(\tau_2) - N(y)(\tau_1)| \le \int_{\tau_1}^{\tau_2} h_q(s)ds.$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $\tau_1 < \tau_2 \le 0$ and $\tau_1 \le 0 \le \tau_2$ is obvious.

As a consequence of Claims 1 to 3, together with the Arzela–Ascoli theorem, we conclude that $N := C([-r,T],\mathbb{R}) \longrightarrow C([-r,T],\mathbb{R})$ is continuous and completely continuous.

Claim 4. *Now it remains to show that the set*

$$\mathcal{E}(N) := \{y \in C([-r,T],\mathbb{R}) : \ y = \lambda N(y) \quad \textit{for some} \quad 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus

$$y(t) = \lambda \left[\phi(0) + \int_0^t f(y_s)\,ds\right], \quad t \in [0,T].$$

This implies by (H_5) that for each $t \in [0,T]$ we have

$$|y(t)| \leq \|\phi\|_{\mathcal{D}} + \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds. \tag{3.2}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by inequality (3.2) we have for $t \in J$

$$\mu(t) \leq \|\phi\|_{\mathcal{D}} + \int_0^t p(s)\psi(\mu(s))ds. \tag{3.3}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the inequality (3.3) holds. Let us take the right-hand side of the inequality (3.3) as $v(t)$. Then we have

$$c = v(0) = \|\phi\|_{\mathcal{D}}, \quad \mu(t) \leq v(t), \quad t \in J,$$

and

$$v'(t) = p(t)\psi(\mu(t)), \quad t \in [0,T].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq p(t)\psi(v(t)), \quad t \in [0,T].$$

By using (H_5) this implies for each $t \in [0,T]$ that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\psi(\tau)} \leq \int_0^T p(s)ds < +\infty.$$

This inequality implies that there exists a constant K such that $v(t) \leq K$, $t \in J$, and hence $\mu(t) \leq K$, $t \in [0,T]$. Since for every $t \in [0,T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\|_\infty \leq K' := \max\{\|\phi\|_{\mathcal{D}}, K\},$$

where K' depends only on T and on the functions p and ψ. This shows that $\mathcal{E}(N)$ is bounded.

Set $X := C([-r,T], \mathbb{R})$. As a consequence of Schaefer's theorem (see [21, p. 29]), we deduce that N has a fixed point y which is a solution to problem (3.1). Denote this solution by y_1.

Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t \quad \text{for} \quad t \geq 0.$$

(H_3) implies that

$$r_{k,1}(0) \neq 0 \quad \text{for} \quad k = 1, \ldots, m.$$

If

$$r_{k,1}(t) \neq 0 \quad \text{on} \quad [0,T] \quad \text{for} \quad k = 1, \ldots, m,$$

i.e.,

$$t \neq \tau_k(y_1(t)) \quad \text{on} \quad [0,T] \quad \text{and for} \quad k = 1, \ldots, m,$$

then y_1 is a solution of the problem (1.1).

It remains to consider the case when

$$r_{1,1}(t) = 0 \quad \text{for some} \quad t \in [0,T].$$

Now since

$$r_{1,1}(0) \neq 0$$

and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0, \quad \text{and} \quad r_{1,1}(t) \neq 0 \quad \text{for all} \quad t \in [0, t_1).$$

Thus by (H_3) we have

$$r_{k,1}(t) \neq 0 \quad \text{for all} \quad t \in [0, t_1), \quad \text{and} \quad k = 1, \ldots, m.$$

Step 2. Consider now the following problem:

$$y'(t) = f(y_t), \quad \text{a.e.} \quad t \in [t_1, T], \tag{3.4}$$

$$y(t) = y_1(t), \quad t \in [t_1 - r, t_1], \quad y(t_1^+) = I_1(y_1(t_1^-)).$$

Set

$$C^* = C([t_1 - r, t_1], \mathbb{R}) \cap C_1,$$

where

$$C_1 = \left\{ y \in C((t_1, T], \mathbb{R}) : \ y(t_1^+) \quad \text{exists} \right\}.$$

Transform the problem (3.4) into a fixed point problem. Consider the operator $N_1 : C^* \to C^*$ defined by

$$N_1(y)(t) := \begin{cases} y_1(t) & \text{if } t \in [t_1 - r, t_1], \\ I_1(y_1(t_1)) + \int\limits_{t_1}^{t} f(y_s)\, ds & \text{if } t \in (t_1, T]. \end{cases}$$

As in Step 1 we can show that N_1 is continuous and completely continuous. Now we prove only that the set

$$\mathcal{E}(N_1) := \{ y \in C^* : \ y = \lambda N_1(y) \quad \text{for some} \quad 0 < \lambda < 1 \}$$

is bounded.

