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VARIATION FORMULAS FOR SOLUTION OF DELAY DIFFERENTIAL EQUATIONS WITH MIXED INITIAL CONDITION AND DELAY PERTURBATION*

ВАРІАЦІЙНІ ФОРМУЛИ ДЛЯ РОЗВ'ЯЗКУ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ІЗ ЗАПІЗНЕННЯМ, МІШАНИМИ ПОЧАТКОВИМИ УМОВАМИ ТА ЗБУРЕННЯМИ В ЗАПІЗНЕННІ

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Variation formulas for solution are proved for a nonlinear differential equation with constant delays. In this work, the essential novelty is an effect of delay perturbation in the variation formulas. The mixed initial condition means that at the initial moment, some coordinates of the trajectory do not coincide with the corresponding coordinates of the initial function, whereas the others coincide.Variation formulas are used in the proof of necessary optimality conditions.

Доведено варіаційні формули для нелінійних диференціальних рівнянь зі сталими запізненнями. Суттєвою новизною є ефект збурення в запізненні у варіаційних формулах. Мішана початкова умова означає, що в початковий момент одні координати траєкторії не збігаються, а інші збігаються з відповідними координатами початкової функції. Варіаційні формули використовуються в доведенні необхідних оптимальних умов.

1. Formulation of the main results. Let R^n_x be the n-dimensional vector space of points $x = (x^1, \ldots, x^n)^T$, where T means transpose; $P \subset R^k_p$ and $Z \subset R^m_z$ be open sets and $O = (P, Z)^T = \{x = (p, z)^T \in R^n_x : p \in P, z \in Z\}$, where $k + m = n$; E_f be the space of functions $f : J \times O \times P \times Z \to R^n_x$, satisfying the following conditions: for almost all $t \in J = [a, b]$ the function $f(t, \cdot) : O \times P \times Z \to R^n_x$ is continuously differentiable; for any $(x, p, z) \in O \times P \times Z$, the functions $f(t, x, p, z)$, $f_x(\cdot)$, $f_p(\cdot)$, $f_z(\cdot)$ are measurable on J; for any function $f \in E_f$ and any compact set $K \subset O$ there exists a function $m_{f,K} \in L(J, [0, \infty))$ such that for any $x \in K$, $(p, z)^T \in K$ and for almost all $t \in J$ we have

$$|f(t, x, p, z)| + |f_x(\cdot)| + |f_p(\cdot)| + |f_z(\cdot)| \le m_{f,K}(t).$$

Further, let $0 < \tau_1 < \tau_2$, $0 < \sigma_1 < \sigma_2$ be given numbers and $E_\varphi = E_\varphi(J_1, R^k_p)$ be the space of continuous functions $\varphi : J_1 \to R^k_p$, where $J_1 = [\hat{\tau}, b]$, $\hat{\tau} = a - \max\{\tau_2, \sigma_2\}$; let $\Phi = \{\varphi \in E_\varphi : \varphi(t) \in P\}$ and $G = \{g \in E_g = E_g(J_1, R^m_z) : g(t) \in Z\}$ be sets of initial functions.

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To any element $\mu = (t_0, \tau, \sigma, p_0, \varphi, g, f) \in \lambda = (a, b) \times (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times P \times \Phi \times G \times E_f$, we assign the delay differential equation

$$\dot{x}(t) = (\dot{p}(t), \dot{z}(t))^T = f(t, x(t), p(t - \tau), z(t - \sigma)) \tag{1.1}$$

with a mixed initial condition

$$x(t) = (\varphi(t), g(t))^T, \quad t \in [\hat{\tau}, t_0), \quad x(t_0) = (p_0, g(t_0))^T. \tag{1.2}$$

The condition (1.2) is said to be a mixed initial condition; it consists of two parts. The first part is $p(t) = \varphi(t), t \in [\hat{\tau}, t_0), p(t_0) = p_0$, the discontinuous part since in general $p(t_0) \neq \varphi(t_0)$; the second part is $z(t) = g(t), t \in [\hat{\tau}, t_0]$, the continuous part since always $z(t_0) = g(t_0)$.

Definition 1.1. *Let $\mu = (t_0, \tau, \sigma, p_0, \varphi, g, f) \in \lambda$. A function $x(t) = x(t; \mu) = (p(t; \mu), z(t; \mu))^T \in \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b)$, is called a solution of Eq. (1.1) with the initial condition (1.2) or a solution corresponding to the element μ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (1.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies Eq. (1.1) almost everywhere on $[t_0, t_1]$.*

Let $\mu_0 = (t_{00}, \tau_0, \sigma_0, p_{00}, \varphi_0, g_0, f_0) \in \lambda$ be a fixed element. In the space $E_\mu = R^1_{t_0} \times R^1_\tau \times \times R^1_\sigma \times R^k_p \times E_\varphi \times E_g \times E_f$ we introduce the set of variations

$$V = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta\sigma, \delta p_0, \delta\varphi, \delta g, \delta f) \in E_\mu - \mu_0 : |\delta t_0| \le \alpha,$$

$$|\delta\tau| \le \alpha, |\delta\sigma| \le \alpha, |\delta p_0| \le \alpha, \delta\varphi = \sum_{i=1}^{\nu} \lambda_i \delta\varphi_i, \delta g = \sum_{i=1}^{\nu} \lambda_i \delta g_i,$$

$$\delta f = \sum_{i=1}^{\nu} \lambda_i \delta f_i, |\lambda_i| \le \alpha, i = \overline{1, \nu} \Big\}, \tag{1.3}$$

where $\delta\varphi_i \in E_\varphi - \varphi_0, \delta g_i \in E_g - g_0, \delta f_i \in E_f - f_0, i = \overline{1, \nu}$, are fixed functions; $\alpha > 0$ is a fixed number.

Let $x_0(t)$ be a solution corresponding to the element μ_0 and defined on the interval $[\hat{\tau}, t_{10}]$, with $t_{10} < b$. There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \lambda$. In addition, to this element, there is a corresponding solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset J_1$ (see Lemma 2.2).

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$. Therefore, in the sequel the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10} + \delta_1]$.

Let us define an increment of the solution $x_0(t) = x(t; \mu_0)$,

$$\Delta x(t) = (\Delta p(t), \Delta z(t))^T = \Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \tag{1.4}$$

$$(t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_1] \times [0, \varepsilon_1] \times V.$$

Theorem 1.1. *Let the following conditions hold:*

1) $t_{00} + \tau_0 < t_{10}$;

2) the functions $\varphi_0(t), g_0(t), t \in J_1$, *are absolutely continuous and* $\dot{\varphi}_0(t), \dot{g}_0(t)$ *are bounded; there exists a compact set* $K_0 \subset O$, *containing a neighborhood of the set* $(\varphi_0(J_1), g_0(J_1))^T \cup \cup x_0([t_{00}, t_{10}])$, *such that the function* $f_0(t, x, p, z), t \in J, x \in K_0, (p, z)^T \in K_0$ *is bounded;*

3) there exist the limits

$$\lim_{t \to t_{00}-} \dot{g}_0(t) = \dot{g}_0^-, \quad \lim_{w \to w_0} f_0(w) = f_0^-, \quad w \in (a, t_{00}] \times O \times P \times Z;$$

4) there exists the limit

$$\lim_{(w_1, w_2) \to (w_{01}, w_{02})} [f_0(w_1) - f_0(w_2)] = f_{01}^-, \quad w_1, w_2 \in (a, t_{00} + \tau_0] \times O \times P \times Z,$$

where $w = (t, x, p, z)$, $w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_0), g_0(t_{00} - \sigma_0))$, $x_{00} = (p_{00}, g_0(t_{00}))^T$, $w_{01} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), p_{00}, z_0(t_{00} + \tau_0 - \sigma_0))$, $w_{02} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), \varphi_0(t_{00}), z_0(t_{00} + \tau_0 - \sigma_0))$.

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu)^1 \tag{1.5}$$

for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-$, *where* $V^- = \{\delta\mu \in V : \delta t_0 \leq 0, \delta\tau \leq \leq 0, \delta\sigma \leq 0\}$ *and*

$$\delta x(t; \delta\mu) = \Big\{Y(t_{00}; t)[(\Theta_{k\times 1}, \dot{g}_0^-)^T - f_0^-] - Y(t_{00} + \tau_0; t) f_{01}^-\Big\}\delta t_0 - \\ - Y(t_{00} + \tau_0; t) f_{01}^- \delta\tau + \beta(t; \delta\mu), \tag{1.6}$$

$$\beta(t; \delta\mu) = Y(t_{00}; t)(\delta p_0, \delta g(t_{00}))^T - \left\{\int_{t_{00}}^{t} Y(\xi; t) f_{0p}[\xi]\dot{p}_0(\xi - \tau_0) d\xi\right\}\delta\tau - \\ - \left\{\int_{t_{00}}^{t} Y(\xi; t) f_{0z}[\xi]\dot{z}_0(\xi - \sigma_0) d\xi\right\}\delta\sigma + \int_{t_{00} - \tau_0}^{t_{00}} Y(\xi + \tau_0; t) f_{0p}[\xi + \tau_0]\delta\varphi(\xi) d\xi + \\ + \int_{t_{00} - \sigma_0}^{t_{00}} Y(\xi + \sigma_0; t) f_{0z}[\xi + \sigma_0]\delta g(\xi) d\xi + \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi] d\xi; \tag{1.7}$$

$\Theta_{k\times 1}$ *is the* $(k \times 1)$*-zero matrix,* $Y(\xi; t)$ *is an* $(n \times n)$*-matrix function satisfying on the interval* $[t_{00}, t]$ *the conditions*

$$Y_\xi(\xi; t) = -Y(\xi; t) f_{0x}[\xi] - (Y(\xi + \tau_0; t) f_{0p}[\xi + \tau_0], Y(\xi + \sigma_0; t) f_{0z}[\xi + \sigma_0]), \tag{1.8}$$

[1] Here and throughout the following, the symbols $O(t; \varepsilon\delta\mu)$, $o(t; \varepsilon\delta\mu)$ stand for quantities (scalar or vector) that have the corresponding order of smallness with respect to ε uniformly with respect to $(t, \delta\mu)$.

and

$$Y(\xi;t) = \begin{cases} I_{n\times n} & \textit{for} \quad \xi = t, \\ \Theta_{n\times n} & \textit{for} \quad \xi > t. \end{cases} \tag{1.9}$$

Here, $I_{n\times n}$ is the identity $(n\times n)$-matrix, $f_{0x} = \dfrac{\partial}{\partial x} f_0$, $f_{0x}[\xi] = f_{0x}(\xi, x_0(\xi), p_0(\xi-\tau_0), z_0(\xi-\sigma_0))$, $\delta f[\xi] = \delta f(\xi, x_0(\xi), p_0(\xi-\tau_0), z_0(\xi-\sigma_0))$; $\dot p_0(\xi-\tau_0) = \dot p_0(s)|_{s=\xi-\tau_0}$, where $\dot p_0(s)$ denotes the derivative of the function $p_0(s)$ on the set $[\hat\tau, t_{00})\cup(t_{00}, t_{10}+\delta_2]$.

