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OSCILLATION OF FOURTH-ORDER DELAY DIFFERENTIAL EQUATIONS*

КОЛИВАННЯ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ЧЕТВЕРТОГО ПОРЯДКУ З ЗАГАЮВАННЯМ

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This article is concerned with oscillation of a certain class of fourth-order delay differential equations. Some new oscillation criteria are presented which include Hille and Nehari type. The results obtained improve some results obtained earlier in [Zhang C., Li T., Sun B., Thandapani E. On the oscillation of higher-order half-linear delay differential equations // Appl. Math. Lett. — 2011. — **24***. — P. 1618 – 1621]. Two examples are considered to illustrate the main results.*

Розглянуто коливання в деякому класі диференціальних рівнянь четвертого порядку з загаюванням. Знайдено нові критерії коливання, які включають в себе критерії типу Хілле та Нехарі. Отримані результати покращують деякі результати з [Zhang C., Li T., Sun B., Thandapani E. On the oscillation of higher-order half-linear delay differential equations // Appl. Math. Lett. — 2011. — **24***. — P. 1618 – 1621]. Розглянуто два приклади, які ілюструють основні результати.*

1. Introduction. In this paper, we are concerned with oscillation of the fourth-order quasilinear delay differential equation

$$\left(r(t)\left(x'''(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}(\tau(t)) = 0, \quad \text{for} \quad t \geq t_0. \tag{1.1}$$

We will assume that the following assumptions hold:

(H_1) α is a quotient of odd positive integers;

(H_2) $r \in \mathrm{C}^1[t_0,\infty)$, $r'(t) \geq 0$, $r(t) > 0$, q, $\tau \in \mathrm{C}[t_0,\infty)$, $q(t) \geq 0$, $\tau(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \infty$.

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By a solution of (1.1), we mean a function $x \in C^3[T_x, \infty)$, $T_x \geq t_0$, which has the property $r(x''')^\alpha \in C^1[T_x, \infty)$ and satisfies (1.1) on $[T_x, \infty)$. We consider only those solutions x of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In recent decades, the oscillation of second-order and third-order differential equations have been deeply studied in the literature, we refer the reader to the related books [1, 3 – 5, 13, 15, 21] and the papers [2, 6 – 12, 14, 16 – 20, 22]. In the following, we present some related results that serve and motivate the contents of this paper. Agarwal et al. [2], Kamo and Usami [11, 12], and Kusano et al. [14] considered the oscillation of the fourth-order nonlinear differential equation

$$\left(r(t)\left(x''(t)\right)^\alpha\right)'' + q(t)x^\beta(t) = 0.$$

Grace et al. [10] examined the oscillation behavior of the fourth-order nonlinear differential equation

$$\left(r(t)\left(x'(t)\right)^\alpha\right)''' + q(t)f(x(g(t))) = 0.$$

Agarwal et al. [7] and Zhang et al. [22] studied the oscillatory properties of the higher-order differential equation

$$\left(r(t)\left(x^{(n-1)}(t)\right)^\alpha\right)' + q(t)x^\beta(\tau(t)) = 0, \tag{1.2}$$

under the conditions

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\, dt = \infty,$$

and

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\, dt < \infty. \tag{1.3}$$

Zhang et al. [22] obtained some results which ensure that every solution x of (1.2) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$ for the case where (1.3) holds. As a special case when $n = 4$, they proved the following result: Let (H_1), (H_2), and (1.3) hold, and $\tau(t) < t$. Further, assume that for some constant $\lambda_0 \in (0, 1)$, the delay differential equation

$$y'(t) + q(t)\left(\frac{\lambda_0 \tau^3(t)}{6r^{1/\alpha}(\tau(t))}\right)^\alpha y(\tau(t)) = 0 \tag{1.4}$$

is oscillatory. If

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[q(s)\left(\frac{\lambda_1}{2}\tau^2(s)\right)^\alpha \delta^\alpha(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\delta(s) r^{1/\alpha}(s)}\right] ds = \infty \tag{1.5}$$

for some constant $\lambda_1 \in (0,1)$, where $\delta(t) := \int_t^{\infty} r^{-(1/\alpha)}(s)\,ds$, then every solution of (1.1) is either oscillatory or converges to zero as $t \to \infty$.