Let $y \in \mathcal{E}(N_1)$. Then $y = \lambda N_1(y)$ for some $0 < \lambda < 1$. Thus

$$y(t) = \lambda \left[I_1(y(t_1^-)) + \int_{t_1}^{t} f(y_s)\, ds \right], \quad t \in [t_1, T].$$

This implies by (H$_4$) and (H$_5$) that for each $t \in [t_1, T]$ we have

$$|y(t)| \leq c_1 + \int_{t_1}^{t} p(s)\psi(\|y_s\|_{\mathcal{D}})ds. \tag{3.5}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : \ t_1 - r \leq s \leq t\}, \quad t_1 \leq t \leq T.$$

Let $t^* \in [r - t_1, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [t_1 - r, T]$, by inequality (3.5) we have for $t \in [t_1 - r, T]$

$$\mu(t) \leq c_1 + \int_{t_1}^{t} p(s)\psi(\mu(s))ds. \tag{3.6}$$

If $t^* \in [t_1 - r, t_1]$ then $\mu(t) = \|y_1\|_\infty$ and the inequality (3.6) holds. Let us take the right-hand side of the inequality (3.6) as $v(t)$. Then we have

$$c_* = v(t_1) = c_1, \quad \mu(t) \leq v(t), \quad t \in [t_1, T],$$

and

$$v'(t) = p(t)\psi(\mu(t)), \quad t \in [t_1, T].$$

Using the nondecreasing character of ψ we get

$$v'(t) \quad \leq p(t)\psi(v(t)), \quad t \in [t_1, T].$$

By using (H$_5$) this implies for each $t \in [t_1, T]$ that

$$\int_{v(t_1)}^{v(t)} \frac{d\tau}{\psi(\tau)} \leq \int_{0}^{T} p(s)ds < +\infty.$$

This inequality implies that there exists a constant K such that $v(t) \leq K_1, \ t \in [t_1, T]$, and hence $\mu(t) \leq K_1, t \in [t_1, T]$. Since for every $t \in [t_1, T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\|_\infty \leq K_2 := \max\{\|y_1\|_\infty, K_1\},$$

where K_2 depends only on T and on the functions p, ψ_1 and ψ. This shows that $\mathcal{E}(N_1)$ is bounded.

Set $X := C([t_1 - r, T], \mathbb{R})$. As a consequence of Schaefer's theorem we deduce that N_1 has a fixed point y which is a solution to problem (3.4). Denote this solution by y_2. Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t \quad \text{for} \quad t \geq t_1.$$

If

$$r_{k,2}(t) \neq 0 \quad \text{on} \quad (t_1, T] \quad \text{and for all} \quad k = 1, \ldots, m$$

then

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [0, t_1], \\ y_2(t) & \text{if } t \in (t_1, T], \end{cases}$$

is a solution of the problem (1.1).

It remains to consider the case when

$$r_{2,2}(t) = 0 \quad \text{for some} \quad t \in (t_1, T].$$

By (H_7) we have

$$r_{2,2}(t_1^+) = \tau_2(y_2(t_1^+)) - t_1 = \tau_2(I_1(y_1(t_1))) - t_1 > \tau_1(y_1(t_1)) - t_1 = r_{1,1}(t_1) = 0.$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that

$$r_{2,2}(t_2) = 0,$$

and

$$r_{2,2}(t) \neq 0 \quad \text{for all} \quad t \in (t_1, t_2).$$

It is clear by (H_3) that

$$r_{k,2}(t) \neq 0 \ \text{ for all } \ t \in (t_1, t_2), \ \ k = 2, \ldots, m.$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that

$$r_{1,2}(\bar{s}) = 0.$$

From (H_7) it follows that

$$r_{1,2}(t_1^+) = \tau_1(y_2(t_1^+)) - t_1 = \tau_1(I_1(y_1(t_1))) - t_1 \leq \tau_1(y_1(t_1)) - t_1 = r_{1,1}(t_1) = 0.$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, T]$. Since

$$y_2'(t) = f(y_{2_t}),$$

we have

$$r'_{1,2}(s_1) = \tau'_1(y_2(s_1))y'_2(s) - 1 = 0.$$

Therefore

$$\tau'_1(y_2(s_1)).f(y_{2_{s_1}}) = 1,$$

which contradicts (H_6).

