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OSCILLATION OF SECOND ORDER NONLINEAR IMPULSIVE DIFFERENCE EQUATIONS WITH CONTINUOUS VARIABLES

ОСЦИЛЯЦІЯ НЕЛІНІЙНИХ ІМПУЛЬСНИХ РІЗНИЦЕВИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ З НЕПЕРЕРВНИМ АРГУМЕНТОМ

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This paper is concerned with a second order nonlinear impulsive difference equation with continuous variable. By using a nonimpulsive inequality sufficient conditions for the oscillation of impulsive difference equation are obtained.

Розглянуто нелінійні імпульсні різницеві рівняння другого порядку з неперервним аргументом. Використовуючи неімпульсну нерівність, отримано достатні умови осциляції імпульсних різницевих рівнянь.

1. Introduction In many applied mathematics problems, it is considered difference equations of the form

$$x_n = f(n, x_{n-j}), \quad n = 1, 2, \ldots, \quad j \in \mathbb{N},$$

which is a discrete equation and it is a special case of the following difference equation with continuous variable

$$x(t) = f(t, x(t-k)), \quad t \geq 0, \quad k \quad \text{is a constant.}$$

Recently, there has been an increasing interest in the study of oscillation of difference equations with continuous variables [1–6]. On the other hand, it is well known that impulsive equations appear as a natural description of the observed evolution phenomena of several real world problems [7, 8]. There has been rich literature on the oscillation of impulsive differential equations. The monographs [9, 10] and the survey papers [11, 12] include many results on the oscillation of impulsive differential equations. But, to the best of our knowledge there has been only a few works on the oscillation of impulsive difference equations with continuous variables [13, 14], and there is no paper on the second order nonlinear impulsive difference equations with continuous variables.

In this paper, our aim is to establish sufficient conditions for the oscillation of second order nonlinear impulsive difference equation with continuous variable. We shall construct a nonimpulsive inequality and using it we shall obtain sufficient conditions for the oscillation. This technique has been used in [15] and it can be applied to higher order impulsive difference equations with continuous variable.

Let $0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots$ be fixed points with $\lim_{n\to\infty} t_n = +\infty$.

We consider second order nonlinear impulsive difference equations of the type

$$\Delta_\tau^2 x(t) + \Delta_\tau x(t) + x(t) + f(x(t-\sigma)) = 0, \quad t \neq t_n, \tag{1}$$

$$x(t_n^+) - x(t_n^-) = g(x(t_n^-)), \quad n \in \mathbb{N} = \{1, 2, \ldots\}, \tag{2}$$

where $\Delta_\tau x(t) = x(t+\tau) - x(t)$, τ, σ are positive constants; $x(t_n^+) = \lim_{t\to t_n^+} x(t)$, and $x(t_n^-) = \lim_{t\to t_n^-} x(t)$.

Throughout this paper we shall assume that the following conditions are satisfied:

(i) $f \in C(\mathbb{R}, \mathbb{R})$, $f(u)/u \geq K$, $K > 0$ is a constant, for $u \neq 0$;

(ii) $g \in C(\mathbb{R}, \mathbb{R})$, $ug(u) > 0$ for $u \neq 0$.

Definition 1. *A function $x : [-\sigma, \infty) \to \mathbb{R}$ is called a solution of (1), (2) if*

(a) for $t \neq t_n$, $n \in \mathbb{N}$, x is continuous and satisfies (1),

(b) for $t = t_n$, $x(t_n^+)$ and $x(t_n^-)$ exist and satisfy (2) with $x(t_n^-) = x(t_n)$.

Definition 2. *If a function $x(t)$ is positive (negative) for all large values of t, then it is said that $x(t)$ is eventually positive (negative). A solution $x(t)$ of (1), (2) is called oscillatory if it is neither eventually positive nor eventually negative.*

2. Main results. In this section, first we introduce some functions. Denote

$$F(u) = \frac{u}{u + g(u)}, \quad u \in \mathbb{R}\backslash\{0\}.$$

From the condition (ii) we have $u + g(u) \neq 0$ and $0 < F(u) < 1$ for $u \neq 0$.