Some comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t)$ on the interval $[t_{10}-\delta_2, t_{10}+\delta_2]$, and expression (1.6) is called the variation formula.

c$_1$) Theorem 1.1 corresponds to the case where the variations at the points t_{00}, τ_0, σ_0 are performed simultaneously on the left.

c$_2$) The term

$$-\left\{Y(t_{00}+\tau_0;t)f_{01}^{-}+\int\limits_{t_{00}}^{t} Y(\xi;t)f_{0p}[\xi]\dot p_0(\xi-\tau_0)d\xi\right\}\delta\tau-\left\{\int\limits_{t_{00}}^{t} Y(\xi;t)f_{0z}[\xi]\dot z_0(\xi-\sigma_0)d\xi\right\}\delta\sigma$$

is the effect of perturbations of the delays τ_0 and σ_0 (see (1.6) and (1.7)).

c$_3$) The expression

$$Y(t_{00};t)(\delta p_0, \delta g(t_{00}))^T + \left\{Y(t_{00};t)[(\Theta_{k\times 1}, \dot g_0^{-})^T - f_0^{-}] - Y(t_{00}+\tau_0;t)f_{01}^{-}\right\}\delta t_0$$

is the effect of mixed initial condition (1.2) under perturbations of the initial moment t_{00}, the initial vector p_{00}, and the function $g_0(t)$.

c$_4$) The expression

$$\int\limits_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\delta\varphi(\xi)d\xi + \int\limits_{t_{00}-\sigma_0}^{t_{00}} Y(\xi+\sigma_0;t)f_{0z}[\xi+\sigma_0]\delta g(\xi)d\xi +$$
$$+\int\limits_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]d\xi$$

in formula (1.7) is the effect of perturbations of the initial functions $\varphi_0(t)$, $g_0(t)$ and the right-hand side of the equation

$$\dot x(t) = f_0(t, x(t), p(t-\tau_0), z(t-\sigma_0)).$$

Theorem 1.2. *Let the conditions 1 and 2 of Theorem 1.1 hold. Moreover,*

5) there exist the limits

$$\lim_{t\to t_{00}+} \dot g_0(t) = \dot g_0^{+}, \quad \lim_{w\to w_0} f_0(w) = f_0^{+}, \quad w\in[t_{00}, b)\times O\times P\times Z,$$

6) there exists the limit

$$\lim_{(w_1,w_2)\to(w_{01},w_{02})}[f_0(w_1)-f_0(w_2)] = f_{01}^+, \quad w_1,w_2\in[t_{00}+\tau_0,b)\times O\times P\times Z.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+$, *where* $V^+=\{\delta\mu\in V:\delta t_0\geq 0,\delta\tau\geq 0,\delta\sigma\geq 0\}$ *formula (1.5) holds, where*

$$\delta x(t;\delta\mu) = \left\{Y(t_{00};t)[(\Theta_{k\times 1},\dot g_0^+)^T - f_0^+] - Y(t_{00}+\tau_0;t)f_{01}^+\right\}\delta t_0 - \\ - Y(t_{00}+\tau_0;t)f_{01}^+\delta\tau + \beta(t;\delta\mu). \qquad (1.10)$$

Theorem 1.2 corresponds to the case where the variations at the points t_{00}, τ_0, σ_0 are performed simultaneously on the right. Theorems 1.1 and 1.2 are proved by a method given in [1]. The following assertion is a corollary of Theorems 1.1 and 1.2.

Theorem 1.3. *Let the conditions of Theorems 1.1 and 1.2 hold. Moreover,* $(\Theta_{k\times 1},\dot g_0^-)^T - f_0^- = (\Theta_{k\times 1},\dot g_0^+)^T - f_0^+ =: \hat f_0, f_{01}^- = f_{01}^+ =: \hat f_{01}$. *Then there exist numbers* $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V$, *formula (1.5) holds, where* $\delta x(t;\delta\mu)=\{Y(t_{00};t)\hat f_0 - Y(t_{00}+\tau_0;t)\hat f_{01}\}\delta t_0 - Y(t_{00}+\tau_0;t)\hat f_{01}\delta\tau+\beta(t;\delta\mu)$.

Theorem 1.3 corresponds to the case where at the points t_{00}, τ_0, σ_0, two-sided variations are simultaneously performed. If $t_{00}+\tau_0 > t_{10}$ then Theorems 1.1 – 1.3 also are valid. In this case the number δ_2 is so small that $t_{00}+\tau_0 > t_{10}+\delta_2$, therefore in the variation formulas we have $Y(t_{00}+\tau_0;t)=\Theta_{n\times n}$, $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$. If $t_{00}+\tau_0=t_{10}$ then Theorem 1.1 is valid on the interval $[t_{10},t_{10}+\delta_2]$ and Theorem 1.2 is valid on the interval $[t_{10}-\delta_2,t_{10}]$.

Finally, we note that variation formulas for solution of various classes of functional-differential equations without perturbation of delay are given [1 – 8].

2. Some auxiliary statements. To each element $\mu=(t_0,\tau,\sigma,p_0,\varphi,g,f)\in\lambda$, we assign the functional-differential equation

$$\dot y(t) = (\dot u(t),\dot v(t))^T = f(t,y(t),h(t_0,\varphi,u)(t-\tau),h(t_0,g,v)(t-\sigma)) = \\ = (Y_1 f(t,y(t),h(t_0,\varphi,u)(t-\tau),h(t_0,g,v)(t-\sigma)), \\ Y_2 f(t,y(t),h(t_0,\varphi,u)(t-\tau),h(t_0,g,v)(t-\sigma))^T \qquad (2.1)$$

with the initial condition

$$y(t_0) = (p_0, g(t_0))^T, \qquad (2.2)$$

where the operator $h(t_0,\varphi,u)(t)$ is defined by the formula

$$h(t_0,\varphi,u)(t) = \begin{cases} \varphi(t), & t\in[\hat\tau,t_0), \\ u(t), & t\in[t_0,b], \end{cases} \qquad (2.3)$$

$Y_1=(I_{k\times k},\Theta_{k\times m})^T$, $Y_2=(\Theta_{m\times k},I_{m\times m})^T$.

Definition 2.1. *Let* $\mu = (t_0, \tau, \sigma, p_0, \varphi, g, f) \in \lambda$. *An absolutely continuous function* $y(t) = y(t;\mu) = (u(t;\mu), v(t;\mu))^T \in O$, $t \in [r_1, r_2] \subset J$, *is called a solution of Eq. (2.1) with the initial condition (2.2) or a solution corresponding to the element* μ *and defined on the interval* $[r_1, r_2]$, *if* $t_0 \in [r_1, r_2]$, $y(t_0) = (p_0, g(t_0))^T$ *and the function* $y(t)$ *satisfies Eq.(2.1) almost everywhere on* $[r_1, r_2]$.

Remark 2.1. Let $y(t;\mu)$, $t \in [r_1, r_2]$, be a solution corresponding to the element $\mu = (t_0, \tau, \sigma, p_0, \varphi, g, f) \in \lambda$. Then the function

$$x(t;\mu) = (h(t_0, \varphi, u(\cdot;\mu))(t), h(t_0, g, v(\cdot;\mu))(t))^T \tag{2.4}$$

on the interval $[\hat{\tau}, r_2]$ is a solution of equation (1.1) with the initial condition (1.2) (see Definition 1.1).

Lemma 2.1. *Let* $y_0(t)$ *be a solution corresponding to the element* $\mu_0 = (t_{00}, \tau_0, \sigma_0, p_{00}, \varphi_0, g_0, f_0) \in \lambda$ *and defined on* $[r_1, r_2] \subset (a, b)$, $K_0 \subset O$ *be a compact set containing a neighborhood of the set* $(\varphi_0(J_1), g_0(J_1))^T \cup y_0([r_1, r_2])$. *Then there exist numbers* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *such that, for any* $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$, *we have* $\mu_0 + \varepsilon\delta\mu \in \lambda$. *In addition, to this element there is a corresponding solution* $y(t; \mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[r_1 - \delta_1, r_2 + \delta_1] \subset J$. *Moreover,*

$$\begin{aligned} &(\varphi(t), g(t))^T \in K_0, \quad t \in J_1, \quad y(t;\mu_0 + \varepsilon\delta\mu) \in K_0, \quad t \in [r_1 - \delta_1, r_2 + \delta_1], \\ &\lim_{\varepsilon \to 0} y(t;\mu_0 + \varepsilon\delta\mu) = y(t;\mu_0) \quad \textit{uniformly for} \quad (t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times V. \end{aligned} \tag{2.5}$$

This lemma is a result of Theorem 5.3 in [9, p. 111]. Here and in what follows we shall assume that $\varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t)$, $g(t) = g_0(t) + \varepsilon\delta g(t)$.

Lemma 2.2. *Let* $x_0(t)$ *be the solution corresponding to the element* $\mu_0 = (t_{00}, \tau_0, \sigma_0, p_{00}, \varphi_0, g_0, f_0) \in \lambda$ *and defined on* $[\hat{\tau}, t_{10}]$ *(see Definition 1.1),* $K_0 \subset O$ *be a compact set containing a neighborhood of the set* $(\varphi_0(J_1), g_0(J_1))^T \cup x_0([t_{00}, t_{10}])$. *Then there exist numbers* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *such that, for any* $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$, *we have* $\mu_0 + \varepsilon\delta\mu \in \lambda$. *In addition, to this element, there is a corresponding solution* $x(t;\mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[\hat{\tau}, t_{10} + \delta_1] \subset J_1$. *Moreover,*

$$x(t;\mu_0 + \varepsilon\delta\mu) \in K_0, \quad t \in [\hat{\tau}, t_{10} + \delta_1]. \tag{2.6}$$

It is easy to see that, if in Lemma 2.1 we put $r_1 = t_{00}$, $r_2 = t_{10}$, then $x_0(t) = y_0(t)$, $t \in [t_{00}, t_{10}]$, $x(t;\mu_0 + \varepsilon\delta\mu) = (h(t_0, \varphi, u(\cdot;\mu_0 + \varepsilon\delta\mu))(t), h(t_0, g, v(\cdot;\mu_0 + \varepsilon\delta\mu))(t))^T$, $(t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_1] \times [0, \varepsilon_1] \times V$ (see (2.4)). Thus, Lemma 2.2 is a simple corollary of Lemma 2.1.

Remark 2.2. Due to the uniqueness, the solution $y(t;\mu_0)$, $t \in [r_1 - \delta_1, r_2 + \delta_1]$, is a continuation of the solution $y_0(t)$. Therefore, we can assume that the solution $y_0(t)$ is defined on the interval $[r_1 - \delta_1, r_2 + \delta_1]$.

Lemma 2.1 allows one for $(t, \varepsilon, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times [0, \varepsilon_1] \times V$, to define the increment of the solution $y_0(t) = y(t;\mu_0)$,

$$\Delta y(t) = (\Delta u(t), \Delta v(t))^T = \Delta y(t;\varepsilon\delta\mu) = y(t;\mu_0 + \varepsilon\delta\mu) - y_0(t). \tag{2.7}$$

Uniformly with respect to $(t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times V$ we have

$$\lim_{\varepsilon\to 0}\Delta y(t;\varepsilon\delta\mu)=0 \tag{2.8}$$

(see Lemma 2.1).

Lemma 2.3. *Let $t_{00}+\tau_0\le r_2$ and the conditions 2 and 3 of Theorem 1.1 hold. Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ and $\delta_2\in(0,\delta_1]$ such that*

$$\max_{t\in[t_{00},r_2+\delta_2]}|\Delta y(t)|\le O(\varepsilon\delta\mu) \tag{2.9}$$

for arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^-$. Moreover,

$$\Delta y(t_{00})=\varepsilon\left\{(\delta p_0,\delta g(t_{00}))^T+[(\Theta_{k\times 1},\dot g_0^-)^T-f_0^-]\delta t_0\right\}+o(\varepsilon\delta\mu). \tag{2.10}$$

Proof. Let a number $\varepsilon_1^{'}\in(0,\varepsilon_1]$ be sufficiently small so that for arbitrary $(t,\varepsilon,\delta\mu)\in$ $\in[t_0,t_{00}]\times(0,\varepsilon_1^{'}]\times V^-$ the following inequalities hold:

$$t-\tau\le t_0,\quad t+\tau\ge t_{00},\quad t-\sigma\le t_0,\quad t+\sigma\ge t_{00}. \tag{2.11}$$

Here and in what follows we shall assume that $t_0=t_{00}+\varepsilon\delta t_0$, $\tau=\tau_0+\varepsilon\delta\tau$, $\sigma=\sigma_0+\varepsilon\delta\sigma$. If $t_{00}+\sigma_0\le r_2$, then $\varepsilon_2=\varepsilon_1^{'}$ and $\delta_2=\delta_1$; if $t_{00}+\sigma_0>r_2$, then $\varepsilon_2=\varepsilon_1^{''}$ and $\delta_2=\delta_1^{'}$, where the numbers $\varepsilon_1^{''}\in(0,\varepsilon_1^{'}]$, $\delta_1^{'}\in(0,\delta_1]$ are so small that $t+\sigma>r_2+\delta_2$ for arbitrary $(t,\varepsilon,\delta\mu)\in[t_0,t_{00}]\times(0,\varepsilon_2]\times V^-$.