Step 3. We continue this process and taking into account that $y_{m+1} := y\Big|_{[t_m,T]}$ is a solution to the problem

$$y'(t) = f(y_t), \quad \text{a.e.} \quad t \in (t_m, T),$$

$$y(t) = y_m(t), \quad t \in [t_m - r, t_m], \quad y(t_m^+) = I_m(y_m(t_m^-)).$$

The solution y of the problem (1.1) is then defined by

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ y_{m+1}(t) & \text{if } t \in (t_m, T]. \end{cases}$$

4. Second order impulsive FDIs. In this section we give an existence result for the IVP (1.2). Let us start by defining what we mean by a solution of problem (1.2).

Definition 4.1. *A function $y \in PC \cap \cup_{k=0}^{m} AC^1((t_k, t_{k+1}), \mathbb{R})$ is said to be a solution of (1.2) if there exists $v(t) \in F(t, y_t)$ a.e. $t \in [0, T]$ such that $y''(t) = v(t)$ a.e. on $[0, T]$, $t \neq \tau_k(y(t))$, $y(t^+) = I_k(y(t^-))$, $t = \tau_k(y(t))$, $y'(t^+) = \overline{I}_k(y(t^-))$, $t = \tau_k(y(t))$, $k = 1, \ldots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$ and $y'(0) = \eta$.*

Theorem 4.1. *Assume (H_1) – (H_7) and the condition*
(H_8) there exist constants $\overline{d}_k > 0$ such that

$$|\overline{I}_k(y)| \le \overline{d}_k \quad for\ each \quad y \in \mathbb{R},\ k = 1, \ldots, m,$$

are satisfied. Then the IVP (1.2) has at least one solution.

Proof. (H_1) and (H_2) imply, by Lemma 2.1, that F is of lower semi-continuous type. Then from Theorem 2.1 there exists a continuous function $f : C([-r, T], \mathbb{R}) \to L^1([0, T], \mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C([-r, T], \mathbb{R})$.

Step 1. Consider the following problem:

$$\begin{gathered} y''(t) = f(y_t), \quad t \in [0, T], \\ y(t) = \phi(t), \ t \in [-r, 0], \ y'(0) = \eta. \end{gathered} \tag{4.1}$$

Consider the operator $\overline{N}: C([-r,T],\mathbb{R}) \to C([-r,T],\mathbb{R})$ defined by

$$\overline{N}(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + t\eta + \int\limits_0^t (t-s)f(y_s)\,ds & \text{if } t \in [0,T]. \end{cases}$$

As in Theorem 3.1 we can show that $\overline{N}$ is continuous and completely continuous. Now we prove only that the set

$$\mathcal{E}(\overline{N}) := \{y \in C([-r,T],\mathbb{R}) : y = \lambda\overline{N}(y) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $y \in \mathcal{E}(\overline{N})$. Then $y = \lambda\overline{N}(y)$ for some $0 < \lambda < 1$. Thus

$$y(t) = \lambda\left[\phi(0) + t\eta + \int\limits_0^t (t-s)f(y_s)ds\right].$$

This implies by (H_5) and (H_8) that for each $t \in J$ we have

$$|y(t)| \le \|\phi\|_{\mathcal{D}} + T|\eta| + \int\limits_0^t (T-s)p(s)\psi(\|y_s\|)ds. \tag{4.2}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le T.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the inequality (4.2) we have for $t \in J$

$$|\mu(t)| \le \|\phi\|_{\mathcal{D}} + T|\eta| + \int\limits_0^t (T-s)p(s)\psi(\mu(s))ds. \tag{4.3}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|$ and the inequality (4.3) holds. Let us take the right-hand side of inequality (4.3) as $v(t)$. Then we have

$$v(0) = \|\phi\|_{\mathcal{D}} + T|\eta| \text{ and } v'(t) = (T-t)p(t)\psi(\mu(t)),\ t \in [0,T].$$

Using the nondecreasing character of ψ we get

$$v'(t) \le (T-t)p(t)\psi(v(t)),\ t \in [0,T].$$

This implies together with (H_5) for each $t \in [0,T]$ that

$$\int\limits_{v(0)}^{v(t)} \frac{d\tau}{\psi(\tau)} \le \int\limits_0^T (T-s)p(s)ds \le T\int\limits_0^T p(s)ds < +\infty.$$

This inequality implies that there exists a constant b such that $v(t) \leq b,\, t \in [0,T]$, and hence $\mu(t) \leq b,\, t \in J$. Since for every $t \in [0,T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\|_\infty \leq \max\{\|\phi\|_{\mathcal{D}}, b\},$$

where b depends only on T and on the functions p and ψ. This shows that $\mathcal{E}(\overline{N})$ is bounded.

