УДК 517 . 958

ON A RELATION BETWEEN MEMORY EFFECTS BY MAXWELL – BOLTZMANN AND KELVIN – VOIGT IN LINEAR VISCOELASTIC THEORY*

ПРО ЗВ'ЯЗОК МІЖ ЕФЕКТАМИ ПАМ'ЯТІ ЗА МАКСВЕЛЛОМ – БОЛЬЦМАНОМ ТА КЕЛЬВІНОМ – ВОЙГТОМ В ЛІНІЙНІЙ ТЕОРІЇ ПРУЖНОСТІ

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We study the smoothness properties of relaxation function such that a linear viscoelastic material system by Maxwell – Boltzmann can be considered of Kelvin – Voigt type; assuming that the relaxation function and its derivative decrease rapidly, and that the infinitesimal strain history is an analytical function, the Cauchy stress tensor of the linear viscoelasticity is well approximated by a constitutive functional of rate type.

Вивчаються властивості гладкості релаксаційної функції для випадку, коли лінійно пружна за Максвеллом – Больцманом матеріальна система може розглядатись як система типу Кельвіна – Войгта. У припущенні, що релаксаційна функція та її похідна швидко спадають, а інфінітезімальна функція деформації є аналітичною, показано, що тензор напруження Коші в лінійній теорії пружності добре апроксимується складовим (конститутивним) функціоналом коефіцієнтного типу.

1. As is posed in evidence in the study of the quasistatic problem in linear viscoelasticity theory [1,2] the crucial point for a good and exaustive formulation of viscoelastic materials theories [3,4] consists in the determination of general and physically admissible conditions [5] so that materials with more fading or negligible memory effects can be classified by a good approximation as particular viscoelastic materials; these conditions must be in accordance with the structural properties [6, 7] of viscoelastic materials and with the pattern that describes them.

This problem is resolved partially in [5, 6], were have been formulated conditions, with the above properties, so that materials of linear elastic type can be considered as particular linear viscoelastic materials.

Purpose of the present paper is to prove that materials of linear rate type can be considered as particular viscoelastic materials; if stronger smoothness hypotheses of relaxation and Boltzmann functions are verified and if the infinitesimal strain history is an analytic function, it is possible to approximate the constitutive functional of linear viscoelasticity theory by a particular constitutive equation of Kelvi – Voigt type.

It is interesting to observe that the coefficient of the memory term of this constitutive

* This research has been performed under the auspices of the G.N.F.M. of the C.N.R. and has been partially supported by Italian Ministry of University and Technology Research M.U.R.S.T.

relation is equal to the value of the dynamic viscosity tensor, when the frequency w approaches to zero, where this tensor has an eliminable discontinuity in virtue of the assumed hypotheses [5].

Finally we conclude proving an existence and uniqueness theorem for the quasistatic problem of material systems expounded by the above functional; the solution is determined as limit of a solution of the quasistatic problem for a strictly viscoelastic material system when $w \to 0$ [6].

2. Let β be a linear viscoelastic and homogeneous material system described by the following constitutive functional:

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{x},t) &= \boldsymbol{G}_0(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},t) + \int\limits_0^{+\infty} \boldsymbol{G}'(\boldsymbol{x},s)\boldsymbol{E}^t(\boldsymbol{x},s)ds = \\ &= \boldsymbol{G}_\infty(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},t) + \int\limits_0^{+\infty} \left[\boldsymbol{G}(\boldsymbol{x},s) - \boldsymbol{G}_\infty(\boldsymbol{x})\right] \dot{E}^t(\boldsymbol{x},s)ds, \end{aligned} \tag{1}$$

$$\boldsymbol{T}(\boldsymbol{x},t) = \boldsymbol{T}^T(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in \Omega \times [0,+\infty) = Q,$$