On the interval $[t_{00},r_2+\delta_2]$, the function $\Delta y(t)=y(t)-y_0(t)$ satisfies the equation

$$\begin{aligned}\dot\Delta y(t)=\;&f_0(t,y_0(t)+\Delta y(t),h(t_0,\varphi,u_0+\Delta u)(t-\tau),h(t_0,g,v_0+\Delta v)(t-\sigma))+\\&+\varepsilon\delta f(t,y_0(t)+\Delta y(t),h(t_0,\varphi,u_0+\Delta u)(t-\tau),h(t_0,g,v_0+\Delta v)(t-\sigma))-\\&-f_0(t,y_0(t),h(t_{00},\varphi_0,u_0)(t-\tau_0),h(t_{00},g_0,v_0)(t-\sigma_0))=\\=\;&a(t;\varepsilon\delta\mu)+\varepsilon b(t;\varepsilon\delta\mu),\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}a(t;\varepsilon\delta\mu)=\;&f_0(t,y_0(t)+\Delta y(t),h(t_0,\varphi,u_0+\Delta u)(t-\tau),h(t_0,g,v_0+\Delta v)(t-\sigma))-\\&-f_0(t,y_0(t),h(t_{00},\varphi_0,u_0)(t-\tau_0),h(t_{00},g_0,v_0)(t-\sigma_0)),\end{aligned} \tag{2.13}$$

$$b(t;\varepsilon\delta\mu)=\delta f(t,y_0(t)+\Delta y(t),h(t_0,\varphi,u_0+\Delta u)(t-\tau),h(t_0,g,v_0+\Delta v)(t-\sigma)). \tag{2.14}$$

We rewrite Eq. (2.12) in the integral form,

$$\Delta y(t)=\Delta y(t_{00})+\int_{t_{00}}^{t}[a(\xi;\varepsilon\delta\mu)+\varepsilon b(\xi;\varepsilon\delta\mu)]d\xi.$$

Hence it follows that

$$|\Delta y(t)| \le |\Delta y(t_{00})| + a_1(t_{00}, t; \varepsilon\delta\mu) + \varepsilon b(\varepsilon\delta\mu), \tag{2.15}$$

where

$$a_1(t_{00}, t; \varepsilon\delta\mu) = \int\limits_{t_{00}}^{t} |a(\xi; \varepsilon\delta\mu)| d\xi, \quad b(\varepsilon\delta\mu) = \int\limits_{t_{00}}^{r_2+\delta_2} |b(\xi; \varepsilon\delta\mu)| d\xi.$$

Let us prove formula (2.10). By Lemma 2.1 for $\xi \in [t_{00}, r_2 + \delta_2]$ we have $y_0(\xi) + \Delta y(\xi) \in K_0$, moreover there exist compact sets $K_{01} \subset P$ and $K_{02} \subset Z$ such that $(K_{01}, K_{02})^T \subset K_0$ and

$$h(t_0, \varphi, u_0 + \Delta u)(\xi - \tau) \in K_{01}, \quad h(t_0, g, v_0 + \Delta v)(\xi - \sigma) \in K_{02}, \quad \xi \in [t_{00}, r_2 + \delta_2].$$

Therefore,

$$|b(\xi; \varepsilon\delta\mu)| \le \alpha \sum_{i=1}^{\nu} m_{\delta f_i, K_0}(\xi), \quad \xi \in [t_{00}, r_2 + \delta_2] \tag{2.16}$$

(see (1.3) and (2.14)). For $\varepsilon \in (0, \varepsilon_2]$ we get

$$\Delta y(t_{00}) = y(t_{00}) - y_0(t_{00}) = y(t_0) + \int\limits_{t_0}^{t_{00}} \dot{y}(\xi) d\xi - y_0(t_{00}) =$$

$$= \int\limits_{t_0}^{t_{00}} f_0(\xi, y_0(\xi) + \Delta y(\xi), \varphi(\xi - \tau), g(\xi - \sigma)) d\xi + o(\varepsilon\delta\mu) - y_0(t_{00}) \tag{2.17}$$

(see (2.11), (2.3) and (2.16)). Further, $y(t_0) - y_0(t_{00}) = (p_{00} + \varepsilon\delta p_0, g_0(t_0) + \varepsilon\delta g(t_0))^T - (p_{00}, g_0(t_{00}))^T = (\varepsilon\delta p_0, g_0(t_0) + \varepsilon\delta g(t_0) - g_0(t_{00}))^T$. Since

$$g_0(t_0) - g_0(t_{00}) = \varepsilon\dot{g}_0^-\delta t_0 + \int\limits_{t_{00}}^{t_0} [\dot{g}_0(\xi) - \dot{g}_0^-] d\xi = \varepsilon\dot{g}_0^-\delta t_0 + o(\varepsilon\delta\mu)$$

and $\lim_{\varepsilon\to 0} \delta g(t_0) = \delta g(t_{00})$ uniformly with respect to $\delta\mu \in V^-$ (see (1.3)), it follows that $g_0(t_0) - g_0(t_{00}) + \varepsilon\delta g(t_0) = \varepsilon\dot{g}_0^-\delta t_0 + o(\varepsilon\delta\mu) + \varepsilon\delta g(t_{00}) + \varepsilon[\delta g(t_0) - \delta g(t_{00})] = \varepsilon[\dot{g}_0^-\delta t_0 + \delta g(t_{00})] + o(\varepsilon\delta\mu)$. Thus,

$$y(t_0) - y_0(t_{00}) = \varepsilon\left[(\delta p_0, \delta g(t_{00}))^T + (\Theta_{k\times 1}, \dot{g}_0^-)^T \delta t_0\right] + o(\varepsilon\delta\mu). \tag{2.18}$$

Taking into account $y_0(t_{00}) = x_0(t_{00}) = x_{00}$ and condition (2.8), for $\xi \in [t_0, t_{00}]$ we have $\lim_{\varepsilon\to 0}(\xi, y_0(\xi) + \Delta y(\xi), \varphi(\xi - \tau), g(\xi - \sigma)) = \lim_{\xi\to t_{00}-}(\xi, y_0(\xi), \varphi_0(\xi - \tau_0), g_0(\xi - \sigma_0)) = (t_{00}, y_0(t_{00}), \varphi_0(t_{00} - \tau_0), g_0(t_{00} - \sigma_0)) = w_0$. Consequently, $\lim_{\varepsilon\to 0} \sup_{t\in[t_0, t_{00}]} |f_0(\xi, y_0(\xi) +$

$+\Delta y(\xi), \varphi(\xi-\tau), g(\xi-\sigma)) - f_0^-| = \lim_{w\to w_0} |f_0(w) - f_0^-| = 0$, $w \in (a, t_{00}] \times O \times P \times Z$. This relation implies that

$$\int_{t_0}^{t_{00}} f_0(\xi, y_0(\xi) + \Delta y(\xi), \varphi(\xi-\tau), g(\xi-\sigma))d\xi = -\varepsilon f_0^- \delta t_0 + o(\varepsilon\delta\mu). \tag{2.19}$$

From (2.17), by virtue of (2.18) and (2.19) we obtain (2.10). Now let us prove inequality (2.9). First of all, we note that for any $f \in E_f$, there exists a function $L_{f,K_0} \in L(J, (0,\infty))$ such that

$$|f(t, x_1, p_1, z_1) - f(t, x_2, p_2, z_2)| \le L_{f,K_0}(t)(|x_1 - x_2| + |p_1 - p_2| + |z_1 - z_2|)$$

for almost all $t \in J$ and for any $x_i \in K_0$, $(p_i, z_i)^T \in K_0$, $i = 1, 2$ (see Lemma 3.1 in [1, p. 35]). For $t \in [t_{00}, t_0 + \tau]$, we have

$$a_1(t_{00}, t; \varepsilon\delta\mu) \le \int_{t_{00}}^{t} L_{f_0,K_0}(\xi)|\Delta y(\xi)|d\xi + a_2(t_{00}, t; \varepsilon\delta\mu) + a_3(t_{00}, t; \varepsilon\delta\mu), \tag{2.20}$$

where

$$a_2(t_{00}, t; \varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{f_0,K_0}(\xi)|h(t_0, \varphi, u_0 + \Delta u)(\xi-\tau) - h(t_{00}, \varphi_0, u_0)(\xi-\tau_0)|d\xi,$$

$$a_3(t_{00}, t; \varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{f_0,K_0}(\xi)|h(t_0, g, v_0 + \Delta v)(\xi-\sigma) - h(t_{00}, g_0, v_0)(\xi-\sigma_0)|d\xi$$

(see (2.13)).

If $\xi \in [t_{00}, t_0+\tau] \subset [t_{00}, t_{00}+\tau_0]$, then $\xi-\tau \le t_0$ and $\xi-\tau_0 \le t_{00}$, therefore for $a_2(t_{00}, t; \varepsilon\delta\mu)$ we obtain

$$a_2(t_{00}, t; \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int_{t_{00}}^{b} L_{f_0,K_0}(\xi)|\varphi_0(\xi-\tau) - \varphi_0(\xi-\tau_0)|d\xi.$$

From boundedness of the function $\dot\varphi_0(t)$, $t \in J_1$, it follows that

$$|\varphi_0(\xi-\tau) - \varphi_0(\xi-\tau_0)| = \int_{\xi-\tau_0}^{\xi-\tau} |\dot\varphi_0(s)|ds \le O(\varepsilon\delta\mu). \tag{2.21}$$

Thus, for $t \in [t_{00}, t_0 + \tau]$ we get

$$a_2(t_{00}, t; \varepsilon\delta\mu) \le O(\varepsilon\delta\mu). \tag{2.22}$$

Now we estimate $a_3(t_{00},t;\varepsilon\delta\mu)$ on the whole interval $[t_{00},r_2+\delta_2]$.

Let $t_{00}+\sigma_0>r_2$ then for $t\in[t_{00},r_2+\delta_2]$ we obtain

$$a_3(t_{00},t;\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{b}L_{f_0,K_0}(\xi)|g_0(\xi-\sigma)-g_0(\xi-\sigma_0)|d\xi. \tag{2.23}$$

From boundedness of the function $\dot g_0(t)$, $t\in J_1$, it follows that

$$|g_0(\xi-\sigma)-g_0(\xi-\sigma_0)|=\int\limits_{\xi-\sigma_0}^{\xi-\sigma}|\dot g_0(s)|ds\le O(\varepsilon\delta\mu). \tag{2.24}$$

Consequently, in this case we have

$$a_3(t_{00},t;\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)\quad\forall(t,\varepsilon,\delta\mu)\in[t_{00},r_2+\delta_2]\times(0,\varepsilon_2]\times V^-. \tag{2.25}$$

Let $t_{00}+\sigma_0\le r_2$. If $\xi\in[t_{00},t_0+\sigma]$, then $\xi-\sigma\le t_0$ and $\xi-\sigma_0\le t_{00}$, therefore we have (2.23). On the basis of (2.24) we conclude that the estimate (2.25) is valid on the interval $[t_{00},t_0+\sigma]$.

Let $t\in[t_0+\sigma,t_{00}+\sigma_0]$, then

$$a_3(t_{00},t;\varepsilon\delta\mu)\le a_3(t_{00},t_0+\sigma;\varepsilon\delta\mu)+\int\limits_{t_0+\sigma}^{t_{00}+\sigma_0}L_{f_0,K_0}(\xi)|v(\xi-\sigma)-g_0(\xi-\sigma)|d\xi+$$

$$+\int\limits_{t_0+\sigma}^{t_{00}+\sigma_0}L_{f_0,K_0}(\xi)|g_0(\xi-\sigma)-g_0(\xi-\sigma_0)|d\xi\le O(\varepsilon\delta\mu)+$$

$$+\int\limits_{t_0}^{t_{00}+\sigma_0-\sigma}L_{f_0,K_0}(\xi)|v(\xi)-g_0(\xi)|d\xi$$

(see (2.24)). Taking into account boundedness of the functions $f_0(t,x,p,z)$, $(t,x)\in J\times K_0$, $(p,z)^T\in K_0$ and $\dot g_0(t), t\in J_1$, for $\xi\in[t_0,t_{00}+\sigma_0-\sigma]$ we have

$$|v(\xi)-g_0(\xi)|=\Bigg|g(t_0)+\int\limits_{t_0}^{\xi}Y_2\Big[f_0(s,y_0(s)+\Delta y(s),h(t_0,\varphi,u_0+\Delta u)(s-\tau),$$

$$h(t_0,g,v_0+\Delta v)(s-\sigma))+\varepsilon b(s;\varepsilon\delta\mu)\Big]ds-g_0(\xi)\Bigg|\le O(\varepsilon\delta\mu) \tag{2.26}$$

(see (2.5) and (2.16)). Consequently, for $t\in[t_{00},t_{00}+\sigma_0]$ the condition (2.25) holds. If $\xi\in$