Set $X := C([-r,T], \mathbb{R})$. As a consequence of Schaefer's theorem we deduce that $\overline{N}$ has a fixed point y which is a solution to problem (4.1). Denote this solution by y_1 and continue as in Theorem 3.1.

Step 2. Consider the following problem:

$$\begin{gathered} y''(t) = f(y_t), \quad \text{a.e.} \quad t \in [t_1, T], \\ y(t) = y_1(t), \quad t \in [t_1 - r, t_1], \quad y(t_1^+) = I_1(y_1(t_1^-)), \\ y'(t_1^+) = \overline{I}_1(y_1(t_1^-)). \end{gathered} \tag{4.4}$$

Consider the operator, $\overline{N}_1 : C^* \to C^*$ defined by

$$\overline{N}_1(y)(t) := \begin{cases} y_1(t) & \text{if } t \in [t_1 - r, t_1], \\ I_1(y_1(t_1)) + t\overline{I}_1(y_1(t_1)) + \int\limits_{t_1}^{t} (t-s) f(y_s)\, ds & \text{if } t \in (t_1, T]. \end{cases}$$

As in Theorem 3.1 we can show that $\overline{N}_1$ is continuous and completely continuous. Now we prove only that the set

$$\mathcal{E}(\overline{N}_1) := \{y \in C^* : y = \lambda \overline{N}_1(y) \quad \text{for some } \ 0 < \lambda < 1\}$$

is bounded. Let $y \in \mathcal{E}(\overline{N}_1)$. Then $y = \lambda \overline{N}_1(y)$ for some $0 < \lambda < 1$. Thus

$$y(t) = \lambda \left[I_1(y_1(t_1^-)) + t\overline{I}_1(y_1(t_1^-)) + \int\limits_{t_1}^{t} (t-s) f(y_s) ds \right].$$

This implies by (H_4), (H_5) and (H_8) that for each $t \in J$ we have

$$|y(t)| \leq c_1 + T\overline{d}_1 + \int\limits_{t_1}^{t} (T-s) p(s) \psi(\|y_s\|)\, ds. \tag{4.5}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : t_1 - r \leq s \leq t\}, \ \ t_1 \leq t \leq T.$$

Let $t^* \in [t_1 - r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [t_1, T]$, by the inequality (4.5) we have for $t \in [t_1, T]$

$$|\mu(t)| \leq c_1 + T\overline{d}_1 + \int_0^t (T - s)p(s)\psi(\mu(s))ds. \tag{4.6}$$

If $t^* \in [t_1 - r, t_1]$ then $\mu(t) = \|y_1\|_\infty$ and the inequality (4.6) holds. Let us take the right-hand side of inequality (4.6) as $v(t)$. Then we have

$$v(t_1) = c_1 + T\overline{d}_1, \quad \text{and} \quad v'(t) = (T - t)p(t)\psi(\mu(t)), \quad t \in [t_1, T].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq (T - t)p(t)\psi(v(t)), \quad t \in [t_1, T].$$

This implies together with (H_5) for each $t \in [t_1, T]$ that

$$\int_{v(t_1)}^{v(t)} \frac{d\tau}{\psi(\tau)} \leq \int_{t_1}^{T} (T - s)p(s)ds \leq T\int_{t_1}^{T} p(s)ds < +\infty.$$

This inequality implies that there exists a constant b such that $v(t) \leq b$, $t \in [t_1, T]$, and hence $\mu(t) \leq b$, $t \in [t_1, T]$. Since for every $t \in [t_1, T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|y\|_\infty \leq \max\{\|y_1\|_\infty, b\},$$

where b depends only on T and on the functions p, and ψ.

Set $X := C^*$. As a consequence of Schaefer's theorem [21] we deduce that $\overline{N}_1$ has a fixed point y which is a solution to problem (4.4). Denote this one by y_2 and continue as in Step 2 of the Theorem 3.1.