Our aim in this paper is to employ the Riccati technique to establish some new conditions for oscillation of all solutions of (1.1). The results not only differ from the results obtained in [22], but also improve some of them. Some examples are considered to illustrate the main results.

2. Main results. In this section, we will derive some new criteria for oscillation of (1.1). To prove the main results we will need the following lemma.

Lemma 2.1 (see [3], Lemma 2.2.3). *Let $f \in \mathrm{C}^n([t_0,\infty),\mathbb{R}^+)$. Assume that $f^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$, and there exists a $t_1 \ge t_0$ such that $f^{(n-1)}(t)f^{(n)}(t) \le 0$ for all $t \ge t_1$. If $\lim_{t\to\infty} f(t) \ne 0$, then for every $k \in (0,1)$, there exists $t_k \in [t_1,\infty)$ such that*

$$f(t) \ge \frac{k}{(n-1)!} t^{n-1} |f^{(n-1)}(t)|, \quad \textit{for} \quad t \in [t_k,\infty).$$

Now, we are ready to state and prove the main results. For convenience, we denote

$$R(t) := \int_t^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)}\,ds, \quad \rho'_+(t) := \max\{0,\rho'(t)\}, \quad \text{and} \quad \theta'_+(t) := \max\{0,\theta'(t)\}.$$

In the sequel, all occurring functional inequalities considered in this section are assumed to hold eventually, that is, they are satisfied for all t large enough.

Theorem 2.1. *Let (H_1), (H_2), and (1.3) hold. Assume that there exists a positive function $\rho \in \mathrm{C}^1[t_0,\infty)$ such that*

$$\int_{t_0}^{\infty} \left[q(s) \left(\frac{\tau^2(s)}{s^2} \right)^{\alpha} \rho(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r(s)(\rho'_+(s))^{\alpha+1}}{(k_1\rho(s)s^2)^{\alpha}} \right] ds = \infty, \tag{2.1}$$

for some constant $k_1 \in (0,1)$. Assume further that there exists a positive function $\theta \in \mathrm{C}^1[t_0,\infty)$ such that

$$\int_{t_0}^{\infty} \left[\theta(s) \int_s^{\infty} \left[\frac{1}{r(\vartheta)} \int_{\vartheta}^{\infty} q(\varsigma) \left(\frac{\tau^2(\varsigma)}{\varsigma^2} \right)^{\alpha} d\varsigma \right]^{\frac{1}{\alpha}} d\vartheta - \frac{(\theta'_+(s))^2}{4\theta(s)} \right] ds = \infty. \tag{2.2}$$

If

$$\int_{t_0}^{\infty} \left[q(s) \left(\int_s^{\infty}\int_u^{\infty} R(v)\,dv\,du \right)^{\alpha} - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{\int_s^{\infty} R(v)dv}{\int_s^{\infty}\int_u^{\infty} R(v)dvdu} \right] ds = \infty, \tag{2.3}$$

and

$$\int_{t_0}^{\infty} \left[q(s) \left(\frac{k_2}{2} \tau^2(s) \right)^{\alpha} R^{\alpha}(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1} R(s) r^{1/\alpha}(s)} \right] ds = \infty, \tag{2.4}$$

for some constant $k_2 \in (0,1)$, *then every solution of (1.1) is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution x. Without loss of generality we may assume that x is eventually positive. It follows from (1.1) that there exist four possible cases for $t \geq t_1$, where $t_1 \geq t_0$ is large enough:

$$Case\ 1: \ x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) > 0, x^{(4)}(t) \leq 0, (r(x''')^{\alpha})'(t) \leq 0.$$

$$Case\ 2: \ x(t) > 0, x'(t) > 0, x''(t) < 0, x'''(t) > 0, x^{(4)}(t) \leq 0, (r(x''')^{\alpha})'(t) \leq 0.$$

$$Case\ 3: \ x(t) > 0, x'(t) < 0, x''(t) > 0, x'''(t) < 0, (r(x''')^{\alpha})'(t) \leq 0.$$

$$Case\ 4: \ x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) < 0, (r(x''')^{\alpha})'(t) \leq 0.$$