$\in [t_{00}+\sigma_0, r_2+\delta_2]$, then $\xi-\sigma \geq t_{00}$ and $\xi-\sigma_0 \geq t_{00}$, therefore for $t \in [t_{00}+\sigma_0, r_2+\delta_2]$ we get

$$a_3(t_{00},t;\varepsilon\delta\mu) = a_3(t_{00},t_{00}+\sigma_0;\varepsilon\delta\mu) + \int\limits_{t_{00}+\sigma_0-\sigma}^{t-\sigma} L_{f_0,K_0}(\xi+\sigma)|\Delta v(\xi)|d\xi +$$

$$+ \int\limits_{t_{00}+\sigma_0}^{t} L_{f_0,K_0}(\xi)|v_0(\xi-\sigma)-v_0(\xi-\sigma_0)|d\xi \leq O(\varepsilon\delta\mu) +$$

$$+ \int\limits_{t_{00}}^{t} \chi(\xi+\sigma)L_{f_0,K_0}(\xi+\sigma)|\Delta v(\xi)|d\xi +$$

$$+ \int\limits_{t_{00}+\sigma_0}^{r_2+\delta_2} L_{f_0,K_0}(\xi)|v_0(\xi-\sigma)-v_0(\xi-\sigma_0)|d\xi, \tag{2.27}$$

where $\chi(\xi)$ is the characteristic function of the interval J. Further,

$$|v_0(\xi-\sigma)-v_0(\xi-\sigma_0)| \leq |Y_2| \int\limits_{\xi-\sigma_0}^{\xi-\sigma} |f_0(s,y_0(s),h(t_{00},\varphi_0,u_0)(s-\tau_0),h(t_{00},g_0,v_0)(s-\sigma_0))|ds \leq$$

$$\leq O(\varepsilon\delta\mu).$$

From (2.27) it follows that

$$a_3(t_{00},t;\varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \chi(\xi+\sigma)L_{f_0,K_0}(\xi+\sigma)|\Delta v(\xi)|d\xi \leq O(\varepsilon\delta\mu) +$$

$$+ \int\limits_{t_{00}}^{t} \chi(\xi+\sigma)L_{f_0,K_0}(\xi+\sigma)|\Delta y(\xi)|d\xi, \quad t \in [t_{00}, r_2+\delta_2]. \tag{2.28}$$

Thus, for $t \in [t_{00}, t_0+\tau]$ we have

$$a_1(t_{00},t;\varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} [L_{f_0,K_0}(\xi) + \chi(\xi+\sigma)L_{f_0,K_0}(\xi+\sigma)]|\Delta y(\xi)|\, d\xi \tag{2.29}$$

(see (2.20), (2.22), (2.28)). Now we continue estimation of the expression $a_1(t_{00},t;\varepsilon\delta\mu)$ on the interval $[t_0+\tau, t_{00}+\tau_0]$. If $t \in [t_0+\tau, t_{00}+\tau_0]$, then $a_1(t_{00},t;\varepsilon\delta\mu) = a_1(t_{00},t_0+\tau;\varepsilon\delta\mu) + a_1(t_0+\tau,t;\varepsilon\delta\mu)$. By condition 2 of Theorem 1.1, the function $|a(\xi;\varepsilon\delta\mu)|$ is bounded, i.e., for

$t \in [t_0+\tau, t_{00}+\tau_0]$, $a_1(t_0+\tau, t; \varepsilon\delta\mu) \le O(\varepsilon; \delta\mu)$. Therefore, the condition (2.29) is valid on the whole interval $t \in [t_{00}, t_{00}+\tau_0]$. Let $t \in [t_{00}+\tau_0, r_2+\delta_2]$, then

$$a_1(t_{00}, t; \varepsilon\delta\mu) = a_1(t_{00}, t_{00}+\tau_0; \varepsilon\delta\mu) + a_1(t_{00}+\tau_0, t; \varepsilon\delta\mu).$$

It is clear that

$$a_1(t_{00}+\tau_0, t; \varepsilon\delta\mu) \le \int\limits_{t_{00}+\tau_0}^{t} L_{f_0,K_0}(\xi)|\Delta y(\xi)|d\xi + a_2(t_{00}+\tau_0, t; \varepsilon\delta\mu) + a_3(t_{00}+\tau_0, t; \varepsilon\delta\mu) \le$$

$$\le \int\limits_{t_{00}}^{t} L_{f_0,K_0}(\xi)|\Delta y(\xi)|d\xi + a_2(t_{00}+\tau_0, t; \varepsilon\delta\mu) + a_3(t_{00}, t; \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) +$$

$$+ \int\limits_{t_{00}}^{t} [L_{f_0,K_0}(\xi) + \chi(\xi+\sigma)L_{f_0,K_0}(\xi+\sigma)]|\Delta y(\xi)|d\xi + a_2(t_{00}+\tau_0, t; \varepsilon\delta\mu).$$

For $\xi \in [t_{00}+\tau_0, r_2+\delta_2]$ we get $\xi-\tau \ge t_0$, $\xi-\tau_0 \ge t_{00}$. Thus,

$$a_2(t_{00}+\tau_0, t; \varepsilon\delta\mu) = \int\limits_{t_{00}+\tau_0}^{t} L_{f_0,K_0}(\xi)|u_0(\xi-\tau) + \Delta u(\xi-\tau) - u_0(\xi-\tau_0)|d\xi \le$$

$$\le \int\limits_{t_{00}+\tau_0-\tau}^{t-\tau} L_{f_0,K_0}(\xi+\tau)|\Delta u(\xi)|d\xi +$$

$$+ \int\limits_{t_{00}+\tau_0}^{t} L_{f_0,K_0}(\xi)|u_0(\xi-\tau) - u_0(\xi-\tau_0)|d\xi \le$$

$$\le \int\limits_{t_{00}}^{t} \chi(\xi+\tau)L_{f_0,K_0}(\xi+\tau)|\Delta u(\xi)|d\xi +$$

$$+ \int\limits_{t_{00}+\tau_0}^{r_2+\delta_2} L_{f_0,K_0}(\xi)|u_0(\xi-\tau) - u_0(\xi-\tau_0)|d\xi. \qquad (2.30)$$

Further,

$$|u_0(\xi-\tau) - u_0(\xi-\tau_0)| \le |Y_1| \int\limits_{\xi-\tau_0}^{\xi-\tau} |f_0(s, y_0(s), h(t_{00}, \varphi_0, u_0)(s-\tau_0),$$

$$h(t_{00}, g_0, v_0)(s-\sigma_0))|ds \le O(\varepsilon\delta\mu).$$

Consequently, from (2.30) it follows that

$$a_2(t_{00}+\tau_0,t;\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\chi(\xi+\tau)L_{f_0,K_0}(\xi+\tau)|\Delta y(\xi)|d\xi$$

and on the interval $[t_{00}+\tau_0,r_2+\delta_2]$ we have

$$a_1(t_{00},t;\varepsilon\delta\mu)\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}[L_{f_0,K_0}(\xi)+\chi(\xi+\sigma)L_{f_0,K_0}(\xi+\sigma)+L_{f_0,K_0}(\xi+\tau)]|\Delta y(\xi)|d\xi.$$

It is clear that

$$\varepsilon b(\varepsilon\delta\mu)\le O(\varepsilon\delta\mu) \tag{2.31}$$

(see (2.16)). According to (2.10), (2.31) and estimates of the expression $a_1(t_{00},t;\varepsilon\delta\mu)$ (see (2.29)) inequality (2.15) implies

$$|\Delta y(t)|\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}[L_{f_0,K_0}(\xi)+\chi(\xi+\sigma)L_{f_0,K_0}(\xi+\sigma)+\chi(\xi+\tau)L_{f_0,K_0}(\xi+\tau)]|\Delta y(\xi)|d\xi.$$

By the Gronwall lemma, (2.9) follows.

Lemma 2.3 is proved.

Let $r_1=t_{00}$ and $r_2=t_{10}$ in Lemma 2.1, then

$$x_0(t)=\begin{cases}(\varphi_0(t),g_0(t))^T, & t\in[\hat{\tau},t_{00}),\\ y_0(t), & t\in[t_{00},t_{10}],\end{cases}$$

and for arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_1]\times V^-$

$$x(t;\mu_0+\varepsilon\delta\mu)=\begin{cases}(\varphi(t),g(t))^T, & t\in[\hat{\tau},t_0),\\ y(t;\mu_0+\varepsilon\delta\mu), & t\in[t_0,t_{10}+\delta_1],\end{cases}$$

(see (2.4)). We note that $\delta\mu\in V^-$, i.e., $t_0\le t_{00}$, therefore we obtain

$$\Delta x(t)=\begin{cases}\varepsilon(\delta\varphi(t),\delta g(t))^T & \text{for}\quad t\in[\hat{\tau},t_0),\\ y(t;\mu_0+\varepsilon\delta\mu)-(\varphi_0(t),g_0(t)^T & \text{for}\quad t\in[t_0,t_{00}),\\ \Delta y(t) & \text{for}\quad t\in[t_{00},t_{10}+\delta_1].\end{cases}$$

Thus,

$$\Delta p(t)=\begin{cases}\varepsilon\delta\varphi(t) & \text{for}\quad t\in[\hat{\tau},t_0),\\ u(t;\mu_0+\varepsilon\delta\mu)-\varphi_0(t) & \text{for}\quad t\in[t_0,t_{00}),\\ \Delta u(t) & \text{for}\quad t\in[t_{00},t_{10}+\delta_1],\end{cases}$$

$$\Delta z(t) = \begin{cases} \varepsilon\delta g(t) & \text{for} \quad t \in [\hat{\tau}, t_0), \\ v(t; \mu_0 + \varepsilon\delta\mu) - g_0(t) & \text{for} \quad t \in [t_0, t_{00}), \\ \Delta v(t) & \text{for} \quad t \in [t_{00}, t_{10} + \delta_1], \end{cases}$$

(see (1.4) and (2.7)). Here we assume that $[t_{00}, t_{00}) = \varnothing$. By Lemma 2.3 we get

$$|\Delta x(t)| \le O(\varepsilon\delta\mu) \quad \forall (t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-, \tag{2.32}$$

$$\Delta x(t_{00}) = \varepsilon \left\{ (\delta p_0, \delta g(t_{00}))^T + [(\Theta_{k\times 1}, \dot{g}_0^-)^T - f_0^-]\delta t_0 \right\} + o(\varepsilon\delta\mu). \tag{2.33}$$

According to (2.32) and relation (2.26) we conclude that

$$|\Delta p(t)| \le O(\varepsilon\delta\mu) \quad \forall (t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-, \tag{2.34}$$

$$|\Delta z(t)| \le O(\varepsilon\delta\mu) \quad \forall (t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-. \tag{2.35}$$

The functions $\Delta p(t)$ and $\Delta z(t)$ satisfy, respectively, the equations

$$\dot{\Delta} p(t) = Y_1[\theta(t; \varepsilon\delta\mu) + \varepsilon\vartheta(t; \varepsilon\delta\mu)] \tag{2.36}$$

and

$$\dot{\Delta} z(t) = Y_2[\theta(t; \varepsilon\delta\mu)\varepsilon\vartheta(t; \varepsilon\delta\mu)], \tag{2.37}$$

where

$$\begin{aligned} &\theta(t; \varepsilon\delta\mu) = f_0(t, x_0(t) + \Delta x(t), p_0(t-\tau) + \delta p(t-\tau), z_0(t-\sigma) + \Delta z(t-\sigma)) - f_0[t], \\ &f_0[t] = f_0(t, x_0(t), p_0(t-\tau_0), z_0(t-\sigma_0)), \end{aligned} \tag{2.38}$$

$$\vartheta(t; \varepsilon\delta\mu) = \delta f(t, x_0(t) + \Delta x(t), p_0(t-\tau) + \Delta p(t-\tau), z_0(t-\sigma) + \Delta z(t-\sigma)). \tag{2.39}$$

Lemma 2.4. *Let the conditions of Theorem 1.1 hold. Then*

$$\alpha_1(\varepsilon\delta\mu) = \int\limits_{t_{00}+\tau_0}^{t_{10}+\delta_2} \zeta(t) \left[|\Delta p(t-\tau) - \Delta p(t-\tau_0)| \right] dt \le o(\varepsilon\delta\mu), \tag{2.40}$$

$$\alpha_2(\varepsilon\delta\mu) = \int\limits_{t_{00}}^{t_{10}+\delta_2} \zeta(t) \left[|\Delta z(t-\sigma) - \Delta z(t-\sigma_0)| \right] dt \le o(\varepsilon\delta\mu), \tag{2.41}$$

for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2] \times V^-$, *where* $\zeta \in L(J, [0, \infty))$ *and the numbers* ε_2 *and* δ_2 *are as in Lemma 2.3.*