Let $x(t)$ be a solution of (1), (2). Define

$$z(t) = x(t) \prod_{t_0 \leq t_m < t} F(x(t_m)), \quad t \geq t_0 > 0.$$

As usual, the symbol $\prod_{a \leq t_m < b} a_m$ denotes the product of members of the sequence $\{a_m\}$ over m such that $t_m \in [a, b) \cap \{t_n : n \in \mathbb{N}\}$.

If $[a, b) \cap \{t_n : n \in \mathbb{N}\} = \varnothing$ or $a > b$, then we use the convention that $\prod_{a \leq t_m < b} a_m = 1$.

It can be seen that the function $z(t)$ is continuous at $t_k \geq t_0$. Indeed,

$$z(t_k^-) = x(t_k^-) \prod_{t_0 \leq t_m < t_k} F(x(t_m)) = z(t_k)$$

and

$$z(t_k^+) = x(t_k^+) \prod_{t_0 \leq t_m < t_k^+} F(x(t_m)) = x(t_k^+) \prod_{t_0 \leq t_m < t_k} F(x(t_m)) F(x(t_k)) = z(t_k),$$

where we have used the impulse condition (2).

Define

$$v(t) = \frac{1}{\tau} \int\limits_{t+\tau}^{t+2\tau} z(u)\, du, \quad t \geq t_0. \tag{3}$$

Lemma 1. *Assume that hypotheses (i), (ii) hold. If $x(t)$ is an eventually positive solution of (1), (2), then $v(t) > 0$, and $v'(t) \leq 0$ eventually.*

Proof. Let $x(t) > 0$, $t \geq t_0$. Then it is clear that $v(t) > 0$ for $t \geq t_0$. From (3) we obtain

$$v'(t) = \frac{1}{\tau}\left[z(t+2\tau) - z(t+\tau)\right] =$$

$$= \frac{1}{\tau} \prod_{t_0 \leq t_m < t+\tau} F(x(t_m)) \left[x(t+2\tau) \prod_{t+\tau \leq t_m < t+2\tau} F(x(t_m)) - x(t+\tau)\right]. \tag{4}$$

Now from Eq. (1), we have

$$x(t+2\tau) - x(t+\tau) < 0.$$

Since $0 < \prod\limits_{t+\tau \leq t_m < t+2\tau} F(x(t_m)) \leq 1$, we also have

$$x(t+2\tau) \prod_{t+\tau \leq t_m < t+2\tau} F(x(t_m)) < x(t+\tau). \tag{5}$$

Using (4) and (5), we obtain $v'(t) < 0$ for $t \geq t_0$, $t \neq t_m$. Since $v(t)$ is continuous, it follows that $v'(t) \leq 0$ for $t \geq t_0$.

Lemma 1 is proved.

Remark 1. Assume that hypotheses (i), (ii) hold. If $x(t)$ is an eventually negative solution of (1), (2), then $v(t) < 0$, and $v'(t) \geq 0$ eventually.

Let $\sigma = k\tau + \theta$, $k \in \mathbb{N}$, $\theta \in [0, \tau)$.

Lemma 2. *Let $x(t)$ be an eventually positive solution of Eqs. (1), (2). Assume that the following conditions hold:*

(H_1) *assumptions (i), (ii) are fulfilled;*

(H_2) *$f(u)$ is convex for $u > 0$, and concave for $u < 0$;*

(H_3) *$ug(u) \leq L_m u^2$ for $u \in \mathbb{R}$, where $L_m \geq 0$, $m = 1, 2, \ldots,$ are constants.*

Then $v(t)$ defined by (3) eventually satisfies the inequality

$$v(t+2\tau) - v(t+\tau) \prod_{t_0 \leq t_m < t+3\tau} (1 + L_m) + v(t) + f(v(t-k\tau)) \leq 0. \tag{6}$$

