UDC 517.9

NEW RESULTS ON PERIODIC SOLUTIONS TO IMPULSIVE NONAUTONOMOUS EVOLUTION EQUATIONS WITH TIME DELAYS*

НОВІ РЕЗУЛЬТАТИ ПРО ПЕРІОДИЧНІ РОЗВ'ЯЗКИ НЕАВТОНОМНИХ ЕВОЛЮЦІЙНИХ РІВНЯНЬ З ІМПУЛЬСАМИ ТА ЗАПІЗНЕННЯМ

Rong-Nian Wang

Guangdong Univ. Foreign Studies
Guangzhou 510420, P.R. China
e-mail: rnwang@mail.ustc.edu.cn

Peng-Xian Zhu

Nanchang Univ.
Nanchang, Jiangxi 330031, P.R. China
e-mail: pxzhuncu@126.com

This paper is devoted to establishing some new results on periodic solutions to a class of nonautonomous impulsive evolution equations with time delays. On making some suitable assumptions such as solutions of equations are ultimately bounded, we obtain the existence theorem of periodic solutions to such equations using the Horn's fixed point theorem. At the end of the paper, an application to a nonautonomous impulsive partial differential equation with finite time delay is given.

Встановлено нові результати про періодичні розв'язки для класу неавтономних еволюційних рівнянь із запізненням. За прийнятними умовами, такими як умова, що розв'язки рівнянь є зрештою обмеженими, отримано теорему про існування періодичних розв'язків таких рівнянь за допомогою теореми Хорна про нерухому точку. Також наведено застосування до неавтономного рівняння з частинними похідними з імпульсами та скінченним запізненням.

1. Introduction. Evolution equations with delays (i.e., with some of the past states of the systems), compared with those without delays, are more realistic to describe many phenomena in nature, and they have a very strong application background. We refer readers to [8, 9, 11] and references therein for more comments. Hence this class of equations has been investigated in various aspects. Among these investigations, several are concerned with the periodic solutions of evolution equations with time delays taking values in infinite-dimensional spaces; see, e.g., [7, 18, 19, 21, 22]. Let us mention, in particular, that Liu [17] considered the existence of periodic solutions to the following evolution equation with time delay

$$u'(t) + Au(t) = f(t, u(t), u_t), \quad t > 0,$$

$$u(s) = \phi(s), \quad s \in [-r, 0],$$

in a Banach space using the boundedness of the solutions, where $r > 0$ is a positive constant, A is an unbounded linear operator, and $u_t(s) = u(t+s)$, $s \in [-r, 0]$, and Li [14] discussed

* Research supported by the NNSF of China (No. 11471083, 11101202).

the existence and asymptotic stability of periodic solution to evolution equation with multiple delays of form

$$u'(t) + Au(t) = f(t, u(t), u(t-\tau_1), \ldots, u(t-\tau_n)), \quad t \in \mathbb{R},$$

in a Hilbert space using the theory of analytic semigroup and an integral inequality with delays, where τ_i, $i = 1, \ldots, n$, are positive constants and A is a positive definite selfadjoint operator.

On the other hand, it is known that impulsive evolution equations are adequate mathematical apparatuses for simulation of numerous evolutionary processes which depend on their prehistory and are subject to abrupt changes of states at certain moments of time between intervals of continuous evolution (these changes can be well approximated as being instantaneous changes as state, that is, in the form of "impulses"). Such processes occur in the theory of optimal control, biotechnologies, industrial robotics, economics, etc (see, e.g., [5, 20]). Since the end of the last century, impulsive evolution equations in infinite-dimensional spaces have been investigated by many authors; see [3, 4, 12, 13, 15] and the references therein. We would like to mention that Ezzinbi et al. [10] studied the periodic solutions to the following impulsive evolution equation:

$$\begin{aligned}
&u'(t) + Au(t) = f(t, u(t)), \quad t > 0, \quad t \neq t_i, \\
&u(0) = u_0, \\
&\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, \quad 0 < t_1 < t_2 < \ldots < \infty,
\end{aligned}$$

in a Banach space using the boundedness of the solutions, where A is an unbounded linear operator and $\triangle u(t_i) = u(t_i^+) - u(t_i^-)$, and in [16], Liang et al. extended the results mentioned in [10, 17] and related papers to the study of periodic solutions to the semilinear impulsive evolution equation with finite time delay in the form

$$\begin{aligned}
&u'(t) + Au(t) = f(t, u(t), u_t), \quad t > 0, \quad t \neq t_i, \\
&u(s) = \phi(s), \quad s \in [-r, 0], \\
&\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, \quad 0 < t_1 < t_2 < \ldots < \infty.
\end{aligned}$$