Proof. It is obvious that $t-\tau \geq t_{00}$ and $t-\tau_0 \geq t_{00}$ for $t \in [t_{00}+\tau_0, t_{10}+\delta_2]$. Therefore,

$$\alpha_1(\varepsilon\delta\mu) \leq \int\limits_{t_{00}+\tau_0}^{t_{10}+\delta_2} \zeta(t) \left[\int\limits_{t-\tau_0}^{t-\tau} |\dot{\Delta} p(\xi)| d\xi \right] dt \leq$$

$$\leq |Y_1| \int\limits_{t_{00}+\tau_0}^{t_{10}+\delta_2} \zeta(t) \left[\int\limits_{t-\tau_0}^{t-\tau} |\theta(\xi;\varepsilon\delta\mu)| d\xi + \varepsilon \int\limits_{t-\tau_0}^{t-\tau} |\vartheta(\xi;\varepsilon\delta\mu)| d\xi \right] dt \leq$$

$$\leq |Y_1| \alpha_{11}(t_{00}+\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu) + \varepsilon\alpha |Y_1| \int\limits_{t_{00}+\tau_0}^{t_{10}+\delta_2} \zeta(t) \left[\sum_{i=1}^{\nu} \int\limits_{t-\tau_0}^{t-\tau} m_{\delta f_i, K_0}(\xi) d\xi \right] dt \leq$$

$$\leq |Y_1| \alpha_{11}(t_{00}+\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu) + o(\varepsilon\delta\mu) \tag{2.42}$$

(see (2.36), (2.39) and (1.3)), here

$$\alpha_{11}(t', t''; \varepsilon\delta\mu) = \int\limits_{t'}^{t''} \zeta(t) \left[\int\limits_{t-\tau_0}^{t-\tau} |\theta(\xi;\varepsilon\delta\mu)| d\xi \right] dt.$$

a) Let $t_{00}+2\tau_0 \leq t_{10}$ and $\varepsilon_2 \in (0,\varepsilon_1]$ be so small that $t_0+2\tau > t_{00}+\tau_0$ $\forall(\varepsilon,\delta\mu) \in (0,\varepsilon_2)\times V^-$, then we have $\alpha_{11}(t_{00}+\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu) = \alpha_{11}(t_{00}+\tau_0, t_0+2\tau;\varepsilon\delta\mu) + \alpha_{11}(t_0+2\tau, t_{00}+2\tau_0;\varepsilon\delta\mu) + \alpha_{11}(t_{00}+2\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu)$. The function $\theta(\xi;\varepsilon\delta\mu)$ is bounded, therefore $\alpha_{11}(t_0+2\tau, t_{00}+2\tau_0;\varepsilon\delta\mu) \leq o(\varepsilon\delta\mu)$. It is not difficult to see that

$$\alpha_{11}(t_{00}+\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu) \leq o(\varepsilon\delta\mu) + \alpha_{12}(t_{00}+\tau_0, t_0+2\tau;\varepsilon\delta\mu) + \\ + \alpha_{12}(t_{00}+2\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu), \tag{2.43}$$

where

$$\alpha_{12}(t', t''; \varepsilon\delta\mu) = \int\limits_{t'}^{t''} \zeta(t) \alpha_{13}(t;\varepsilon\delta\mu) dt,$$

$$\alpha_{13}(t;\varepsilon\delta\mu) = \int\limits_{t-\tau_0}^{t-\tau} L_{f_0,K_0}(\xi) \Big\{ |\Delta x(\xi)| + |p_0(\xi-\tau) - p_0(\xi-\tau_0)| + |\Delta p(\xi-\tau)| + \\ + |z_0(\xi-\sigma) - z_0(\xi-\sigma_0)| + |\Delta z(\xi-\sigma)| \Big\} d\xi.$$

If $t \in [t_{00}+\tau_0, t_0+2\tau]$ and $\xi \in [t-\tau_0, t-\tau]$, then

$$\xi \geq t_{00}, \quad \xi-\tau \leq t_0, \quad \xi-\tau_0 \leq t_0. \tag{2.44}$$

Therefore,

$$|\Delta x(\xi)| \le O(\varepsilon\delta\mu), \quad |p_0(\xi-\tau) - p_0(\xi-\tau_0)| = |\varphi_0(\xi-\tau) - \varphi_0(\xi-\tau_0)| \le O(\varepsilon\delta\mu),$$
$$|\Delta p(\xi-\tau)| = \varepsilon|\delta\varphi(\xi-\tau)| \tag{2.45}$$

(see (2.35) and (2.21)). On the other hand,

$$|z_0(\xi-\sigma) - z_0(\xi-\sigma_0)| \le \int\limits_{\xi-\sigma_0}^{\xi-\sigma} |\dot{z}_0(s)|ds \le O(\varepsilon\delta\mu), \quad |\Delta z(\xi-\sigma)| \le O(\varepsilon\delta\mu) \tag{2.46}$$

(see (2.35) and condition 2 of Theorem 1.1). Further, if $t \in [t_{00}+2\tau_0, t_{10}+\delta_2]$ and $\xi \in [t-\tau_0, t-\tau]$ then

$$\xi \ge t_{00}+\tau_0, \quad \xi-\tau \ge t_{00}, \quad \xi-\tau_0 \ge t_{00}. \tag{2.47}$$

Therefore,

$$|p_0(\xi-\tau) - p_0(\xi-\tau_0)| \le \int\limits_{\xi-\tau_0}^{\xi-\tau} |\dot{p}_0(s)|ds \le O(\varepsilon\delta\mu), \quad |\Delta p(\xi-\tau)| \le O(\varepsilon\delta\mu). \tag{2.48}$$

On the basis of last relations for $t \in [t_{00}+\tau_0, t_{00}+2\tau] \cup [t_{00}+2\tau_0, t_{10}+\delta_2]$ we get

$$\alpha_{13}(t;\varepsilon\delta\mu) \le o(t;\varepsilon\delta\mu). \tag{2.49}$$

Thus, $\alpha_{12}(t_{00}+\tau_0, t_0+2\tau;\varepsilon\delta\mu) + \alpha_{12}(t_{00}+2\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu)$, i.e.,

$$\alpha_{11}(t_{00}+\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu) \tag{2.50}$$

(see (2.43)). According to (2.50) from (2.42) it follows that (2.40) holds.

a$_1$) Let $t_{00}+2\tau_0 > t_{10}$ and the numbers ε_2 and δ_2 be so small that $t_{00}+2\tau > t_{10}+\delta_2$. In a similar way we obtain that $\alpha_{11}(t_{00}+\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu) + \alpha_{12}(t_{00}+\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu)$ (see (2.42), (2.43)). If $t \in [t_{00}+\tau_0, t_{10}+\delta_2]$ and $\xi \in [t-\tau_0, t-\tau]$, then the conditions (2.44) and (2.49) hold, i.e., $\alpha_{12}(t_{00}+\tau_0, t_{10}+\delta_2;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu)$. Thus, the inequality (2.50) is valid. The first part of the lemma is proved. Now we prove inequality (2.41).

b) Let $t_{00}+\sigma_0 > t_{10}$ and the number ε_2 be so small that $t_0+\sigma > t_{10}+\delta_2$. In this case if $t \in [t_{00}, t_{10}+\delta_2]$ then $t-\sigma \le t_0$ and $t-\sigma_0 \le t_0$. Therefore,

$$\alpha_2(\varepsilon\delta\mu) = \varepsilon \int\limits_{t_{00}}^{t_{10}+\delta_2} \zeta(t)\left[|\delta g(t-\sigma) - \delta g(t-\sigma_0)|\right]dt \le o(\varepsilon\delta\mu).$$

b$_1$) Let $t_{00}+\sigma_0 = t_{10}$, then we have

$$\alpha_2(\varepsilon\delta\mu) = \alpha_{21}(t_{00}, t_0+\sigma;\varepsilon\delta\mu) + \alpha_{21}(t_0+\sigma, t_{00}+\sigma_0;\varepsilon\delta\mu) +$$
$$+\,\alpha_{21}(t_{00}+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu), \tag{2.51}$$

where

$$\alpha_{21}(t^{'},t^{''};\varepsilon\delta\mu) = \int\limits_{t^{'}}^{t^{''}} \zeta(t)\,[|\Delta z(t-\sigma)-\Delta z(t-\sigma_0)|]\,dt.$$

It is obvious that

$$\alpha_{21}(t_{00},t_0+\sigma;\varepsilon\delta\mu) = \varepsilon\int\limits_{t_{00}}^{t_0+\sigma} \zeta(t)\,[|\delta g(t-\sigma)-\delta g(t-\sigma_0)|]\,dt \le o(\varepsilon\delta\mu), \qquad (2.52)$$

therefore

$$\alpha_{21}(t_0+\sigma,t_{00}+\sigma_0;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu) \qquad (2.53)$$

(see (1.3), (2.35)). Further,

$$\alpha_{21}(t_{00}+\sigma_0,t_{10}+\delta_2;\varepsilon\delta\mu) \le \int\limits_{t_{00}+\sigma_0}^{t_{10}+\delta_2} \zeta(t)\left[\int\limits_{t-\sigma_0}^{t-\sigma} |\dot{\Delta} z(\xi)|d\xi\right]dt \le$$

$$\le |Y_2|\alpha_{22}(t_{00}+\sigma_0,t_{10}+\delta_2;\varepsilon\delta\mu)+$$

$$+\,\varepsilon\alpha|Y_2|\int\limits_{t_{00}+\sigma_0}^{t_{10}+\delta_2} \zeta(t)\left[\sum_{i=1}^{\nu}\int\limits_{t-\sigma_0}^{t-\sigma} m_{\delta f_i,K_0}(\xi)d\xi\right]dt \le$$

$$\le |Y_2|\alpha_{22}(t_{00}+\sigma_0,t_{10}+\delta_2;\varepsilon\delta\mu)+o(\varepsilon\delta\mu)$$

(see (2.37)), where

$$\alpha_{22}(t^{'},t^{''};\varepsilon\delta\mu) = \int\limits_{t^{'}}^{t^{''}} \zeta(t)\alpha_{23}(t;\varepsilon\delta\mu)dt,$$

$$\alpha_{23}(t;\varepsilon\delta\mu) = \int\limits_{t-\sigma_0}^{t-\sigma} L_{f_0,K_0}(\xi)\,\{|\Delta x(\xi)|+|p_0(\xi-\tau)-p_0(\xi-\tau_0)|+|\Delta p(\xi-\tau)|+$$

$$+|z_0(\xi-\sigma)-z_0(\xi-\sigma_0)|+|\Delta z(\xi-\sigma)|\}\,d\xi.$$

Let the numbers ε_2 and δ_2 be so small that $t_0+\tau+\sigma > t_{10}+\delta_2$, then for $t\in[t_{00}+\sigma_0,t_{10}+\delta_2]$ the conditions (2.44) are satisfied, therefore $\alpha_{23}(t;\varepsilon\delta\mu) \le o(t;\varepsilon\delta\mu)$ for $t\in[t_{00}+\sigma_0,t_{10}+\delta_2]$ (see (2.45) and (2.46)), i.e., $\alpha_{22}(t_{00}+\sigma_0,t_{10}+\delta_2;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu)$. Thus,

$$\alpha_{21}(t_{00}+\sigma_0,t_{10}+\delta_2;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu). \qquad (2.54)$$

According to (2.52), (2.53) and (2.54) from (2.51) it follows that (2.41) holds.

b$_2$) Let $t_{00}+\sigma_0 < t_{10}$. In this case we have the relation (2.51). Moreover, the conditions (2.52) and (2.53) hold. Further,

$$\alpha_{21}(t_{00}+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu) \le \int_{t_{00}+\sigma_0}^{t_{10}+\delta_2} \zeta(t)\left[\int_{t-\sigma_0}^{t-\sigma} |\dot{\Delta} z(\xi)|d\xi\right]dt \le$$

$$\le |Y_2|\hat{\alpha}_{22}(t_{00}+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu)+$$

$$+\varepsilon\alpha|Y_2|\int_{t_{00}+\sigma_0}^{t_{10}+\delta_2} \zeta(t)\left[\sum_{i=1}^{\nu}\int_{t-\sigma_0}^{t-\sigma} m_{\delta f_i,K_0}(\xi)d\xi\right]dt \le$$

$$\le |Y_2|\hat{\alpha}_{22}(t_{00}+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu)+o(\varepsilon\delta\mu),$$

where

$$\hat{\alpha}_{22}(t', t'';\varepsilon\delta\mu) = \int_{t'}^{t''} \zeta(t)\hat{\alpha}_{23}(t;\varepsilon\delta\mu)dt, \quad \hat{\alpha}_{23}(t;\varepsilon\delta\mu)dt = \int_{t-\sigma_0}^{t-\sigma} |\theta(\xi;\varepsilon\delta\mu)|d\xi.$$

We assume that $t_{00}+\tau_0+\sigma_0 \le t_{10}$ and $\varepsilon_2 \in (0,\varepsilon_1)$ is so small that $t_0+\tau+\sigma > t_{00}+\sigma_0$ $\forall(\varepsilon,\delta\mu) \in (0,\varepsilon_2)\times V^-$, then we have $\hat{\alpha}_{22}(t_{00}+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu) = \hat{\alpha}_{22}(t_{00}+\sigma_0, t_0+\tau+ +\sigma;\varepsilon\delta\mu) + \hat{\alpha}_{22}(t_{00}+\tau+\sigma, t_0+\tau_0+\sigma_0;\varepsilon\delta\mu) + \hat{\alpha}_{22}(t_{00}+\tau_0+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu)$. Taking into account boundedness of $|\theta(\xi;\varepsilon\delta\mu)|$ we get $\hat{\alpha}_{22}(t_{00}+\tau+\sigma, t_0+\tau_0+\sigma_0;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu)$.