where $\boldsymbol{T}(\boldsymbol{x},s)$ is the Cauchy stress tensor, $\boldsymbol{G}(\boldsymbol{x},s)$ and $\boldsymbol{G}'(\boldsymbol{x},s)$ are respectively the relaxation and Boltzmann fourth-order Cartesian tensors, $\boldsymbol{G}_0(\boldsymbol{x})$ and $\boldsymbol{G}_\infty(\boldsymbol{x})$ denote respectively the instantaneous and equilibrium elastic moduli, that are so defined:

$$\begin{aligned} \boldsymbol{G}_0(\boldsymbol{x}) &= \lim_{s\to 0} \boldsymbol{G}(\boldsymbol{x},s) = \boldsymbol{G}(\boldsymbol{x},s) - \int\limits_0^s \boldsymbol{G}'(\boldsymbol{x},\tau)d\tau, \\ \boldsymbol{G}_\infty(\boldsymbol{x}) &= \lim_{s\to+\infty} \boldsymbol{G}(\boldsymbol{x},s) = \boldsymbol{G}_0(\boldsymbol{x}) + \int\limits_0^{+\infty} \boldsymbol{G}'(\boldsymbol{x},\tau)d\tau; \end{aligned}$$

$\boldsymbol{E}(\boldsymbol{x},t) = \dfrac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right]$, where $\boldsymbol{u}(\boldsymbol{x},t)$ denotes the displacement vector, is the second-order infinitesimal strain tensor, while $\boldsymbol{E}^t(\boldsymbol{x},s) = \boldsymbol{E}(\boldsymbol{x},t-s)$, $s \in [0,+\infty)$ with respect to every fixed $t \in [0,+\infty)$, denotes the history of the infinitesimal strain tensor at instant t; finally Ω is an open and bounded domain of $\mathbb{R}^3$ with sufficiently regular boundary $\partial\Omega$.

We assume that the following hypotheses are verified $\forall \boldsymbol{x} \in \Omega$:

$$\begin{aligned} & s\boldsymbol{G}'(\boldsymbol{x},s) \in L^1(0,+\infty), \\ & \boldsymbol{G}(\boldsymbol{x},\cdot) - \boldsymbol{G}_\infty(\boldsymbol{x}) = -\int\limits_s^{\infty} \boldsymbol{G}'(\tau)d\tau \in H^{1,1}(0,+\infty) \cap H^{1,2}(0,+\infty), \\ & \lim_{s\to+\infty} s^2\left[\boldsymbol{G}(\boldsymbol{x},s) - \boldsymbol{G}_\infty(\boldsymbol{x})\right] = \boldsymbol{0}; \\ & \left.\begin{aligned} \boldsymbol{G}(\boldsymbol{x},s) &= -\boldsymbol{G}(\boldsymbol{x},-s) \\ \boldsymbol{G}'(\boldsymbol{x},s) &= \boldsymbol{G}'(\boldsymbol{x},-s) \end{aligned}\right. \qquad \forall s \in [0,+\infty). \end{aligned} \tag{2}$$

If and only if $\boldsymbol{G}_0(\boldsymbol{x}) = \boldsymbol{G}_\infty(\boldsymbol{x})$, then, for all $t > 0$

$$\lim_{a\to+\infty} \int\limits_{-at}^{+\infty} a \left[\boldsymbol{G}\left(\boldsymbol{x}, t + \frac{y}{a}\right) - \boldsymbol{G}_\infty(\boldsymbol{x})\right] \left[\frac{\sin y - y\cos y}{y^2}\right] dy = \boldsymbol{0},$$

where $y = a(s - t)$ and $a > 0$.

1. It is assumed that $\boldsymbol{G}'(\boldsymbol{x}, \cdot)$ is continuous $\forall \boldsymbol{x} \in \Omega$ while $\boldsymbol{G}''(\boldsymbol{x}, \cdot)$ is piecewise continuous; furthermore $\boldsymbol{G}'(\boldsymbol{x}, \cdot)$ verifies Dini condition in every point of discontinuity and in a neighbourhood of such points $\boldsymbol{G}''(\boldsymbol{x}, \cdot)$ is bounded.