Assume that Case 1 holds. By Kiguradze Lemma [13], we have $x(t) \geq (t/2)x'(t)$, and so

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\tau^2(t)}{t^2}. \tag{2.5}$$

Define

$$\omega(t) := \rho(t) \frac{r(t)(x''')^{\alpha}(t)}{x^{\alpha}(t)}, \quad t \geq t_1. \tag{2.6}$$

Then $\omega(t) > 0$ for $t \geq t_1$, and

$$\omega'(t) = \rho'(t) \frac{r(t)(x''')^{\alpha}(t)}{x^{\alpha}(t)} + \rho(t) \frac{(r(x''')^{\alpha})'(t)}{x^{\alpha}(t)} - \alpha\rho(t) \frac{x^{\alpha-1}(t)x'(t)r(t)(x''')^{\alpha}(t)}{x^{2\alpha}(t)}. \tag{2.7}$$

From Lemma 2.1, we have

$$x'(t) \geq \frac{k}{2} t^2 x'''(t) \tag{2.8}$$

for every $k \in (0,1)$ and for all sufficiently large t. Hence, we obtain by (2.7) and (2.8) that

$$\omega'(t) \leq \rho'(t) \frac{r(t)(x''')^{\alpha}(t)}{x^{\alpha}(t)} + \rho(t) \frac{(r(x''')^{\alpha})'(t)}{x^{\alpha}(t)} - \frac{\alpha k}{2} t^2 \rho(t) \frac{x'''(t)r(t)(x''')^{\alpha}(t)}{x^{\alpha+1}(t)}.$$

Hence by (1.1), we get

$$\omega'(t) \leq -q(t) \left(\frac{\tau^2(t)}{t^2} \right)^{\alpha} \rho(t) + \frac{\rho'_+(t)}{\rho(t)} \omega(t) - \frac{\alpha k}{2} \frac{t^2}{(r(t)\rho(t))^{\frac{1}{\alpha}}} \omega^{\frac{\alpha+1}{\alpha}}(t). \tag{2.9}$$

Set

$$A := \frac{\alpha k t^2}{2(r(t)\rho(t))^{\frac{1}{\alpha}}}, \quad B := \frac{\rho_+^{'}(t)}{\rho(t)}, \quad y := \omega(t).$$

Using the inequality

$$By - Ay^{\frac{\alpha+1}{\alpha}} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad A, B > 0,$$

we have

$$\frac{\rho_+^{'}(t)}{\rho(t)}\,\omega(t) - \frac{\alpha k t^2}{2(r(t)\rho(t))^{\frac{1}{\alpha}}}\,\omega^{\frac{\alpha+1}{\alpha}}(t) \le \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(t)(\rho_+^{'}(t))^{\alpha+1}}{(k\rho(t)t^2)^\alpha}.$$

Hence, we obtain

$$\omega^{'}(t) \le -q(t)\left(\frac{\tau^2(t)}{t^2}\right)^\alpha \rho(t) + \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(t)(\rho_+^{'}(t))^{\alpha+1}}{(k\rho(t)t^2)^\alpha},$$

which implies that

$$\int_{t_1}^{t} \left[q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha \rho(s) - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(s)(\rho_+^{'}(s))^{\alpha+1}}{(k\rho(s)s^2)^\alpha}\right] ds \le \omega(t_1),$$

for every $k \in (0,1)$ and for all sufficiently large t. This is a contradiction to (2.1). Assume that Case 2 holds. Integrating (1.1) from t to l, we have

$$r(l)(x^{'''})^\alpha(l) - r(t)(x^{'''})^\alpha(t) + \int_t^l q(s)x^\alpha(\tau(s))\,ds = 0.$$

By virtue of $x > 0$, $x^{'} > 0$, and $x^{''} < 0$, we get $x(t) \ge (t/2)x^{'}(t)$, and so (2.5) holds. Then by (2.5), we have

$$r(l)(x^{'''})^\alpha(l) - r(t)(x^{'''})^\alpha(t) + \int_t^l q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha x^\alpha(s)\,ds \le 0,$$

from which follows by $x^{'} > 0$ that

$$r(l)(x^{'''})^\alpha(l) - r(t)(x^{'''})^\alpha(t) + x^\alpha(t)\int_t^l q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds \le 0.$$