It is not difficult to see that $\hat{\alpha}_{22}(t_{00}+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu) + \alpha_{22}(t_{00}+\sigma_0, t_0+\tau+ +\sigma;\varepsilon\delta\mu) + \alpha_{22}(t_{00}+\tau_0+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu)$ (see item b$_1$)). If $t \in [t_{00}+\sigma_0, t_0+\tau+\sigma]$ and $\xi \in [t-\sigma_0, t-\sigma]$, then the relations (2.44), (2.45) and (2.46) hold. If $t \in [t_{00}+\tau_0+\sigma_0, t_{10}+\delta_2]$ and $\xi \in [t-\sigma_0, t-\sigma]$, then the relations (2.47) and (2.48) hold.

Thus, $\alpha_{23}(t;\varepsilon\delta\mu) \le o(t,\varepsilon\delta\mu)$ for $t \in [t_{00}+\sigma_0, t_0+\tau+\sigma] \cup [t_{00}+\tau_0+\sigma_0, t_0+\delta_2]$, i.e., $\alpha_{22}(t_{00}+\sigma_0, t_0+\tau+\sigma;\varepsilon\delta\mu)+\alpha_{22}(t_{00}+\tau_0+\sigma_0, t_{10}+\delta_2;\varepsilon\delta\mu) \le o(\varepsilon\delta\mu)$, therefore (2.54) is valid. The inequality (2.41) is proved. Let $t_{00}+\tau_0+\sigma_0 > t_{10}$ and the numbers ε_2, δ_2 be so small that $t_0+\tau+\sigma > t_{10}+\delta_2$. In this case, if $t \in [t_{00}+\sigma_0, t_{10}+\delta_2]$ and $\xi \in [t-\sigma_0, t-\sigma]$, then (2.44) is valid, therefore the inequality (2.54) holds.

Lemma 2.4 is proved.

The following assertion can be proved by analogy with Lemma 2.3.

Lemma 2.5. *Let $t_{00}+\tau_0 \le r_2$ and the condition 2 of Theorem 1.1 and the condition 5 of Theorem 1.2 hold.Then there exist numbers $\varepsilon_2 \in (0,\varepsilon_1)$ and $\delta_2 \in (0,\delta_1)$ such that*

$$\max_{t\in[t_0, r_2+\delta_2]} |\Delta y(t)| \le O(\varepsilon\delta\mu)$$

for arbitrary $(\varepsilon,\delta\mu) \in [0,\varepsilon_2]\times V^+$. Moreover,

$$\Delta y(t_0) = \varepsilon\left\{(\delta p_0, \delta g(t_{00}))^T + [(\Theta_{k\times 1}, \dot{g}_0^+)^T - f_0^+]\delta t_0\right\} + o(\varepsilon\delta\mu).$$

It is not difficult to see that for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times V^+$ we have

$$\Delta x(t)=\begin{cases}\varepsilon(\delta\varphi(t),\delta g(t))^T & \text{for} \quad t\in[\hat\tau,t_{00}),\\ (\varphi(t),g(t))^T-y_0(t) & \text{for} \quad t\in[t_{00},t_0),\\ \Delta y(t) & \text{for} \quad t\in[t_{00},t_{10}+\delta_1].\end{cases}$$

Thus,

$$\Delta p(t)=\begin{cases}\varepsilon\delta\varphi(t) & \text{for} \quad t\in[\hat\tau,t_{00}),\\ \varphi(t)-u_0(t) & \text{for} \quad t\in[t_{00},t_0),\\ \Delta u(t) & \text{for} \quad t\in[t_0,t_{10}+\delta_1],\end{cases}$$

$$\Delta z(t)=\begin{cases}\varepsilon\delta g(t) & \text{for} \quad t\in[\hat\tau,t_{00}),\\ g_0(t)-v_0(t) & \text{for} \quad t\in[t_{00},t_0),\\ \Delta v(t) & \text{for} \quad t\in[t_0,t_{10}+\delta_1].\end{cases}$$

By Lemma 2.5 we obtain

$$|\Delta x(t)|\le O(\varepsilon\delta\mu) \quad \forall(t,\varepsilon,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+, \tag{2.55}$$

$$\Delta x(t_0)=\varepsilon\left\{(\delta p_0,\delta g(t_{00}))^T+[(\Theta_{k\times1},\dot g_0^+)^T-f_0^+]\delta t_0\right\}+o(\varepsilon\delta\mu). \tag{2.56}$$

According to (2.55) and relation (2.26) we get

$$|\Delta p(t)|\le O(\varepsilon\delta\mu) \quad \forall(t,\varepsilon,\delta\mu)\in[t_0,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+, \tag{2.57}$$

$$|\Delta z(t)|\le O(\varepsilon\delta\mu) \quad \forall(t,\varepsilon,\delta\mu)\in[\hat\tau,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+. \tag{2.58}$$

Lemma 2.6. *Let the conditions 1 and 2 of Theorem 1.1 hold. Then*

$$\int_{t_0+\tau}^{t_{10}+\delta_2}\zeta(t)\left[|\Delta p(t-\tau)-\Delta p(t-\tau_0)|\right]dt\le o(\varepsilon\delta\mu),$$

$$\int_{t_0}^{t_{10}+\delta_2}\zeta(t)\left[|\Delta z(t-\sigma)-\Delta z(t-\sigma_0)|\right]dt\le o(\varepsilon\delta\mu),$$

for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2)\times V^+$.

This lemma is proved analogously to Lemma 2.4.

3. Proof of Theorem 1.1. The function $\Delta x(t)$ satisfies the equation

$$\dot\Delta x(t)=\theta(t;\varepsilon\delta\mu)+\varepsilon\vartheta(t;\varepsilon\delta\mu)=f_{0x}[t]\Delta x(t)+f_{0p}[t]\Delta p(t-\tau_0)+f_{0z}[t]\Delta z(t-\sigma_0)+$$

$$+\varepsilon\delta f[t]+\sum_{i=1}^{2}r_i(t;\varepsilon\delta\mu) \tag{3.1}$$

on the interval $[t_{00}, t_{10} + \delta_2]$, where

$$r_1(t;\varepsilon\delta\mu) = \theta(t;\varepsilon\delta\mu) - f_{0x}[t]\Delta x(t) - f_{0p}[t]\Delta p(t-\tau_0) - f_{0z}[t]\Delta z(t-\sigma_0), \tag{3.2}$$

$$r_2(t;\varepsilon\delta\mu) = \varepsilon[\vartheta(t;\varepsilon\delta\mu) - \delta f[t]]. \tag{3.3}$$

By using the Cauchy formula [1, p. 21], one can represent the solution of Eq. (3.1) in the form

$$\Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon \int_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]d\xi +$$

$$+ \sum_{i=0}^{2}[R_{0i}(t;t_{00},\varepsilon\delta\mu) + R_{1i}(t;t_{00},\varepsilon\delta\mu)], \quad t \in [t_{00}, t_{10} + \delta_2], \tag{3.4}$$

where

$$R_{01}(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\Delta p(\xi)d\xi,$$

$$R_{02}(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}-\sigma_0}^{t_{00}} Y(\xi+\sigma_0;t)f_{0z}[\xi+\sigma_0]\Delta z(\xi)d\xi,$$

$$R_{1i}(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t)r_i(\xi;\varepsilon\delta\mu)d\xi, \quad i = 1,2,$$

and $Y(\xi;t)$ is the matrix function satisfying Eq. (1.8) and condition (1.9). The function $Y(\xi;t)$ is continuous on the set $\Pi = \{(\xi,t) : a \le \xi \le t, t \in J\}$ by Lemma 2.1.7 in [1, p. 22]. Therefore,

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\left\{(\delta p_0, \delta g(t_{00}))^T + [(\Theta_{k\times 1}, \dot{g}_0^-)^T - f_0^-]\delta t_0\right\} + o(t;\varepsilon\delta\mu) \tag{3.5}$$

(see (2.33)). For $R_{0i}(t;t_{00},\varepsilon\delta\mu)$, $i=1,2$, we have

$$R_{01}(t;t_{00},\varepsilon\delta\mu)=\varepsilon\int\limits_{t_{00}-\tau_0}^{t_0}Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\delta\varphi(\xi)d\xi+$$

$$+\int\limits_{t_0}^{t_{00}}Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\Delta p(\xi)d\xi=$$

$$=\varepsilon\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\delta\varphi(\xi)d\xi+$$

$$+\int\limits_{t_0}^{t_{00}}Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\Delta p(\xi)d\xi+o(t;\varepsilon\delta\mu), \tag{3.6}$$

where

$$o(t;\varepsilon\delta\mu)=-\varepsilon\int\limits_{t_0}^{t_{00}}Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\delta\varphi(\xi)d\xi.$$

Further,

$$R_{02}(t;t_{00},\varepsilon\delta\mu)=\varepsilon\int\limits_{t_{00}-\sigma_0}^{t_0}Y(\xi+\sigma_0;t)f_{0z}[\xi+\sigma_0]\delta g(\xi)d\xi+\int\limits_{t_0}^{t_{00}}Y(\xi+\sigma_0;t)f_{0z}[\xi+\sigma_0]\Delta z(\xi)d\xi=$$

$$=\varepsilon\int\limits_{t_{00}-\sigma_0}^{t_{00}}Y(\xi+\sigma_0;t)f_{0z}[\xi+\sigma_0]\delta g(\xi)d\xi+o(t;\varepsilon\delta\mu) \tag{3.7}$$

(see (2.35)). Let the number $\delta_2\in(0,\delta_1)$ be, in addition (see Lemma 2.3), so small that $t_{00}+\tau_0<$ $<t_{10}-\delta_2$. Obviously, for $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$ we have

$$R_{11}(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^{3}R_{2i}(t;\varepsilon\delta\mu), \tag{3.8}$$

where

$$R_{21}(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_0+\tau}r_{11}(\xi;t,\varepsilon\delta\mu)d\xi,\quad R_{22}(t;\varepsilon\delta\mu)=\int\limits_{t_0+\tau}^{t_{00}+\tau_0}r_{11}(\xi;t,\varepsilon\delta\mu)d\xi,$$

$$R_{23}(t;\varepsilon\delta\mu)=\int\limits_{t_{00}+\tau_0}^{t}r_{11}(\xi;t,\varepsilon\delta\mu)d\xi,\quad r_{11}(\xi;t,\varepsilon\delta\mu)=Y(\xi;t)r_1(\xi;\varepsilon\delta\mu).$$

We introduce the notations:

$$f_0[\xi;s,\varepsilon\delta\mu] = f_0\Big(\xi, x_0(\xi)+s\Delta x(\xi), p_0(\xi-\tau_0)+s(p_0(\xi-\tau)-p_0(\xi-\tau_0)+\Delta p(\xi-\tau)),$$

$$z_0(\xi-\sigma_0)+s(z_0(\xi-\sigma)-z_0(\xi-\sigma_0)+\Delta z(\xi-\sigma))\Big),$$

$$\nu(\xi;s,\varepsilon\delta\mu) = f_{0x}[\xi;s,\varepsilon\delta\mu]-f_{0x}[\xi],$$

$$\rho(\xi;s,\varepsilon\delta\mu) = f_{0p}[\xi;s,\varepsilon\delta\mu]-f_{0p}[\xi], \quad \varsigma(\xi;s,\varepsilon\delta\mu) = f_{0z}[\xi;s,\varepsilon\delta\mu]-f_{0z}[\xi],$$

$$\nu_1(\xi;\varepsilon\delta\mu)=\int_0^1\nu(\xi;s,\varepsilon\delta\mu)ds, \quad \rho_1(\xi;\varepsilon\delta\mu)=\int_0^1\rho(\xi;s,\varepsilon\delta\mu)ds, \quad \varsigma_1(\xi;\varepsilon\delta\mu)=\int_0^1\varsigma(\xi;s,\varepsilon\delta\mu)ds.$$