Step 3. We continue this process and taking into account that $y_{m+1} := y\Big|_{[t_m, T]}$ is a solution to the problem

$$y''(t) = f(y_t), \;\text{ a.e. }\; t \in (t_m, T),$$

$$y(t) = y_m(t), \;\; t \in [t_m - r, t_m], \;\; y(t_m^+) = I_m(y_m(t_m^-)),$$

$$y'(t_m^+) = \overline{I}_m(y_{m-1}(t_m^-)).$$

The solution y of the problem (1.2) is then defined by

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ y_{m+1}(t) & \text{if } t \in (t_m, T]. \end{cases}$$

5. First order impulsive neutral FDIs. In this section we are concerned with the existence of solutions for problem (1.3).

Definition 5.1. *A function* $y \in \Omega \cap \cup_{k=0}^{m} AC((t_k, t_{k+1}), \mathbb{R})$ *is said to be a solution of (1.3) if there exists* $v(t) \in F(t, y_t)$ *a.e.* $t \in [0,T]$ *such that* $\frac{d}{dt}[y(t) - g(t, y_t)] = v(t)$ *a.e.* $t \in [0,T]$, $t \neq \tau_k(y(t))$, $y(t^+) = I_k(y(t^-))$, $t = \tau_k(y(t))$, $k = 1, \ldots, m$, *and* $y(t) = \phi(t)$, $t \in [-r, 0]$.

We first list the following hypotheses:

(A_1) the function g is completely continuous and for any bounded set B in $C([-r,T],\mathbb{R})$, the set $\{t \to g(t, y_t) : y \in B\}$ is equicontinuous in $C([0,T],\mathbb{R})$ and there exist constants $0 \leq d_1 < 1$ and $d_2 \geq 0$ such that

$$|g(t,u)| \leq d_1 \|u\|_{\mathcal{D}} + d_2, \quad t \in [0,T],\ u \in \mathcal{D}, \quad k = 1, \ldots, m;$$

(A_2) g is a nonnegative function;

(A_3) τ_k is a nonincreasing function and

$$I_k(x) \leq x \quad \text{for all} \quad x \in \mathbb{R}, k = 1, \ldots, m;$$

(A_4) for all $x \in \mathbb{R}$

$$\tau_k(x) < \tau_{k+1}(I_k(x)) \text{ for } k = 1, \ldots, m;$$

(A_5) for all $t \in [0,T]$ and for all $y_t \in \mathcal{D}$ we have

$$\tau_k'(y(t) - g(t, y_t))v(t) \neq 1 \text{ for } k = 1, \ldots, m \ \text{ for all } \ v \in \mathcal{F}(y).$$

Theorem 5.1. *Assume that hypotheses (H_1)–(H_5) and (A_1)–(A_4) hold, then the IVP (1.3) has at least one solution on* $[-r,T]$.

Proof. (H_1) and (H_2) imply by Lemma 2.1 that F is of lower semi-continuous type. Then from Theorem 2.1 there exists a continuous function $f : C([-r,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C([-r,T],\mathbb{R})$.

Step 1. Consider now the following problem:

$$\begin{aligned} &\frac{d}{dt}[y(t) - g(t, y_t)] = f(y_t), \ \ t \in [0,T], \\ &y(t) = \phi(t), \ \ t \in [-r, 0]. \end{aligned} \tag{5.1}$$

Consider the operator $N_2 : C([-r,T],\mathbb{R}) \to C([-r,T],\mathbb{R})$ defined by

$$N_2(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) - g(0,\phi(0)) + g(t,y_t) + \int\limits_0^t f(y_s)ds & \text{if } t \in [0,T]. \end{cases}$$

Clearly from (A_1) N_2 is a continuous and completely continuous operator. Now it remains to show that the set

$$\mathcal{E}(N_2) := \{y \in C([-r,T], \mathbb{R}) : \ y = \lambda N_2(y) \text{ for some } \lambda \in (0,1)\}$$

is bounded.