Letting $l \to \infty$, we have

$$-r(t)(x^{'''})^\alpha(t) + x^\alpha(t)\int_t^\infty q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds \le 0,$$

i.e.,

$$-x'''(t) + x(t)\left[\frac{1}{r(t)}\int\limits_t^\infty q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds\right]^{\frac{1}{\alpha}} \le 0.$$

Integrating again from t to ∞, we get

$$x''(t) + x(t)\int\limits_t^\infty \left[\frac{1}{r(\vartheta)}\int\limits_\vartheta^\infty q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds\right]^{\frac{1}{\alpha}} d\vartheta \le 0. \tag{2.10}$$

Define

$$\xi(t) := \theta(t)\frac{x'(t)}{x(t)}, \quad t \ge t_1.$$

Then $\xi(t) > 0$ for $t \ge t_1$, and

$$\xi'(t) = \theta'(t)\frac{x'(t)}{x(t)} + \theta(t)\,\frac{x''(t)x(t) - (x')^2(t)}{x^2(t)} =$$

$$= \theta(t)\,\frac{x''(t)}{x(t)} + \frac{\theta'(t)}{\theta(t)}\xi(t) - \frac{\xi^2(t)}{\theta(t)}.$$

Hence by (2.10), we get

$$\xi'(t) \le -\theta(t)\int\limits_t^\infty \left[\frac{1}{r(\vartheta)}\int\limits_\vartheta^\infty q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds\right]^{\frac{1}{\alpha}} d\vartheta + \frac{\theta'_+(t)}{\theta(t)}\xi(t) - \frac{\xi^2(t)}{\theta(t)}. \tag{2.11}$$

Thus, we have

$$\xi'(t) \le -\theta(t)\int\limits_t^\infty \left[\frac{1}{r(\vartheta)}\int\limits_\vartheta^\infty q(s)\left(\frac{\tau^2(s)}{s^2}\right)^\alpha ds\right]^{\frac{1}{\alpha}} d\vartheta + \frac{(\theta'_+(t))^2}{4\theta(t)},$$

which yields

$$\int\limits_{t_1}^t \left[\theta(s)\int\limits_s^\infty \left[\frac{1}{r(\vartheta)}\int\limits_\vartheta^\infty q(\varsigma)\left(\frac{\tau^2(\varsigma)}{\varsigma^2}\right)^\alpha d\varsigma\right]^{\frac{1}{\alpha}} d\vartheta - \frac{(\theta'_+(s))^2}{4\theta(s)}\right] ds \le \xi(t_1),$$

which contradicts (2.2). Assume that Case 3 holds. Recalling that $r(x''')^\alpha$ is nonincreasing, we have

$$r^{1/\alpha}(s)x'''(s) \le r^{1/\alpha}(t)x'''(t), \quad s \ge t \ge t_1.$$

Dividing the above inequality by $r^{1/\alpha}(s)$ and integrating the resulting inequality from t to l, we obtain

$$x^{''}(l) \leq x^{''}(t) + r^{1/\alpha}(t)x^{'''}(t)\int_t^l r^{-1/\alpha}(s)\,ds.$$

Letting $l \to \infty$, we get

$$x^{''}(t) \geq -r^{1/\alpha}(t)x^{'''}(t)R(t). \tag{2.12}$$

Integrating (2.12) from t to ∞, we have

$$-x^{'}(t) \geq \int_t^\infty -r^{1/\alpha}(s)x^{'''}(s)R(s)\,ds \geq -r^{1/\alpha}(t)x^{'''}(t)\int_t^\infty R(s)\,ds. \tag{2.13}$$

Integrating (2.13) from t to ∞, we get

$$x(t) \geq \int_t^\infty -r^{1/\alpha}(u)x^{'''}(u)\int_u^\infty R(s)\,ds\,du \geq -r^{1/\alpha}(t)x^{'''}(t)\int_t^\infty\int_u^\infty R(s)\,ds\,du. \tag{2.14}$$