2. The fourth-order symmetric tensors $\boldsymbol{G}_0(\boldsymbol{x})$ and $\boldsymbol{G}_\infty(\boldsymbol{x})$ are positive definite and continuous in $\overline{\Omega}$; furthermore $\boldsymbol{G}(\boldsymbol{x}, \cdot)$ and $\boldsymbol{G}'(\boldsymbol{x}, \cdot)$ are continuous in $\overline{\Omega}$ with respect to every fixed s.

We remark that conditions $(3)_{1,2,3}$ are all verified if we suppose:

$$\exists\, \alpha \geq 3\colon \lim_{s\to\infty} s^{\alpha+1}\boldsymbol{G}'(s) = \boldsymbol{0}. \tag{3}$$

By the assumed hypotheses we can state the following:

Theorem 1. *If hypoteses (2) hold and if (3) holds by a suitable value of* α *and if* $\boldsymbol{E}(\boldsymbol{x}, t - s) \in \in H^{1,1}(0, +\infty) \cap H^{1,2}(0, +\infty)\ \forall \boldsymbol{x} \in \Omega$ *is an analytic function, then the body* $\boldsymbol{\beta}$ *is of Kelvin – Voigt type, i.e.:*

$$\boldsymbol{T}(\boldsymbol{x}, t) = \boldsymbol{G}_\infty(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{K}(\boldsymbol{x})\dot{\boldsymbol{E}}(\boldsymbol{x}, t) \qquad \forall(\boldsymbol{x}, t) \in \Omega \times [0, dp_\alpha), \quad dp_\alpha < +\infty, \tag{4}$$

where $\boldsymbol{K}(\boldsymbol{x}) = \int\limits_0^{+\infty} [\boldsymbol{G}(\boldsymbol{x}, s) - \boldsymbol{G}_\infty(\boldsymbol{x})]ds$ *is such that:*

$$\begin{gathered}\beta_1\boldsymbol{A}\colon \boldsymbol{A} > -\boldsymbol{A}\colon \int\limits_0^{+\infty} s\boldsymbol{G}'(\boldsymbol{x}, s)dsA = \boldsymbol{A}\colon \boldsymbol{K}(\boldsymbol{x})\boldsymbol{A} > \beta_2\boldsymbol{A}\colon \boldsymbol{A} > 0 \\ \forall\boldsymbol{x} \in \Omega, \quad \forall\boldsymbol{A} \in \mathrm{Sym}(V) \setminus \{\boldsymbol{0}\},\end{gathered} \tag{5}$$

and β_1, β_2 *are positive constants.*

Proof. By Maclaurin formula we have:

$$\boldsymbol{E}^t(\boldsymbol{x}, s) = \boldsymbol{E}(\boldsymbol{x}, t) + (-s)\dot{\boldsymbol{E}}(\boldsymbol{x}, t) + \sigma(\boldsymbol{x}, s^2) \qquad \forall\boldsymbol{x} \in \Omega, \tag{6}$$

where $\lim\limits_{s\to0} \dfrac{\sigma(\boldsymbol{x}, s)}{s^2} = \boldsymbol{0}$.

Using (6) we can rewrite $(1)_1$, by a suitable value of t, in this manner:

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{x}, t) \;=\; &\boldsymbol{G}_0(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}, t) + \int\limits_0^t \boldsymbol{G}'(\boldsymbol{x}, s)\left[\boldsymbol{E}(\boldsymbol{x}, t) - s\dot{\boldsymbol{E}}(\boldsymbol{x}, t)\right] ds + \\ &+ \int\limits_t^{+\infty} \boldsymbol{G}'(\boldsymbol{x}, s)\left[\boldsymbol{E}(\boldsymbol{x}, t) - s\dot{\boldsymbol{E}}(\boldsymbol{x}, t) + \sigma(\boldsymbol{x}, s^2)\right] ds;\end{aligned} \tag{7}$$

remarking that, in the assumed hypotheses, it is possible to replace the limit of the integral with the integral of the limit, and that, within a linear theory, we have that:

$$\lim_{t\to+\infty} \boldsymbol{G}'(\boldsymbol{x},s)\left[\boldsymbol{E}(\boldsymbol{x},t) - s\dot{\boldsymbol{E}}(\boldsymbol{x},t) + \sigma(\boldsymbol{x},s^2)\right] = \boldsymbol{0} \qquad \forall \boldsymbol{x}\in\Omega;$$

if we pass to the limit of the last integral of (7) with $t\to+\infty$, finally by a suitable value of α in (3) and by the analiticity hypothesis of $\boldsymbol{E}^t(\boldsymbol{x},s)$ we have that:

$$\boldsymbol{T}(\boldsymbol{x},t) = \boldsymbol{G}_\infty(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},t) - \int_0^{+\infty} s\boldsymbol{G}'(\boldsymbol{x},s)ds\dot{\boldsymbol{E}}(\boldsymbol{x},t) = \boldsymbol{G}_\infty(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},t) +$$

$$+\int_0^{+\infty}\left[\boldsymbol{G}(\boldsymbol{x},s) - \boldsymbol{G}_\infty(\boldsymbol{x})\right]ds\dot{\boldsymbol{E}}(\boldsymbol{x},t) \;\; \forall\,(\boldsymbol{x},t)\in\Omega\times[0,dp_\alpha), \quad dp_\alpha < +\infty, \qquad (8)$$

that implies (4) setting

$$\boldsymbol{K}(\boldsymbol{x}) = \int_0^{+\infty}\left[\boldsymbol{G}(\boldsymbol{x},s) - \boldsymbol{G}_\infty(\boldsymbol{x})\right]ds;$$

finally (5) is a consequence of theorem IV of [5], because we get $\forall \boldsymbol{x}\in\Omega$

$$\beta_1\boldsymbol{A}\colon\boldsymbol{A} > \boldsymbol{A}\colon \lim_{w\to0}\widehat{\boldsymbol{G}}_c(\boldsymbol{x},w)\boldsymbol{A} = -\boldsymbol{A}\colon\int_0^{+\infty} s\boldsymbol{G}'(\boldsymbol{x},s)ds\boldsymbol{A} =$$

$$= \boldsymbol{A}\colon\int_0^{+\infty}\left[\boldsymbol{G}(\boldsymbol{x},s) - \boldsymbol{G}_\infty(\boldsymbol{x})\right]ds\boldsymbol{A} > \beta_2\boldsymbol{A}\colon\boldsymbol{A},$$

where $\widehat{\boldsymbol{G}}_c(\boldsymbol{x},w) = \int_0^{+\infty}\left[\boldsymbol{G}(\boldsymbol{x},s) - \boldsymbol{G}_\infty(\boldsymbol{x})\right]\cos ws\,ds$ is the dynamic viscosity tensor.

By the above theorem, the theorem I – VI of [5] and the definitions I, II of [6] we are able to complete the examination of viscoelastic materials, that have stronger or weaker or negligible memory, formulating the following definitions:

Definition 1. *A continuum material system expressed by the constitutive functional (1) is said strictly viscoelastic if and only if, in hypotheses (2), the following conditions are verified:*

i) $\boldsymbol{G}(\boldsymbol{x},\cdot) - \boldsymbol{G}_0(\boldsymbol{x}) \notin L^1(0,+\infty)\ \forall\boldsymbol{x}\in\Omega$, $\boldsymbol{G}(\boldsymbol{x},s) = \boldsymbol{G}^T(\boldsymbol{x},s)\ \forall(\boldsymbol{x},s)\in\Omega\times[0,+\infty)$*;*

ii) there exist two constants $\mu_1 > \mu_2 > 0$*, such that:*

$$\mu_1\boldsymbol{A}\colon\boldsymbol{A} > \boldsymbol{A}\colon\left[\boldsymbol{G}_0(\boldsymbol{x}) - \boldsymbol{G}_\infty(\boldsymbol{x})\right]\boldsymbol{A} > \mu_2\boldsymbol{A}\colon\boldsymbol{A} \qquad \forall\boldsymbol{A}\in\mathrm{Sym}(V)\setminus\{\boldsymbol{0}\}\ and\ \forall\boldsymbol{x}\in\Omega,$$