It is easy to see that

$$\theta(\xi;\varepsilon\delta\mu) = \int_0^1\frac{d}{ds}f_0[\xi;s,\varepsilon\delta\mu]ds=\int_0^1\Big\{f_{0x}[\xi;s,\varepsilon\delta\mu]\Delta x(\xi)+f_{0p}[\xi;s,\varepsilon\delta\mu](p_0(\xi-\tau)-$$

$$-p_0(\xi-\tau_0)+\Delta p(\xi-\tau))+f_{0z}[\xi;s,\varepsilon\delta\mu](z_0(\xi-\sigma)-z_0(\xi-\sigma_0)+\Delta z(\xi-\sigma))\Big\}ds=$$

$$=\nu_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)+\rho_1(\xi;\varepsilon\delta\mu)(p_0(\xi-\tau)-p_0(\xi-\tau_0)+\Delta p(\xi-\tau))+$$

$$+\varsigma_1(\xi;\varepsilon\delta\mu)(z_0(\xi-\sigma)-z_0(\xi-\sigma_0)+\Delta z(\xi-\sigma))+f_{0x}[\xi]\Delta x(\xi)+$$

$$+f_{0p}[\xi](p_0(\xi-\tau)-p_0(\xi-\tau_0)+\Delta p(\xi-\tau))+f_{0z}[\xi](z_0(\xi-\sigma)-$$

$$-z_0(\xi-\sigma_0)+\Delta z(\xi-\sigma)).$$

By using the last relation, we have

$$R_{21}(t;\varepsilon\delta\mu) = \sum_{i=1}^{7}R_{3i}(t;\varepsilon\delta\mu),$$

where

$$R_{31} = \int_{t_{00}}^{t_0+\tau}Y(\xi;t)\nu_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)d\xi,$$

$$R_{32} = \int_{t_{00}}^{t_0+\tau}Y(\xi;t)\rho_1(\xi;\varepsilon\delta\mu)(p_0(\xi-\tau)-p_0(\xi-\tau_0)+\Delta p(\xi-\tau))d\xi,$$

$$R_{33} = \int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_{0p}[\xi](\Delta p(\xi-\tau)-\Delta p(\xi-\tau_0))d\xi,$$

$$R_{34} = \int\limits_{t_{00}}^{t_0+\tau} Y(\xi;t)f_{0p}[\xi](p_0(\xi-\tau)-p_0(\xi-\tau_0))d\xi,$$

$$R_{35} = \int\limits_{t_{00}}^{t_0+\tau} Y(\xi;t)\varsigma_1(\xi;\varepsilon\delta\mu)(z_0(\xi-\sigma)-z_0(\xi-\sigma_0)+\Delta z(\xi-\sigma))d\xi,$$

$$R_{36} = \int\limits_{t_{00}}^{t_0+\tau} Y(\xi;t)f_{0z}[\xi](z_0(\xi-\sigma)-z_0(\xi-\sigma_0))d\xi,$$

$$R_{37} = \int\limits_{t_{00}}^{t_0+\tau} Y(\xi;t)f_{0z}[\xi](\Delta z(\xi-\sigma)-\Delta z(\xi-\sigma_0))d\xi$$

(see (3.2)). For $\xi \in [t_{00}, t_0+\tau]$ we get

$$\begin{aligned} \Delta p(\xi-\tau)-\Delta p(\xi-\tau_0) &= \varepsilon[\delta\varphi(\xi-\tau)-\delta\varphi(\xi-\tau_0)], \\ p_0(\xi-\tau)-p_0(\xi-\tau_0) &= \varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0). \end{aligned} \tag{3.9}$$

The function $\varphi_0(t)$ is absolutely continuous, therefore for each fixed Lebesgue point $\xi \in \in (t_{00}, t_{00}+\tau_0)$ of the function $\dot{\varphi}_0(\xi-\tau_0)$ we obtain

$$\varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0) = \int\limits_{\xi}^{\xi-\varepsilon\delta\tau} \dot{\varphi}_0(s-\tau_0)ds = -\varepsilon\dot{\varphi}_0(\xi-\tau_0)\delta\tau+\gamma(\xi;\varepsilon\delta\mu)+\gamma(\xi;\varepsilon\delta\mu) \tag{3.10}$$

with

$$\lim_{\varepsilon\to 0}\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon} = 0 \quad \text{uniformly for} \quad \delta\mu \in V^-. \tag{3.11}$$

Thus, (3.10) is valid for almost all points of the interval $(t_{00}, t_{00}+\tau_0)$. From (3.10) taking into that account boundedness of the function $\dot{\varphi}_0(\xi)$ it follows that

$$|\varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0)| \le O(\varepsilon\delta\mu) \quad \text{and} \quad \left|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\right| \le \text{const.} \tag{3.12}$$

According to (2.32), (3.9), (3.10), (3.12) for the expressions R_{3i}, $i=\overline{1,4}$, we have

$$|R_{31}| \le \| Y \| \, O(\varepsilon\delta\mu)\nu_2(\varepsilon\delta\mu), \quad |R_{32}| \le \| Y \| \, O(\varepsilon\delta\mu)\rho_2(\varepsilon\delta\mu), \quad |R_{33}| \le o(\varepsilon\delta\mu),$$

$$R_{34} = \hat{\gamma}(t;\varepsilon\delta\mu) - \varepsilon\left[\int\limits_{t_{00}}^{t_0+\tau} Y(\xi;t)f_{0p}[\xi]\dot{\varphi}_0(\xi-\tau_0)d\xi\right]\delta\tau,$$

where

$$\nu_2(\varepsilon\delta\mu) = \int\limits_{t_{00}}^{b} \Bigg[\int\limits_0^1 |f_{0x}(\xi, x_0(\xi) + s\Delta x(\xi), \varphi_0(\xi - \tau_0) + s(\varphi_0(\xi - \tau) - \varphi_0(\xi - \tau_0) +$$

$$+ \varepsilon\delta\varphi(\xi - \tau)), z_0(\xi - \sigma_0) + s(z_0(\xi - \sigma) - z_0(\xi - \sigma_0) + \Delta z(\xi - \sigma))) -$$

$$- f_{0x}(\xi, x_0(\xi), \varphi_0(\xi - \tau_0), z_0(\xi - \sigma_0))| ds \Bigg] d\xi,$$

$$\rho_2(\varepsilon\delta\mu) = \int\limits_{t_{00}}^{b} \Bigg[\int\limits_0^1 |f_{0p}(\xi, x_0(\xi) + s\Delta x(\xi), \varphi_0(\xi - \tau_0) + s(\varphi_0(\xi - \tau) - \varphi_0(\xi - \tau_0) +$$

$$+ \varepsilon\delta\varphi(\xi - \tau)), z_0(\xi - \sigma_0) + s(z_0(\xi - \sigma) - z_0(\xi - \sigma_0) + \Delta z(\xi - \sigma))) -$$

$$- f_{0p}(\xi, x_0(\xi), \varphi_0(\xi - \tau_0), z_0(\xi - \sigma_0))| ds \Bigg] d\xi,$$

$$\|Y\| = \sup\{|Y(\xi;t)| : (\xi,t) \in \Pi\}, \quad \hat{\gamma}(t;\varepsilon\delta\mu) = \int\limits_{t_{00}}^{t_0+\tau} Y(\xi;t) f_{0p}[\xi] \gamma(\xi;\varepsilon\delta\mu) d\xi.$$

Obviously,

$$\left| \frac{\hat{\gamma}(t;\varepsilon\delta\mu)}{\varepsilon} \right| \le \|Y\| \int\limits_{t_{00}}^{t_{00}+\tau_0} |f_{0p}[\xi]| \left| \frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon} \right| d\xi.$$

By the Lebesgue convergence theorem, we have

$$\lim_{\varepsilon \to 0} \varsigma_2(\varepsilon\delta\mu) = 0, \quad \lim_{\varepsilon \to 0} \rho_2(\varepsilon\delta\mu) = 0, \quad \lim_{\varepsilon \to 0} \left| \frac{\hat{\gamma}(t;\varepsilon\delta\mu)}{\varepsilon} \right| = 0$$

uniformly for $(t, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times V^-$ (see (2.33), (2.35), (3.11), (3.12)). Thus,

$$R_{3i}(t;\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu), \quad i = \overline{1,3}, \tag{3.13}$$

and

$$R_{34}(t;\varepsilon\delta\mu) = -\varepsilon \left[\int\limits_{t_{00}}^{t_0+\tau} Y(\xi;t) f_{0p}[\xi] \dot{\varphi}_0(\xi - \tau_0) d\xi \right] \delta\tau + o(t;\varepsilon\delta\mu).$$

It is clear that

$$\varepsilon \left[\int\limits_{t_0+\tau}^{t_{00}+\tau_0} Y(\xi;t) f_{0p}[\xi] \dot{\varphi}_0(\xi - \tau_0) d\xi \right] \delta\tau = o(t;\varepsilon\delta\mu),$$

$$\dot{p}_0(\xi-\tau_0) = \dot{\varphi}_0(\xi-\tau_0), \quad \xi \in [t_{00}, t_{00}+\tau_0],$$

therefore,

$$R_{34}(t;\varepsilon\delta\mu) = -\varepsilon \left[\int\limits_{t_{00}}^{t_{00}+\tau_0} Y(\xi;t) f_{0p}[\xi] \dot{p}_0(\xi-\tau_0) d\xi \right] \delta\tau + o(t;\varepsilon\delta\mu). \tag{3.14}$$

The function $z_0(\xi)$, $\xi \in [t_{00}, t_{10}+\delta_2]$, is absolutely continuous, therefore for each fixed Lebesgue point $\xi \in (t_{00}, t_{10}+\delta_2)$ of the function $\dot{z}_0(\xi-\tau_0)$ we obtain

$$z_0(\xi-\sigma) - z_0(\xi-\sigma_0) = \int\limits_{\xi}^{\xi-\varepsilon\delta\sigma} \dot{z}_0(s-\sigma_0) ds = -\varepsilon \dot{z}_0(\xi-\sigma_0)\delta\sigma + \gamma_1(\xi;\varepsilon\delta\mu) \tag{3.15}$$

with

$$\lim_{\varepsilon\to 0} \frac{\gamma_1(\xi;\varepsilon\delta\mu)}{\varepsilon} = 0 \quad \text{uniformly for} \quad \delta\mu \in V^-$$

and

$$\left| \frac{\gamma_1(\xi;\varepsilon\delta\mu)}{\varepsilon} \right| \le \text{const} \quad \forall \xi \in (t_{00}, r_2+\delta_2).$$

Thus, for R_{3i}, $i=5,6$, we get

$$|R_{35}| \le \|Y\| O(\varepsilon\delta\mu) \varsigma_2(\varepsilon\delta\mu), \quad R_{36} = \hat{\gamma}_1(t;\varepsilon\delta\mu) - \varepsilon \left[\int\limits_{t_{00}}^{t_0+\tau} Y(\xi;t) f_{0z}[\xi] \dot{z}_0(\xi-\sigma_0) d\xi \right] \delta\sigma$$

(see (2.35), (3.15)), where

$$\varsigma_2(\varepsilon\delta\mu) = \int\limits_{t_{00}}^{b} \left[\int\limits_0^1 |f_{0z}(\xi, x_0(\xi)+s\Delta x(\xi), \varphi_0(\xi-\tau_0) + s(\varphi_0(\xi-\tau) - \varphi_0(\xi-\tau_0) + \right.$$

$$+ \varepsilon\delta\varphi(\xi-\tau)), z_0(\xi-\sigma_0) + s(z_0(\xi-\sigma) - z_0(\xi-\sigma_0) + \Delta z(\xi-\sigma))) -$$

$$\left. - f_{0z}(\xi, x_0(\xi), \varphi_0(\xi-\tau_0), z_0(\xi-\sigma_0))| ds \right] d\xi,$$

$$\hat{\gamma}_1(t;\varepsilon\delta\mu) = \int\limits_{t_{00}}^{t_{00}+\tau} Y(\xi;t) f_{0z}[\xi] \gamma_1(\xi;\varepsilon\delta\mu) d\xi.$$

It is clear that

$$\lim_{\varepsilon\to 0} \varsigma_2(\varepsilon\delta\mu) = 0, \quad \lim_{\varepsilon\to 0} \left| \frac{\hat{\gamma}_1(t;\varepsilon\delta\mu)}{\varepsilon} \right| = 0$$

and

$$\varepsilon\left[\int_{t_0+\tau}^{t_{00}+\tau_0} Y(\xi;t)f_{0z}[\xi]\dot{z}_0(\xi-\sigma_0)d\xi\right]\delta\sigma = o(t;\varepsilon\delta\mu).$$