We define

$$\varphi(t) := \frac{r(t)(x^{'''})^\alpha(t)}{x^\alpha(t)}, \quad t \geq t_1. \tag{2.15}$$

Then $\varphi(t) < 0$, for $t \geq t_1$, and by (2.13), we have that

$$\varphi^{'}(t) = \frac{(r(x^{'''})^\alpha)^{'}(t)}{x^\alpha(t)} - \alpha\frac{r(t)(x^{'''})^\alpha(t)x^{'}(t)}{x^{\alpha+1}(t)} \leq$$

$$\leq -q(t)\frac{x^\alpha(\tau(t))}{x^\alpha(t)} - \alpha\frac{r^{\frac{\alpha+1}{\alpha}}(t)(x^{'''})^{\alpha+1}(t)}{x^{\alpha+1}(t)}\int_t^\infty R(s)\,ds. \tag{2.16}$$

Hence by (2.15) and (2.16), we obtain

$$\varphi^{'}(t) \leq -q(t) - \alpha\varphi^{\frac{\alpha+1}{\alpha}}(t)\int_t^\infty R(s)\,ds. \tag{2.17}$$

From (2.14), we get

$$\varphi(t)\left(\int_t^\infty\int_u^\infty R(s)\,ds\,du\right)^\alpha \geq -1. \tag{2.18}$$

Multiplying (2.17) by $\left(\int_t^\infty \int_u^\infty R(s)\,ds\,du\right)^\alpha$ and integrating the resulting inequality from t_1 to t, we have

$$\left(\int_t^\infty\int_u^\infty R(s)\,ds\,du\right)^\alpha \varphi(t) - \left(\int_{t_1}^\infty\int_u^\infty R(s)\,ds\,du\right)^\alpha \varphi(t_1)+$$

$$+\alpha\int_{t_1}^t\int_s^\infty R(v)\,dv\left(\int_s^\infty\int_u^\infty R(v)\,dv\,du\right)^{\alpha-1}\varphi(s)\,ds+$$

$$+\int_{t_1}^t q(s)\left(\int_s^\infty\int_u^\infty R(v)\,dv\,du\right)^{\alpha} ds+$$

$$+\alpha\int_{t_1}^t \varphi^{\frac{\alpha+1}{\alpha}}(s)\left(\int_s^\infty\int_u^\infty R(v)\,dv\,du\right)^{\alpha}\int_s^\infty R(v)\,dv\,ds \le 0.$$

Set

$$B := \int_s^\infty R(v)\,dv\left(\int_s^\infty\int_u^\infty R(v)\,dv\,du\right)^{\alpha-1},$$

and

$$A := \left(\int_s^\infty\int_u^\infty R(v)\,dv\,du\right)^{\alpha}\int_s^\infty R(v)\,dv, \quad y := -\varphi(s).$$

Using the inequality

$$-By + Ay^{\frac{\alpha+1}{\alpha}} \ge -\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha}, \quad A,B > 0, \tag{2.19}$$

we have

$$\int_s^\infty R(v)\,dv\left(\int_s^\infty\int_u^\infty R(v)\,dv\,du\right)^{\alpha-1}\varphi(s) + \varphi^{\frac{\alpha+1}{\alpha}}(s)\left(\int_s^\infty\int_u^\infty R(v)\,dv\,du\right)^{\alpha}\int_s^\infty R(v)\,dv \ge$$

$$\ge -\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{\int_s^\infty R(v)\,dv}{\int_s^\infty\int_u^\infty R(v)\,dv\,du}.$$

Hence, we obtain by (2.18) that

$$\int_{t_1}^{t}\left[q(s)\left(\int_{s}^{\infty}\int_{u}^{\infty}R(v)\,dv\,du\right)^{\alpha}-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{\int_s^\infty R(v)\,dv}{\int_s^\infty\int_u^\infty R(v)\,dv\,du}\right]ds\le$$

$$\le\left(\int_{t_1}^{\infty}\int_{u}^{\infty}R(s)\,ds\,du\right)^{\alpha}\varphi(t_1)+1.$$