where $\mathrm{Sym}(V)$ *is the second-order Cartesian symmetric tensor space of* $\mathbb{R}^3$*;*

iii) the dynamic viscosity tensor $\widehat{\boldsymbol{G}}_c(\boldsymbol{x},w) = \int_0^{+\infty}\left[\boldsymbol{G}(\boldsymbol{x},s) - \boldsymbol{G}_\infty(\boldsymbol{x})\right]\cos ws\,ds$ *is positive definite and bounded, i.e. there exist two constants* $\beta_1 > \beta_2 > 0$*, independent of* w*, such that:*

$$\beta_1\boldsymbol{A}\colon\boldsymbol{A} > \boldsymbol{A}\colon\widehat{\boldsymbol{G}}_s(\boldsymbol{x},w)\boldsymbol{A} > \beta_2\boldsymbol{A}\colon\boldsymbol{A} \qquad \forall\,\boldsymbol{A}\in\mathrm{Sym}(V)\setminus\{\boldsymbol{0}\},$$

$$\forall\,w\in(-\infty,+\infty)\ and\ \forall\,\boldsymbol{x}\in\Omega;$$

particularly, $\forall \boldsymbol{x} \in \Omega$, we have:

$$\beta_1 \boldsymbol{A}\colon \boldsymbol{A} > \boldsymbol{A}\colon \lim_{w\to 0} \widehat{\boldsymbol{G}}_c(\boldsymbol{x}, w)\boldsymbol{A} = -\boldsymbol{A}\colon \int\limits_0^{+\infty} s\boldsymbol{G}'(\boldsymbol{x}, s)ds\boldsymbol{A} =$$

$$= \boldsymbol{A}\colon \int\limits_0^{+\infty} [\boldsymbol{G}(\boldsymbol{x}, s) - \boldsymbol{G}_\infty(\boldsymbol{x})]\, ds\boldsymbol{A} > \beta_2 \boldsymbol{A}\colon \boldsymbol{A}$$

and $\lim\limits_{w\to\pm\infty} \widehat{\boldsymbol{G}}_c(\boldsymbol{x}, w) = \boldsymbol{0}$;

iv) $\forall \boldsymbol{x} \in \Omega$, $\forall\ \boldsymbol{A} \in \mathrm{Sym}(V) \setminus \{\boldsymbol{0}\}$, $\exists \nu_1, \nu_2 > 0$ *such that:*

$$\boldsymbol{A}\colon \left[\boldsymbol{G}_0(\boldsymbol{x}) + \widehat{\boldsymbol{G}}_c'(\boldsymbol{x}, w)\right]\boldsymbol{A} = \boldsymbol{A}\colon \left[\boldsymbol{G}_\infty(\boldsymbol{x}) + w\widehat{\boldsymbol{G}}_s(\boldsymbol{x}, w)\right]\boldsymbol{A} \geq \nu_1 \boldsymbol{A}\colon \boldsymbol{A} \qquad \forall\ w \in R,$$

$$w^2 \boldsymbol{A}\colon \widehat{\boldsymbol{G}}_c(\boldsymbol{x}, w)\boldsymbol{A} = -w\widehat{\boldsymbol{G}}_s'(\boldsymbol{x}, w) \geq \nu_2 \boldsymbol{A}\colon \boldsymbol{A} \qquad \forall\ w \neq 0,$$

where

$$\widehat{\boldsymbol{G}}_c'(\boldsymbol{x}, w) = \int\limits_0^{+\infty} \boldsymbol{G}'(\boldsymbol{x}, s)\cos ws\, ds, \qquad \widehat{\boldsymbol{G}}_s(\boldsymbol{x}, w) = \int\limits_0^{+\infty} [\boldsymbol{G}(\boldsymbol{x}, s) - \boldsymbol{G}_\infty(\boldsymbol{x})]\sin ws\, ds,$$