When dealing with some parabolic evolution equations, it is usually assumed that the partial differential operator in the linear part (possibly unbounded) depends on time (i.e., it is the case of equations being nonautonomous), stimulated by the fact that this class of operators appears very often in the applications. Hence, it is natural to ask whether it is possible to study the periodic solutions to impulsive nonautonomous evolution equation with finite time delay. In fact, to the best of our knowledge, this study is a topic not yet considered in the literature. Motivated by the consideration above, in this paper, among others, we are interested in studying the periodic solutions to evolution equation having the form

$$\begin{aligned}
&u'(t) = A(t)u(t) + F(t, u(t), u_t), \quad t > 0, \quad t \neq t_i, \\
&u(s) = \phi(s), \quad s \in [-r, 0], \\
&\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, \quad 0 < t_1 < t_2 < \ldots < \infty,
\end{aligned} \tag{1.1}$$

in the Banach space $(X, \|\cdot\|)$, where $(A(t))_{t\in\mathbb{R}^+}$ (possibly unbounded), depending on time, is a family of closed and densely defined linear operators on X and has the domains $(D(A(t)))_{t\in\mathbb{R}^+}$, $u_t(s) = u(t+s)$, $s \in [-r, 0]$, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ represents the jump of the function u at t_i, and F, I_i, $i = 1, 2, \ldots$, are appropriate functions to be specified later. As one has seen, the equation above is nonautonomous and I_i, $i = 1, 2, \ldots$, constitute impulsive conditions.

This work is a continuation of Liu [17–19], Ezzinbi et al. [10], and Liang et al. [16]. The new results obtained here extend some results in this area for impulsive autonomous evolution equations with time delays. Moreover, even for corresponding nonautonomous evolution equation without impulsive conditions or without delays, the results here are new.

2. Preliminaries. Throughout this paper, $\mathcal{L}(X)$ stands for the Banach space of all bounded linear operators from X to X equipped with its natural topology, $C([-r,0];X)$ is the Banach space of all continuous functions ϕ from $[-r,0]$ to X with the supremum norm

$$\|\phi\|_0 = \sup_{s\in[-r,0]} \|\phi(s)\|,$$

and $PC([-r,0];X)$ is the Banach space of all piecewise continuous functions φ from $[-r,0]$ to X with supremum norm

$$\|\varphi\|_{PC} = \sup_{s\in[-r,0]} \|\varphi(s)\|.$$

That is, $\varphi \in PC([-r,0];X)$ if and only if φ is continuous in $[-r,0]$ except for finite points where φ is left continuous and has right limits.

Definition 2.1. *A family $U = \{U(t,\tau) : t \geq \tau, t, \tau \in \mathbb{R}^+\}$ of bounded liner operators on X is called an evolution family if*

(1) $U(t,r)U(r,\tau) = U(t,\tau)$ and $U(t,t) = I$ for all $t \geq r \geq \tau$ and $t, r, \tau \in \mathbb{R}^+$,

(2) the map $(t,\tau) \mapsto U(t,\tau)\xi$ is continuous for all $\xi \in X$, $t \geq \tau$ and $t, \tau \in \mathbb{R}^+$.

Frow now on, Acquistapace and Terreni conditions (AT_1) and (AT_2) (parabolicity conditions) below will be assumed throughout.

(AT_1) $A(t)$ are linear operators on X and there are constants $\lambda_0 \geq 0$, $\theta \in \left(\dfrac{\pi}{2}, \pi\right)$, and $K_1 \geq 0$ such that $\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0)$ and for all $\lambda \in \Sigma_\theta \cup \{0\}$ and $t \in \mathbb{R}^+$,

$$\|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K_1}{1 + |\lambda|}.$$

(AT_2) There are constants $K_2 \geq 0$ and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that for all $\lambda \in \Sigma_\theta$ and $t, \tau \in \mathbb{R}^+$,

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(\tau))]\| \leq \frac{K_2|t-\tau|^\alpha}{|\lambda|^\beta}.$$

Here $\Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\}; |\lambda| \leq \theta\}$.

Conditions (AT_1) and (AT_2), which are initiated by Acquistapace and Terreni [1, 2] for $\lambda_0 = 0$, are well understood and widely used in the literature. Furthermore, it is known that under conditions (AT_1) and (AT_2), there exists a unique evolution family $U = \{U(t,\tau) : t \geq \tau, t, \tau \in \mathbb{R}^+\}$ on X such that

(i) $U(\cdot,\tau) \in C^1((\tau,\infty),\mathcal{L}(X))$, $\dfrac{\partial U(t,\tau)}{\partial t} = A(t)U(t,\tau)$ for $t > \tau$, and for $0 < t-\tau \le 1$, $k = 0,1$, $\|A(t)^k U(t,\tau)\| \le C(t-\tau)^{-k}$,

(ii) $\dfrac{\partial^+ U(t,\tau)\xi}{\partial \tau} = -U(t,\tau)A(\tau)\xi$ for $t > \tau$ and $\xi \in D(A(\tau))$ with $A(\tau)\xi \in \overline{D(A(\tau))}$.