This is a contradiction to (2.3). Assume that Case 4 holds. In view of the proof of Case 3, we have (2.12). On the other hand, by Lemma 2.1, we get

$$x(t)\ge\frac{k}{2}\,t^2x^{''}(t)\tag{2.20}$$

for every $k\in(0,1)$ and for all sufficiently large t. Now define

$$\phi(t):=\frac{r(t)(x^{'''})^{\alpha}(t)}{(x^{''})^{\alpha}(t)},\quad t\ge t_1.\tag{2.21}$$

Then $\phi(t)<0$ for $t\ge t_1$, and by (2.20) and (2.21), we get that

$$\phi^{'}(t)=-q(t)\frac{x^{\alpha}(\tau(t))}{(x^{''}(\tau(t)))^{\alpha}}\,\frac{(x^{''}(\tau(t)))^{\alpha}}{(x^{''})^{\alpha}(t)}-\alpha\,\frac{\phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}\le-q(t)\left(\frac{k}{2}\tau^2(t)\right)^{\alpha}-\alpha\frac{\phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}.\tag{2.22}$$

Multiplying the above inequality by $R^{\alpha}(t)$ and integrating the resulting inequality from t_1 to t, we have

$$R^{\alpha}(t)\phi(t)-R^{\alpha}(t_1)\phi(t_1)+\alpha\int_{t_1}^{t}r^{-1/\alpha}(s)R^{\alpha-1}(s)\phi(s)\,ds\le$$

$$\le-\int_{t_1}^{t}q(s)\left(\frac{k}{2}\,\tau^2(s)\right)^{\alpha}R^{\alpha}(s)\,ds-\alpha\int_{t_1}^{t}\frac{\phi^{(\alpha+1)/\alpha}(s)}{r^{1/\alpha}(s)}\,R^{\alpha}(s)\,ds.$$

Set $B:=r^{-1/\alpha}(s)R^{\alpha-1}(s)$, $A:=R^{\alpha}(s)/r^{1/\alpha}(s)$, and $y:=-\phi(s)$. Using the inequality (2.19) and (2.12), we have, for every $k\in(0,1)$ and for all sufficiently large t,

$$\int_{t_1}^{t}\left[q(s)\left(\frac{k}{2}\tau^2(s)\right)^{\alpha}R^{\alpha}(s)-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\,\frac{1}{R(s)r^{1/\alpha}(s)}\right]ds\le R^{\alpha}(t_1)\phi(t_1)+1.$$

This is a contradiction to (2.4).

Theorem 2.1 is proved.

It is well known (see [8]) that the differential equation

$$(a(t)(x'(t))^{\alpha})' + q(t)x^{\alpha}(t) = 0, \tag{2.23}$$

where $\alpha > 0$ is a ratio of odd positive integers, $a, q \in \mathrm{C}([t_0, \infty), \mathbb{R}^+)$ is nonoscillatory if and only if there exist a number $T \geq t_0$ and a function $v \in \mathrm{C}^1([T, \infty), \mathbb{R})$ which satisfies the inequality

$$v'(t) + \alpha a^{-1/\alpha}(t)(v(t))^{(1+\alpha)/\alpha} + q(t) \leq 0, \quad \text{on} \quad [T, \infty).$$

In the following, we compare the oscillatory behavior of equation (1.1) with second-order half-linear equations of type (2.23). For the oscillation of equation (2.23), there are many results; see e.g., [1, 3–5, 17, 18, 20, 21] which include Hille and Nehari type, Philos type, etc.

Theorem 2.2. *Let (H_1), (H_2), and (1.3) hold. Assume that the equation*

$$\left(\frac{r(t)}{t^{2\alpha}}(x'(t))^{\alpha}\right)' + q(t)\left(\frac{k_1\tau^2(t)}{2t^2}\right)^{\alpha} x^{\alpha}(t) = 0 \tag{2.24}$$

is oscillatory for some constant $k_1 \in (0, 1)$, the equation

$$x''(t) + x(t)\int_t^{\infty}\left[\frac{1}{r(\vartheta)}\int_{\vartheta}^{\infty} q(s)\left(\frac{\tau^2(s)}{s^2}\right)^{\alpha} ds\right]^{\frac{1}{\alpha}} d\vartheta = 0 \tag{2.25}$$