$$\widehat{\boldsymbol{G}}_s'(\boldsymbol{x}, w) = \int\limits_0^{+\infty} \boldsymbol{G}'(\boldsymbol{x}, s)\sin ws\, ds$$

and ν_1, ν_2 don't depend on w, particularly, $\forall \boldsymbol{x} \in \Omega$, we have:

$$\boldsymbol{A}\colon \lim_{w\to 0}\left[\boldsymbol{G}_0(\boldsymbol{x}) + \widehat{\boldsymbol{G}}_c'(\boldsymbol{x}, w)\right]\boldsymbol{A} = \boldsymbol{A}\colon \lim_{w\to 0}\left[\boldsymbol{G}_\infty(\boldsymbol{x}) + w\widehat{\boldsymbol{G}}_s(\boldsymbol{x}, w)\right]\boldsymbol{A} =$$

$$= \boldsymbol{A}\colon \boldsymbol{G}_\infty(\boldsymbol{x})\boldsymbol{A} \geq \nu_1 \boldsymbol{A}\colon \boldsymbol{A},$$

$$\boldsymbol{A}\colon \lim_{w\to\pm\infty}\left[\boldsymbol{G}_0(\boldsymbol{x}) + \widehat{\boldsymbol{G}}_c'(\boldsymbol{x}, w)\right]\boldsymbol{A} = \boldsymbol{A}\colon \lim_{w\to\pm\infty}\left[\boldsymbol{G}_\infty(\boldsymbol{x}) + w\widehat{\boldsymbol{G}}_s(\boldsymbol{x}, w)\right]\boldsymbol{A} =$$

$$= \boldsymbol{A}\colon \boldsymbol{G}_0(\boldsymbol{x})\boldsymbol{A} > \nu_1 \boldsymbol{A}\colon \boldsymbol{A}.$$

Definition 2. *If and only if $\boldsymbol{G}(\boldsymbol{x}, \cdot) - \boldsymbol{G}_0(\boldsymbol{x}) \notin L^1(0, +\infty)$ $\forall \boldsymbol{x} \in \Omega$, a continuum material system described by the constitutive functional (1) is a linear viscoelastic body of Kelvin – Voigt type, if by hypotheses of theorem 1 the following conditions are verified:*

i) $\boldsymbol{T}(\boldsymbol{x}, t) = \boldsymbol{G}_\infty(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{K}(\boldsymbol{x})\dot{\boldsymbol{E}}(\boldsymbol{x}, t)$ $\forall (\boldsymbol{x}, t) \in \Omega \times [0, dp_\alpha)$, $dp_\alpha < +\infty$, *where* $\boldsymbol{K}(\boldsymbol{x}) = \int\limits_0^{+\infty} [\boldsymbol{G}(\boldsymbol{x}, s) - \boldsymbol{G}_\infty(\boldsymbol{x})]\, ds$ *is such that:*

$$\beta_1 \boldsymbol{A}\colon \boldsymbol{A} > \boldsymbol{A}\colon \lim_{w\to 0} \widehat{\boldsymbol{G}}_c(\boldsymbol{x}, w)\boldsymbol{A} = -\boldsymbol{A}\colon \int\limits_0^{+\infty} s\boldsymbol{G}'(\boldsymbol{x}, s)ds\boldsymbol{A} =$$

$$= \boldsymbol{A}\colon \int\limits_0^{+\infty} [\boldsymbol{G}(\boldsymbol{x}, s) - \boldsymbol{G}_\infty(\boldsymbol{x})]\, ds\boldsymbol{A} > \beta_2 \boldsymbol{A}\colon \boldsymbol{A} \qquad \forall\ \boldsymbol{x} \in \Omega, \forall A \in \mathrm{Sym}(V) \setminus \{\boldsymbol{0}\}$$

and β_1, β_2 are positive constants;

ii) $\boldsymbol{G}(\boldsymbol{x},s)=\boldsymbol{G}^T(\boldsymbol{x},s)\in\Omega\times[0,+\infty)$.