Consequently,

$$|R_{35}| \le o(\varepsilon\delta\mu), \quad R_{36} = -\varepsilon\left[\int_{t_{00}}^{t_{00}+\tau_0} Y(\xi;t)f_{0z}[\xi]\dot{z}_0(\xi-\sigma_0)d\xi\right]\delta\sigma + o(t;\varepsilon\delta\mu). \tag{3.16}$$

Moreover,

$$|R_{37}| \le o(\varepsilon\delta\mu) \tag{3.17}$$

(see (2.41)). Thus,

$$R_{21}(t;\varepsilon\delta\mu) = -\varepsilon\left[\int_{t_{00}}^{t_{00}+\tau_0} Y(\xi;t)f_{0p}[\xi]\dot{p}_0(\xi-\tau_0)d\xi\right]\delta\tau -$$

$$-\varepsilon\left[\int_{t_{00}}^{t_{00}+\tau_0} Y(\xi;t)f_{0z}[\xi]\dot{z}_0(\xi-\sigma_0)d\xi\right]\delta\sigma + o(t;\varepsilon\delta\mu) \tag{3.18}$$

(see (3.13), (3.14), (3.16), (3.17)). Now let us transform $R_{22}(t;\varepsilon\delta\mu)$. We have

$$R_{22}(t;\varepsilon\delta\mu) = \sum_{i=1}^{4} R_{4i}(t;\varepsilon\delta\mu),$$

where

$$R_{41}(t;\varepsilon\delta\mu) = \int_{t_0+\tau}^{t_{00}+\tau_0} Y(\xi;t)\theta(\xi;\varepsilon\delta\mu)d\xi,$$

$$R_{42}(t;\varepsilon\delta\mu) = -\int_{t_0+\tau}^{t_{00}+\tau_0} Y(\xi;t)f_{0x}[\xi]\Delta x(\xi)d\xi,$$

$$R_{43}(t;\varepsilon\delta\mu) = -\int_{t_0+\tau}^{t_{00}+\tau_0} Y(\xi;t)f_{0p}[\xi]\Delta p(\xi-\tau_0)d\xi,$$

$$R_{44}(t;\varepsilon\delta\mu) = -\int_{t_0+\tau}^{t_{00}+\tau_0} Y(\xi;t)f_{0z}[\xi]\Delta z(\xi-\sigma_0)d\xi.$$

If $\xi \in [t_0+\tau, t_{00}+\tau_0]$, we get

$$p_0(\xi-\tau)+\Delta p(\xi-\tau) = u(\xi-\tau;\varepsilon\delta\mu) = u_0(\xi-\tau)+\Delta u(\xi-\tau;\varepsilon\delta\mu)$$

and

$$p_0(\xi-\tau_0) = \varphi_0(\xi-\tau_0).$$

Therefore,

$$\lim_{\varepsilon\to 0}(\xi, x_0(\xi)+\Delta x(\xi), p_0(\xi-\tau)+\Delta p(\xi-\tau), z_0(\xi-\sigma)+\Delta z(\xi-\sigma)) =$$

$$= \lim_{\xi\to t_{00}+\tau_0-}(\xi, x_0(\xi), u_0(\xi-\tau_0), z_0(\xi-\sigma_0)) = w_{01},$$

$$\lim_{\varepsilon\to 0}(\xi, x_0(\xi), p_0(\xi-\tau_0), z_0(\xi-\sigma_0)) = \lim_{\xi\to t_{00}+\tau_0-}(\xi, x_0(\xi)\varphi_0(\xi-\tau_0), z_0(\xi-\sigma_0)) = w_{02}.$$

On the basis of the last relations we obtain

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[t_0+\tau, t_{00}+\tau_0]}|\theta(\xi;\varepsilon\delta\mu)-f_{01}^-| = 0$$

(see the condition 4)). The function $Y(\xi;t)$ is continuous on the set $[t_{00}, t_{00}+\tau_0]\times[t_{10}-\delta_2, t_{10}+ +\delta_2] \subset \Pi$. Thus, $R_{41}(t;\varepsilon\delta\mu) = -\varepsilon Y(t_{00}+\tau_0;t)f_{01}^-(\delta t_0+\delta\tau)+o(t;\varepsilon\delta\mu)$. Further, if $\xi \in \in [t_0+\tau, t_0+\tau_0]$, then $\Delta p(\xi-\tau_0) = \varepsilon\delta\varphi(\xi-\tau_0)$, therefore

$$R_{43}(t;\varepsilon\delta\mu) = -\varepsilon\int_{t_0+\tau}^{t_0+\tau_0} Y(\xi;t)f_{0p}[\xi]\delta\varphi(\xi-\tau_0)d\xi - \int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_{0p}[\xi],$$

$$\Delta p(\xi-\tau_0)d\xi = -\int_{t_0}^{t_{00}} Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\Delta p(\xi)d\xi + o(t;\varepsilon\delta\mu).$$

Moreover, we note that $|R_{42}(t;\varepsilon\delta\mu)| \le o(t;\varepsilon\delta\mu)$, $|R_{44}(t;\varepsilon\delta\mu)| \le o(t;\varepsilon\delta\mu)$ (see (2.32), (2.35)). Consequently, we have

$$R_{22}(t;\varepsilon\delta\mu) = -\varepsilon Y(t_{00}+\tau_0;t)f_{01}^-(\delta t_0+\delta\tau)-$$

$$+\int_{t_0}^{t_{00}} Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\Delta p(\xi)d\xi + o(t;\varepsilon\delta\mu). \tag{3.19}$$

By similar transformations (see the proof of formula (3.18)), on the basis of (2.40) and (2.41), one can prove

$$R_{23}(t;\varepsilon\delta\mu) = -\varepsilon\left[\int_{t_{00}+\tau_0}^{t} Y(\xi;t)f_{0p}[\xi]\dot{p}_0(\xi-\tau_0)d\xi\right]\delta\tau-$$

$$-\varepsilon\left[\int_{t_{00}+\tau_0}^{t} Y(\xi;t)f_{0z}[\xi]\dot{z}_0(\xi-\sigma_0)d\xi\right]\delta\sigma + o(t;\varepsilon\delta\mu). \tag{3.20}$$

Taking into consideration (3.18) – (3.20), from (3.8) we obtain

$$R_{11}(t;t_{00},\varepsilon\delta\mu) = -\varepsilon\Bigg\{\Bigg[\int\limits_{t_{00}+\tau_0}^{t} Y(\xi;t)f_{0p}[\xi]\dot{p}_0(\xi-\tau_0)d\xi\Bigg]\delta\tau+$$

$$+\,\varepsilon\Bigg[\int\limits_{t_{00}+\tau_0}^{t} Y(\xi;t)f_{0z}[\xi]\dot{z}_0(\xi-\sigma_0)d\xi\Bigg]\delta\sigma+\varepsilon Y(t_{00}+\tau_0;t)f_{01}^{-}(\delta t_0+\delta\tau)\Bigg\}-$$

$$-\int\limits_{t_0}^{t_{00}} Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\Delta p(\xi)d\xi+o(t;\varepsilon\delta\mu). \tag{3.21}$$

For the $R_{12}(t;t_{00},\varepsilon\delta\mu)$ we have

$$|R_{12}(t;t_{00},\varepsilon\delta\mu)|\le\varepsilon\alpha\sum_{i=1}^{\nu}\Big[w_i(t_{00},t_{00}+\tau;\varepsilon\delta\mu)+w_i(t_{00}+\tau,t_{00}+\tau_0;\varepsilon\delta\mu)+w_i(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)\Big],$$

where

$$w_i(t',t'';\varepsilon\delta\mu) = \int\limits_{t'}^{t''} L_{\delta f_i,K_0}(t)\Big\{|\Delta x(t)|+|p_0(t-\tau)-p_0(t-\tau_0)|+$$

$$+\,|\Delta p(t-\tau)|+|z_0(t-\sigma)-z_0(t-\sigma_0)|+|\Delta z(t-\sigma)|\Big\}\,dt$$

(see (3.3)). Owing to (2.32) – (2.35), (2.45) and (2.46) we get

$$w_i(t_{00},t_{00}+\tau;\varepsilon\delta\mu)|\le O(\varepsilon\delta\mu),\quad w_i(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)\le O(\varepsilon\delta\mu).$$

On the other hand, $\varepsilon w_i(t_{00}+\tau,t_{00}+\tau_0;\varepsilon\delta\mu)\le o(\varepsilon\delta\mu)$. Consequently,

$$|R_{12}(t;t_{00},\varepsilon\delta\mu)|\le o(\varepsilon\delta\mu). \tag{3.22}$$

From (3.4) according to (3.5) – (3.7), (3.21) and (3.22) we obtain the desired formula (1.5), where $\delta x(t;\delta\mu)$ has the form (1.6).

4. Proof of Theorem 1.2. The function $\Delta x(t)$ on the interval $[t_0,t_{10}+\delta_2]$ satisfies Eq. (3.1) and therefore it can be represented by the Cauchy formula

$$\Delta x(t) = Y(t_0;t)\Delta x(t_0)+\varepsilon\int\limits_{t_0}^{t} Y(\xi;t)\delta f[\xi]d\xi+$$

$$+\sum_{i=0}^{2}[R_{0i}(t;t_0,\varepsilon\delta\mu)+R_{1i}(t;t_0,\varepsilon\delta\mu)],\quad t\in[t_0,t_{10}+\delta_2]. \tag{4.1}$$

The matrix function $Y(\xi;t)$ is continuous on the set $[t_{00}, t_{00}+\tau_0]\times[t_{10}-\delta_2, t_{10}+\delta_2]\subset\Pi$, therefore

$$Y(t_0;t)\Delta x(t_0)=\varepsilon Y(t_{00};t)\left\{(\delta p_0,\delta g(t_{00}))^T+[(\Theta_{k\times1},\dot g_0^+)^T-f_0^+]\delta t_0\right\}+o(t;\varepsilon\delta\mu) \tag{4.2}$$

(see (2.56)).

Let a number $\varepsilon_2\in(0,\varepsilon_1)$ be sufficiently small so that for any $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^+$

$$t_0-\tau_0\le t_{00},\quad t_0-\sigma_0\le t_{00}.$$

Now let us transform $R_{0i}(t;t_0,\varepsilon\delta\mu)$, $i=1,2$. We have

$$R_{01}(t;t_0,\varepsilon\delta\mu)=\varepsilon\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\delta\varphi(\xi)d\xi+$$

$$+\int\limits_{t_{00}}^{t_0}Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\Delta p(\xi)d\xi+o(t;\varepsilon\delta\mu), \tag{4.3}$$

$$R_{02}(t;t_0,\varepsilon\delta\mu)=\varepsilon\int\limits_{t_{00}-\sigma_0}^{t_{00}}Y(\xi+\sigma_0;t)f_{0z}[\xi+\sigma_0]\delta g(\xi)d\xi+o(t;\varphi\delta\mu). \tag{4.4}$$

In a similar way (see the proof of Theorem 1.1), on the basis of (2.55), (2.57) and (2.58), one can prove that

$$R_{11}(t;t_0,\varepsilon\delta\mu)=-\varepsilon\left\{\left[\int\limits_{t_{00}+\tau_0}^{t}Y(\xi;t)f_{0p}[\xi]\dot p_0(\xi-\tau_0)d\xi\right]\delta\tau+\right.$$

$$\left.+\left[\int\limits_{t_{00}+\tau_0}^{t}Y(\xi;t)f_{0z}[\xi]\dot z_0(\xi-\sigma_0)d\xi\right]\delta\sigma+Y(t_{00}+\tau_0;t)f_{01}^+(\delta t_0+\delta\tau)\right\}-$$

$$-\int\limits_{t_{00}}^{t_0}Y(\xi+\tau_0;t)f_{0p}[\xi+\tau_0]\Delta p(\xi)d\xi+o(t;\varepsilon\delta\mu),$$

$$|R_{12}(t;t_{00},\varepsilon\delta\mu)|\le o(\varepsilon\delta\mu). \tag{4.5}$$

Finally, we note that

$$\varepsilon\int\limits_{t_0}^{t}Y(\xi;t)\delta f[\xi]d\xi=\varepsilon\int\limits_{t_{00}}^{t}Y(\xi;t)\delta f[\xi]d\xi+o(t;\varepsilon\delta\mu). \tag{4.6}$$

From (4.1) according to (4.2) – (4.6) we obtain the desired formula (1.5), where $\delta x(t; \delta\mu)$ has the form (1.10).

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