is oscillatory, and the equation

$$\left(\left(\int_t^{\infty} R(s)\, ds\right)^{-\alpha}(x'(t))^{\alpha}\right)' + q(t)x^{\alpha}(t) = 0 \tag{2.26}$$

is oscillatory, and the equation

$$\left(r(t)(x'(t))^{\alpha}\right)' + q(t)\left(\frac{k_2}{2}\tau^2(t)\right)^{\alpha} x^{\alpha}(t) = 0 \tag{2.27}$$

is oscillatory for some constant $k_2 \in (0, 1)$. Then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.1, we have (2.9), (2.11), (2.17), and (2.22). Letting $\rho(t) = 1$ in (2.9), we have

$$\omega'(t) + \frac{\alpha k t^2}{2(r(t))^{\frac{1}{\alpha}}}\omega^{\frac{\alpha+1}{\alpha}}(t) + q(t)\left(\frac{\tau^2(t)}{t^2}\right)^{\alpha} \leq 0$$

for every constant $k \in (0, 1)$. Then we can see that equation (2.24) is nonoscillatory for every constant $k_1 \in (0, 1)$, which is a contradiction. Letting $\theta(t) = 1$ in (2.11), we have

$$\xi'(t) + \xi^2(t) + \int_t^{\infty}\left[\frac{1}{r(\vartheta)}\int_{\vartheta}^{\infty} q(s)\left(\frac{\tau^2(s)}{s^2}\right)^{\alpha} ds\right]^{\frac{1}{\alpha}} d\vartheta \leq 0.$$

Then equation (2.25) is nonoscillatory, which is a contradiction. From (2.17), we have

$$\varphi^{'}(t)+\alpha\varphi^{\frac{\alpha+1}{\alpha}}(t)\int\limits_t^\infty R(s)\,ds+q(t)\,\le\,0.$$

Then we can find that equation (2.26) is nonoscillatory, which is a contradiction. From (2.22), we have

$$\phi^{'}(t)+\alpha\,\frac{\phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}+q(t)\left(\frac{k}{2}\,\tau^2(t)\right)^\alpha\,\le 0$$

for every constant $k\,\in\,(0,1)$. Then we can see that equation (2.27) is nonoscillatory for every constant $k_2\,\in\,(0,1)$, which is a contradiction.

Theorem 2.2 is proved.

It is well known (see [18]) that if

$$\int\limits_{t_0}^\infty \frac{1}{a(t)}\,dt\,=\,\infty,\quad \text{and}\quad \liminf_{t\to\infty}\left(\int\limits_{t_0}^t \frac{1}{a(s)}\,ds\right)\int\limits_t^\infty q(s)\,ds\,>\,\frac{1}{4},$$

then equation (2.23) with $\alpha\,=\,1$ is oscillatory. Also, it is well known (see [20], Theorem 3.3) that if

$$\int\limits_{t_0}^\infty \frac{1}{a(t)}\,dt\,<\,\infty,\quad \text{and}\quad \liminf_{t\to\infty}\left(\int\limits_t^\infty \frac{1}{a(s)}\,ds\right)^{-1}\int\limits_t^\infty\left(\int\limits_s^\infty \frac{1}{a(v)}\,dv\right)^2 q(s)\,ds\,>\,\frac{1}{4},$$

then equation (2.23) with $\alpha\,=\,1$ is oscillatory.

Based on the above results and Theorem 2.2, we can easily obtain the following Hille and Nehari type oscillation criteria for (1.1) when $\alpha\,=\,1$.