Definition 3. *If hypotheses (2) hold, and if and only if* $\boldsymbol{G}(\boldsymbol{x},\cdot)-\boldsymbol{G}_\infty(\boldsymbol{x}),\boldsymbol{G}(\boldsymbol{x},\cdot)-\boldsymbol{G}_0(\boldsymbol{x})\in$ $\in L^1(0,+\infty)$ $\forall\boldsymbol{x}\in\Omega$, *body* $\boldsymbol{\beta}$ *is of linear elastic type, i.e.*

$$\boldsymbol{T}(\boldsymbol{x},t)-\boldsymbol{G}_0(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},t)=\boldsymbol{G}_\infty(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},t)\quad\forall\, t\in[0,T_c)\quad where\quad T_c<+\infty;$$

if and only if

$$\boldsymbol{T}(\boldsymbol{x})=\boldsymbol{G}_0(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x})=\boldsymbol{G}_\infty(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x})$$

then body $\boldsymbol{\beta}$ *is linear elastic.*

3. The quasistatic problem for a strictly viscoelastic body expressed by definition 1 is formulated by the following Dirichlet problem:

$$\nabla\cdot\left\{\boldsymbol{G}_\infty(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t)+\int\limits_0^{+\infty}[\boldsymbol{G}(\boldsymbol{x},s)-\boldsymbol{G}_\infty(\boldsymbol{x})]\,\nabla\dot{\boldsymbol{u}}^t(\boldsymbol{x},s)\,ds\right\}+\boldsymbol{b}(\boldsymbol{x},t)=$$

$$=\nabla\cdot\left\{\boldsymbol{G}_\infty(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t)+\int\limits_0^{+\infty}\boldsymbol{G}'(\boldsymbol{x},s)\nabla\boldsymbol{u}^t(\boldsymbol{x},s)\,ds\right\}+\boldsymbol{b}(\boldsymbol{x},t)=\boldsymbol{0},\quad(\boldsymbol{x},t)\in Q,\tag{9}$$

$$\boldsymbol{u}(\boldsymbol{x},t)\big|_{\partial\Omega}=\boldsymbol{0},$$

where

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{u}_\infty(\boldsymbol{x}),\quad\lim_{t\to+\infty}\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{u}_\infty(\boldsymbol{x}),\quad\boldsymbol{b}(\boldsymbol{x},t)=\boldsymbol{b}(\boldsymbol{x},t)-\boldsymbol{b}_\infty(\boldsymbol{x})$$

and

$$\lim_{t\to+\infty}\boldsymbol{b}(\boldsymbol{x},t)=\boldsymbol{b}_\infty(\boldsymbol{x}).$$

Relating to this problem we have proved [6] the following

Theorem 2. *If and only if body* $\boldsymbol{\beta}$ *is strictly viscoelastic according to definition 1, if* $\boldsymbol{b}(\boldsymbol{x},t)\in$ $\in L^1(R;H^{1,2}(\Omega))\cap L^2(R;H^{1,2}(\Omega))$, $\boldsymbol{b}(\boldsymbol{x},\cdot)\in S_\infty(R)$ *and has compact support in* R, *there exists one and only one solution with compact support* $\boldsymbol{u}(\boldsymbol{x},t)\in H^{1,1}(R;H^{1,2}(\Omega))\cap H^{1,2}(R;H^{1,2}(\Omega))$, $\boldsymbol{u}(\boldsymbol{x},\cdot)\in S_\infty(R)$, *such that:*

$$\int\limits_{\Omega'}\left\{\boldsymbol{G}_\infty(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t)+\int\limits_0^{+\infty}[\boldsymbol{G}(\boldsymbol{x},s)-\boldsymbol{G}_\infty]\,\nabla\dot{\boldsymbol{u}}^t(\boldsymbol{x},s)\,ds\right\}:\nabla\boldsymbol{H}(\boldsymbol{x},\boldsymbol{x}',t)d\boldsymbol{x}'=$$