Let $y \in \mathcal{E}(N_2)$. Then $\lambda N_2(y) = y$ for some $0 < \lambda < 1$ and

$$y(t) = \lambda \left[\phi(0) - g(0,\phi(0)) + g(t,y_t) + \int_0^t f(y_s)ds \right].$$

This implies by (H_5) and (A_1) that for each $t \in [0,T]$ we have

$$|y(t)| \leq |\phi(0)| + |g(0,\phi(0))| + |g(t,y_t)| + \int_0^t p(s)\psi(\|y_s\|)ds$$

or

$$|y(t)| \leq (1+d_1)\|\phi\|_{\mathcal{D}} + 2d_2 + d_1\|y_t\| + \int_0^t p(s)\psi(\|y_s\|)ds. \tag{5.2}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : -r \leq s \leq t\}, \ \ t \in [0,T].$$

Let $t^* \in [-r,t]$ be such that $\mu = |y(t^*)|$. If $t^* \in [0,T]$, by the inequality (5.2), we have for $t \in [0,T]$

$$\mu(t) \leq (1+d_1)\|\phi\|_{\mathcal{D}} + 2d_2 + d_1\mu(t) + \int_0^t p(s)\psi(\mu(s))ds.$$

Thus

$$\mu(t) \leq \frac{1}{1-d_1}\left[(1+d_1)\|\phi\|_{\mathcal{D}} + 2d_2 + \int_0^t p(s)\psi(\mu(s))ds \right]. \tag{5.3}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the inequality (5.3) holds. Let us take the right-hand side of the inequality (5.3) as $v(t)$, then we have

$$v(0) = \frac{1}{1-d_1}\left[(1+d_1)\|\phi\|_{\mathcal{D}} + 2d_2\right] \text{ and } v'(t) = p(t)\psi(\mu(t)).$$

Since ψ is nondecreasing we have

$$v'(t) = p(t)\psi(\mu(t)) \leq p(t)\psi(v(t)) \quad \text{for all} \quad t \in [0,T].$$

From this inequality, it follows that

$$\int_0^t \frac{v'(s)}{\psi(v(s))}\,ds \le \int_0^t p(s)ds.$$

By using (H_5) we then have

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le \int_0^t p(s)ds \le \int_0^T p(s)\,ds < +\infty.$$

This inequality implies that there exists a constant b depending only on T and on the function p such that

$$|y(t)| \le b \quad \text{for each} \quad t \in [0,T].$$

Hence

$$\|y\|_\infty \le b.$$

This shows that $\mathcal{E}(N_2)$ is bounded, and hence N_2 has a fixed point y which is a solution to problem (5.1).

Denote this solution by y_1. Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t \quad \text{for} \ \ t \ge 0.$$

(H_3) implies that

$$r_{k,1}(0) \ne 0 \quad \text{for} \quad k = 1,\dots,m.$$

If

$$r_{k,1}(t) \ne 0 \quad \text{on} \quad [0,T] \quad \text{for} \quad k = 1,\dots,m,$$

i.e.,

$$t \ne \tau_k(y_1(t)) \quad \text{on} \ \ [0,T] \ \ \text{and for} \quad k = 1,\dots,m,$$

then y_1 is a solution of the problem (1.3).

It remains to consider the case when

$$r_{1,1}(t) = 0 \quad \text{for some} \quad t \in [0,T].$$

Now since

$$r_{1,1}(0) \ne 0$$

and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0 \quad \text{and} \quad r_{1,1}(t) \ne 0 \quad \text{for all} \quad t \in [0,t_1).$$

Thus by (H_3) we have

$$r_{k,1}(t) \ne 0 \quad \text{for all} \quad t \in [0,t_1), \quad k = 1,\dots,m.$$

Step 2. Consider now the following problem:

$$\frac{d}{dt}[y(t) - g(t, y_t)] = f(y_t), \text{ a.e. } t \in [t_1, T],$$
$$y(t) = y_1(t), \ \ t \in [t_1 - r, t_1], \ \ y(t_1^+) = I_1(y_1(t_1^-)). \tag{5.4}$$

Consider the operator $\overline{N}_2 : C^* \to C^*$ defined by

$$\overline{N}_2(y)(t) := \begin{cases} y_1(t) & \text{if } t \in [t_1 - r, t_1], \\ I_1(y_1(t_1)) - g(t_1, y_{1t_1}) + g(t, y_t) + \int\limits_{t_1}^{t} f(y_s)ds & \text{if } t \in [t_1, T]. \end{cases}$$

As in Step 1 we can show that $\overline{N}_2$ is continuous and completely continuous, and the set

$$\mathcal{E}(\overline{N}_2) := \{y \in C^* : y = \lambda \overline{N}_2(y) \ \text{ for some } \ 0 < \lambda < 1\}$$

is bounded.