Theorem 2.3. *Let* $\alpha\,=\,1$, (H_1), (H_2), *and (1.3) hold. Assume that*

$$\int\limits_{t_0}^\infty \frac{t^2}{r(t)}\,dt\,=\,\infty,\quad \textit{and}\quad \liminf_{t\to\infty}\left(\int\limits_{t_0}^t \frac{s^2}{r(s)}\,ds\right)\int\limits_t^\infty q(s)\frac{\tau^2(s)}{s^2}\,ds\,>\,\frac{1}{2k_1}$$

for some constant $k_1\,\in\,(0,1)$, *and*

$$\liminf_{t\to\infty} t\int\limits_t^\infty\int\limits_\eta^\infty \frac{1}{r(\vartheta)}\int\limits_\vartheta^\infty q(s)\frac{\tau^2(s)}{s^2}\,ds\,d\vartheta\,d\eta\,>\,\frac{1}{4}, \tag{2.28}$$

and

$$\int\limits_{t_0}^\infty\int\limits_t^\infty R(s)\,ds\,dt\,=\,\infty,\quad \textit{and}\quad \liminf_{t\to\infty}\left(\int\limits_{t_0}^t\int\limits_s^\infty R(v)\,dv\,ds\right)\int\limits_t^\infty q(s)\,ds\,>\,\frac{1}{4},$$

and

$$\liminf_{t\to\infty}\left(\int_t^\infty \frac{1}{r(s)}\,ds\right)^{-1}\int_t^\infty\left(\int_s^\infty \frac{1}{r(v)}\,dv\right)^2 q(s)\tau^2(s)\,ds > \frac{1}{2k_2} \tag{2.29}$$

for some constant $k_2 \in (0,1)$. *Then every solution of (1.1) with* $\alpha = 1$ *is oscillatory.*

Theorem 2.4. *Let* $\alpha = 1$, (H_1), (H_2), *(1.3), and (2.28) hold. Assume that*

$$\int_{t_0}^\infty \frac{t^2}{r(t)}\,dt < \infty,$$

$$\liminf_{t\to\infty}\left(\int_t^\infty \frac{s^2}{r(s)}\,ds\right)^{-1}\int_t^\infty\left(\int_s^\infty \frac{v^2}{r(v)}\,dv\right)^2 q(s)\,\frac{\tau^2(s)}{s^2}\,ds > \frac{1}{2k_1}$$

for some constant $k_1 \in (0,1)$, *and*

$$\int_{t_0}^\infty\int_t^\infty R(s)\,ds\,dt < \infty,$$

$$\liminf_{t\to\infty}\left(\int_t^\infty\int_s^\infty R(v)\,dv\,ds\right)^{-1}\int_t^\infty\left(\int_s^\infty\int_u^\infty R(v)\,dv\,du\right)^2 q(s)\,ds > \frac{1}{4},$$

and (2.29) holds for some constant $k_2 \in (0,1)$. *Then every solution of (1.1) with* $\alpha = 1$ *is oscillatory.*

3. Examples. In this section, we give two examples to illustrate the main results.

Example 3.1. Consider the differential equation

$$\left(t^5 x'''(t)\right)' + \beta t x(t) = 0, \quad t \ge 1. \tag{3.1}$$

Here $\beta > 0$ is a constant. Let

$$\alpha = 1, \quad r(t) = t^5, \quad q(t) = \beta t, \quad \tau(t) = t.$$

Then, we have

$$R(t) = \frac{1}{4t^4}, \quad \int_s^\infty R(v)\,dv = \frac{1}{12s^3}, \quad \int_s^\infty\int_u^\infty R(v)\,dv\,du = \frac{1}{24s^2}.$$

Letting $\rho(t) = \theta(t) = 1$, then we have that (2.1) and (2.2) are satisfied. By calculating, we see that (2.3) and (2.4) hold when $\beta > 12$. Hence by Theorem 2.1, every solution of (3.1) is oscillatory if $\beta > 12$. However, results of [22] cannot give this conclusion.

Example 3.2. Consider the delay differential equation

$$\left(e^t x'''(t)\right)' + 2\sqrt{10}\, e^{t+\arcsin\frac{\sqrt{10}}{10}} x\left(t - \arcsin\frac{\sqrt{10}}{10}\right) = 0, \quad t \geq 1. \tag{3.2}$$

It is easy to see that every solution of (3.2) is oscillatory due to Theorem 2.1. One such solution is $x(t) = e^t \sin t$. However, [22] (Corollary 2.1) implies that (3.2) may exist nonoscillatory solutions x which satisfy $\lim_{t\to\infty} x(t) = 0$. Hence our results supplement and improve those in [22].

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