$$=\int\limits_{\Omega'}\left\{\boldsymbol{G}_0(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t)+\int\limits_0^{+\infty}\boldsymbol{G}'(\boldsymbol{x},s)\nabla\boldsymbol{u}^t(\boldsymbol{x},s)\,ds\right\}\nabla\boldsymbol{H}(\boldsymbol{x},\boldsymbol{x}',t)\,d\boldsymbol{x}'=$$

$$=\int\limits_{\Omega'}\boldsymbol{b}(\boldsymbol{x},t)\boldsymbol{H}(\boldsymbol{x},\boldsymbol{x}',t)d\boldsymbol{x}'\tag{10}$$

$$\forall\,\boldsymbol{H}(\boldsymbol{x},\boldsymbol{x}',t)\in L^\infty(-\infty,+\infty;H^{1,2}(\Omega)\times H^{1,2}(\Omega')):\ \boldsymbol{H}(\boldsymbol{x},\boldsymbol{x}',t)\big|_{\partial\Omega}=\boldsymbol{0},$$

where $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{x}',t)$ is strongly measurable, if $\boldsymbol{x}'\neq\boldsymbol{x}$, and $S_\infty(R)$ denote the class of infinitely many time differentiable functions $\boldsymbol{u}(\boldsymbol{x},t)$ with respect to t for which there exists a set of constants C_{pq}, dependent on same function $\boldsymbol{u}(\boldsymbol{x},t)$ and on numbers p and q, such that:

$$\int_\Omega \left|t^p\partial_t^{(q)}\boldsymbol{u}(\boldsymbol{x},t)\right|^2 d\boldsymbol{x} < C_{pq}^2, \qquad \int_\Omega \left|t^p\partial_t^{(q)}\nabla\boldsymbol{u}(\boldsymbol{x},t)\right|^2 d\boldsymbol{x} < C_{pq}^2.$$

In [6] we verify that (10) holds if we inversely transform by Fourier the solution of the Fourier transformed problem of (9); this solution is null outside at a compact interval of the time origin, because of a condition of compatibility with the meaning itself of the quasistatic problem.

This consideration and definition 2 imply that a solution of the quasistatic problem for a viscoelastic body descrebed by definition 2 must be determined as limit of this Fourier inverse transformed solution when $w\to 0$.

Consequently we can state relating to the Dirichlet problem:

$$\begin{aligned}
&\nabla\cdot\left\{\boldsymbol{G}_\infty(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t)+\int_0^{+\infty}[\boldsymbol{G}(\boldsymbol{x},s)-\boldsymbol{G}_\infty(\boldsymbol{x})]\,ds\nabla\dot{\boldsymbol{u}}(\boldsymbol{x},t)\right\}+\boldsymbol{b}(\boldsymbol{x},t)=\\
&=\nabla\cdot\left\{\boldsymbol{G}_\infty(\boldsymbol{x})\nabla\boldsymbol{u}(\boldsymbol{x},t)-\int_0^{+\infty}s\boldsymbol{G}'(\boldsymbol{x},s)ds\nabla\dot{\boldsymbol{u}}(\boldsymbol{x},t)\right\}+\boldsymbol{b}(\boldsymbol{x},t)=\boldsymbol{0},\\
&\qquad(\boldsymbol{x},t)\in\Omega\times[0,dp_\alpha),\ dp_\alpha<\infty,\\
&\boldsymbol{u}(\boldsymbol{x},t)\big|_{\partial\Omega}=\boldsymbol{0},
\end{aligned}\tag{11}$$

the following

Theorem 3. *If body $\boldsymbol{\beta}$ is a linear viscoelastic material system according to definition 2, if $\boldsymbol{b}(\boldsymbol{x},t)\in L^2(I_\alpha;H^{1,2}(\Omega))$, where $I_\alpha=[0,dp_\alpha)$, is analytic and has compact support, only null solution solves the problem (11).*

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Received 22.03.99