Proof. Let $x(t) > 0$, $t \geq t_0$. By using $(H_1)-(H_3)$ and employing the Jensen's inequality, we get

$$v(t+2\tau) - v(t+\tau) \prod_{t_0 \leq t_m < t+3\tau} (1 + L_m) + v(t) + f(v(t-\sigma)) \leq$$

$$\leq \frac{1}{\tau}\left\{\int\limits_{t+\tau}^{t+2\tau} x(u+2\tau)\,du - \int\limits_{t+\tau}^{t+2\tau} x(u+\tau)\,du + \int\limits_{t+\tau}^{t+2\tau} x(u)\,du + \int\limits_{t+\tau}^{t+2\tau} f(x(u-\sigma))\,du\right\} = 0. \tag{7}$$

Note that from condition (i) and (H_2) f is nondecreasing. On the other hand, in view of Lemma 1, we have

$$v(t-\sigma) \geq v(t-k\tau). \tag{8}$$

Using (8) and the fact that $f(u)$ is nondecreasing, we easily obtain (6) from (7).

Lemma 2 is proved.

Remark 2. Let $x(t)$ be an eventually negative solution of Eqs. (1), (2). Under the hypotheses of Lemma 2 it is shown that $v(t)$ defined by (3) eventually satisfies the inequality

$$v(t+2\tau) - v(t+\tau) \prod_{t_0 \leq t_m < t+3\tau} (1+L_m) + v(t) + f(v(t-k\tau)) \geq 0.$$

Theorem 1. *In addition to (H_1) – (H_3), assume that*

(H_4) $\limsup\limits_{t\to\infty} \prod\limits_{t_0 \leq t_m < t+3\tau} (1+L_m) = L < \infty.$

If

$$K > L^{k+2} \frac{(k+1)^{k+1}}{(k+2)^{k+2}}, \tag{9}$$

then every solution of (1), (2) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1), (2). We may assume without any loss of generality that $x(t)$ is eventually positive. From (6), we have

$$\frac{v(t+2\tau)}{v(t+\tau)} - \prod_{t_0 \leq t_m < t+3\tau} (1+L_m) \leq -\frac{f(v(t-k\tau))}{v(t+\tau)} = -\frac{f(v(t-k\tau))}{v(t-k\tau)} \prod_{j=0}^{k} \frac{v(t-j\tau)}{v(t-(j-1)\tau)}. \tag{10}$$

Define

$$\alpha(t) = \frac{v(t)}{v(t+\tau)}, \quad t \geq t_0.$$

Since $v'(t) \leq 0$, it is clear that $\alpha(t) \geq 1$. From (ii) and (10), we have

$$\frac{1}{\alpha(t+\tau)} + K \prod_{j=0}^{k} \alpha(t-j\tau) \leq \prod_{t_0 \leq t_m < t+3\tau} (1+L_m). \tag{11}$$

In view of (H_4) inequality (11) implies that $\alpha(t)$ is bounded. Let $\beta = \liminf_{t\to\infty} \alpha(t)$. Taking the inferior limit on both sides of (11), we obtain

$$1 + K\beta^{k+2} \leq \beta L.$$

This inequality implies that

$$\beta > \frac{1}{L} \quad \text{and} \quad \frac{K\beta^{k+2}}{\beta L - 1} \leq 1. \tag{12}$$

Using the fact that

$$\min_{\beta > \frac{1}{L}} \frac{\beta^{k+2}}{\beta L - 1} = \frac{1}{L^{k+2}} \frac{(k+2)^{k+2}}{(k+1)^{k+1}},$$

we obtain from (12) that

$$\frac{1}{L^{k+2}} \frac{(k+2)^{k+2}}{(k+1)^{k+1}} \leq \frac{1}{K},$$

which however contradicts (9). If $x(t)$ is an eventually negative solution of Eqs. (1), (2), we are lead to a contradiction by a similar argument.

Theorem 1 is proved.

Theorem 2. *In addition to (H_1) – (H_3) assume that the following conditions are satisfied:*

(H_5) $\sum_{m=1}^{\infty} L_m < \infty$,

(H_6) $\limsup_{u \to 0} \dfrac{f(u)}{u} > 1$.