Set $X := C^*$. As a consequence of Schaefer's theorem we deduce that $\overline{N}_2$ has a fixed point y which is a solution to problem (5.4). Denote this solution by y_2. Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t \quad \text{for } t \geq t_1.$$

If

$$r_{k,2}(t) \neq 0 \ \text{ on } \ (t_1, T] \ \text{ and for all } \ k = 1, \ldots, m$$

then

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [0, t_1], \\ y_2(t) & \text{if } t \in (t_1, T], \end{cases}$$

is a solution of the problem (1.3). It remains to consider the case when

$$r_{2,2}(t) = 0 \ \text{ for some } \ t \in (t_1, T].$$

By (A_4) we have

$$r_{2,2}(t_1^+) = \tau_2(y_2(t_1^+)) - t_1 = \tau_2(I_1(y_1(t_1))) - t_1 > \tau_1(y_1(t_1)) - t_1 = r_{1,1}(t_1) = 0.$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that

$$r_{2,2}(t_2) = 0,$$

and

$$r_{2,2}(t) \neq 0 \ \text{ for all } \ t \in (t_1, t_2).$$

It is clear by (H_3) that

$$r_{k,2}(t) \neq 0 \ \text{ for all } \ t \in (t_1, t_2), \ \ k = 2, \ldots, m.$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that

$$r_{1,2}(\bar{s}) = 0.$$

Consider the function $L_1(t) = \tau_1(y_2(t) - g(t, y_{2_t})) - t$.

From (A_2) – (A_4) it follows that

$$L_1(\bar{s}) = \tau_1(y_2(\bar{s}) - g(\bar{s}, y_{2_{\bar{s}}})) - \bar{s} \geq \tau_1(y_2(\bar{s})) - \bar{s} = r_{1,2}(\bar{s}) = 0.$$

Thus the function L_1 attains a nonnegative maximum at some point $s_1 \in (t_1, T]$. Since

$$\frac{d}{dt}[y_2(t) - g(t, y_{2_t})] = f(y_{2_t}),$$

it follows that

$$L_1'(s_1) = \tau_1'(y_2(s_1) - g(s_1, y_{2_{s_1}}))\frac{d}{dt}[y_2(s_1) - g(s_1, y_{2_{s_1}})] - 1 = 0.$$

Therefore

$$[\tau_1'(y_2(s_1) - g(s_1, y_{2_{s_1}}))]f(y_{2_{s_1}}) = 1,$$

which contradicts (A_5).

Step 3. We continue this process and taking into account that $y_{m+1} := y\Big|_{[t_m, T]}$ is a solution to the problem

$$\frac{d}{dt}[y(t) - g(t, y_t)] = f(y_t), \quad \text{a.e.} \quad t \in (t_m, T),$$

$$y(t) = y_m(t), \ \ t \in [t_{m-1} - r, t_{m-1}], \quad y(t_m^+) = I_m(y_m(t_m^-)).$$

The solution y of the problem (1.3) is then defined by

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ y_{m+1}(t) & \text{if } t \in (t_m, T]. \end{cases}$$

6. Second order impulsive neutral FDIs. In this section we study the initial value problem (1.4). Its solution is defined in a similar maner. Let us introduce the following hypotheses:

(A_6) for all $(t,\bar{s},x) \in [0,T]\times[0,T]\times \mathbb{R}$ and for all $y_t \in \mathcal{D}$ we have

$$\tau_k'(x)\left[\overline{I}_k(y(\bar{s})) - g(\bar{s},\overline{y}_s) + g(t,y_t) + \int\limits_{\bar{s}}^{t} v(s)ds\right] \neq 1 \text{ for } k=1,\ldots,m, \text{ for all } v\in\mathcal{F}(y);$$

(A_7) there exists a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1([0,T],\mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t)\psi(\|u\|_{\mathcal{D}}) \text{ for a.e. } t\in[0,T] \text{ and each } u\in D$$

with

$$\int\limits_1^{\infty} \frac{d\gamma}{\gamma+\psi(\gamma)} = \infty.$$

Theorem 6.1. *Assume that hypotheses (H_1) – (H_4), (H_7) – (H_8), (A_1), and (A_6) – (A_7) are satisfied. Then the IVP (1.4) has at least one solution on* $[-r,T]$.

Proof. The details of the proof are left to the reader.

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