Then every solution of Eqs. (1), (2) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1), (2). We may assume without any loss of generality that $x(t)$ is eventually positive. From (6), we have

$$v(t+2\tau) \leq v(t+\tau) \prod_{t_0 \leq t_m < t+3\tau} (1+L_m). \tag{13}$$

Using (13) we obtain

$$v(t+\tau) \leq v(t-k\tau) \prod_{i=1}^{k+1} \prod_{t_0 \leq t_m < t-(i-3)\tau} (1+L_m).$$

Now using the above inequality from (6), we get

$$f(v(t-k\tau)) \leq v(t+\tau) \prod_{t_0 \leq t_m < t+3\tau} (1+L_m) \leq v(t-k\tau) \prod_{t_0 \leq t_m < t+3\tau} (1+L_m)^{k+2}.$$

From the last inequality, we have

$$\frac{f(v(t-k\tau))}{v(t-k\tau)} \leq \prod_{t_0 \leq t_m < t+3\tau} (1+L_m)^{k+2}. \tag{14}$$

Since $v(t) > 0$ is a continuous function and $v'(t) \leq 0$, $\lim_{t\to\infty} v(t) = v_0 \geq 0$. We claim that $v_0 = 0$. If $v_0 > 0$, then from (4) we get

$$v(t+2\tau) + v(t) < v(t+\tau) \prod_{t_0 \leq t_m < t+3\tau} (1+L_m).$$

Taking the limit on both sides of last inequality we obtain $2v_0 \leq v_0$ which is a contradiction. So, $\lim_{t\to\infty} v(t) = 0$.

Now, taking the superior limit on both sides of (14), we obtain

$$\limsup_{t\to\infty}\frac{f(v(t-k\tau))}{v(t-k\tau)}\le 1,$$

which however contradicts (H_6). If $x(t)$ is an eventually negative solution of Eqs. (1), (2), we are lead to a contradiction by a similar argument.

Theorem 2 is proved.

Remark 3. If $x(t_n^+)=x(t_n^-)$ for all $n\in\mathbb{N}$, then $L=1$ and the assertions of Theorems 1 and 2 are valid for nonimpulsive equation.

Corollary 1. *Assume that (H_1) – (H_3) and (H_5) are satisfied. If*

$$\liminf_{u\to 0}\frac{f(u)}{u}>\frac{(k+1)^{k+1}}{(k+2)^{k+2}},$$

then every solution of Eqs. (1), (2) is oscillatory.

Example 1. Consider the linear impulsive difference equation with continuous variable

$$\begin{gathered}\Delta_{1/2}^2x(t)+\Delta_{1/2}x(t)+x(t)+Ax\left(t-\frac{3}{2}\right)=0,\quad t\ne t_n,\\ x(t_n^+)-x(t_n^-)=\frac{1}{n(n+1)}\,x(t_n),\quad t_n=n,\quad n\in\mathbb{N},\end{gathered}\tag{15}$$

where $\tau=1/2$, $\sigma=3/2$, $A>0$ is a constant, $L_n=\dfrac{1}{n(n+1)}$, $n\in\mathbb{N}$. If $A>\dfrac{4^4}{5^5}$, then by Corollary 1 every solution of equation (15) is oscillatory.

Example 2. Consider the nonlinear impulsive difference equation with continuous variable

$$\begin{gathered}x(t+2)-x(t+1)+x(t)+x(t-1)(1+x^2(t-1))=0,\quad t\ne t_n,\\ x(t_n^+)-x(t_n^-)=\frac{1}{3^n}\,\frac{x(t_n)}{1+x^2(t_n)},\quad t_n=n,\quad n=1,2,\ldots,\end{gathered}\tag{16}$$

where $\tau=\sigma=1$, $f(u)=u(1+u^2)$, $g(u)=\dfrac{1}{3^n}\dfrac{u}{1+u^2}$. Equation (16) satisfy the conditions (H_1) – (H_3), (H_5). Moreover, $\liminf\limits_{u\to 0}\dfrac{f(u)}{u}>\dfrac{2^2}{3^3}$. So, by Corollary 1, every solution of equation (16) is oscillatory.

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