Definition 2.2. *A piecewise continuous function u is called a mild solution of Eqs. (1.1) with $u_0 = \phi \in C([-r,0];X)$ if it satisfies the integral equation of form*

$$u(t) = U(t,0)\phi(0) + \int\limits_0^t U(t,\tau)F(\tau,u(\tau),u_\tau)d\tau + \sum_{0<t_i<t} U(t,t_i)I_i(u(t_i)), \quad t \ge 0.$$

The following fixed point theorem plays a key role in the proofs of our main results, which can be found in many books.

Lemma 2.1 [6]. *Let $E_0 \subset E_1 \subset E_2$ be convex subsets of a Banach space Z, with E_0 and E_2 compact subsets and E_1 open relative to E_2. Let $P : E_2 \to Z$ be a continuous operator such that for some integer m, one has*

$$P^j(E_1) \subset E_2, \quad 1 \le j \le m-1,$$

$$P^j(E_1) \subset E_0, \quad m \le j \le 2m-1,$$

then P has a fixed point in E_2.

Let $T > 0$ be a constant. We end this section, by introducing the following assumptions:

(H_1) The evolution family $U = \{U(t,\tau) : t \ge \tau, t,\tau \in \mathbb{R}^+\}$ is exponentially stable, i.e., there exist constants $M > 0, \omega > 0$ such that

$$\|U(t,\tau)\| \le Me^{-\omega(t-\tau)}$$

for all $t \ge \tau$.

(H_2) $U(t+T,\tau+T) = U(t,\tau)$ for all $t \ge \tau$ and $U(t,\tau)$ is compact for all $t > \tau$.

(H_3) $F(t+T,v,w) = F(t,v,w)$, $t \ge 0$, $F(t,v,w)$ is continuous in (t,v,w) and Lipschitz continuous in (v,w), and maps a bounded set into a bounded set.

(H_4) I_i is Lipschitz continuous and maps a bounded set into a bounded set for each $i = 1,2,\ldots$, and there is a $p \in \mathbb{N}^+$ such that $0 < t_1 < t_2 < \ldots < t_p < T-r$, $T < t_{p+1}$, and $t_{p+k} = t_k + T$, $I_{p+k} = I_k$, $k \ge 1$.

3. Periodic solutions. In the present work, we would follow [16, 19] and other related papers and name “mild solutions” as “solutions”. To show the existence of periodic solutions, we also assume that solutions of Eqs. (1.1) exist on $[0,\infty)$ and are unique. Denote by $u(t;\phi)$ the unique solution with the initial function $\phi \in C([-r,0];X)$.

Definition 3.1. *Solutions of Eqs. (1.1) are bounded if for each $B_1 > 0$ there is a $B_2 > 0$ such that $\{\|\phi\|_0 \le B_1, t \ge 0\}$ implies $\|u(t;\phi)\| < B_2$.*

Definition 3.2. *Solutions of Eqs. (1.1) are ultimately bounded if there is a bound $B_2' > 0$ such that for each $B_1' > 0$, there is a $K > 0$ such that $\{\|\phi\|_0 \le B_1', t \ge K\}$ implies $\|u(t;\phi)\| < B_2'$.*

Now, we introduce an operator P on $C([-r,0];X)$ as

$$(P\phi)(s) = u_T(\phi)(s) = u(T+s;\phi), \quad s \in [-r,0].$$

From (H_4) one finds that P maps $C([-r,0];X)$ into itself. Moreover, we have the following results.

Lemma 3.1. *Let the hypotheses* (H_1) – (H_4) *hold. Then*
(a) P *is continuous,*
(b) if the solutions of Eqs. (1.1) are ultimately bounded, then P *is compact.*

Proof. (a) Given $\phi_1, \phi_2 \in C([-r,0];X)$. Assume that $v(t) = u(t;\phi_1)$ and $w(t) = u(t;\phi_2)$ are two solutions of Eqs. (1.1) corresponding to ϕ_1 and ϕ_2, respectively. Let $M_0 > 0$ be a constant such that $\|U(t,\tau)\| \le M_0$ for all $0 \le \tau \le t \le T$. Since F is Lipschitz continuous with respect to the second and third variables and I_i is Lipschitz continuous for each $i = 1,2,\ldots$, a direct calculation yields that for $t \in [0,T]$,

$$\|v(t)-w(t)\| \le \|U(t,0)(\phi_1(0)-\phi_2(0))\| + \int\limits_0^t \|U(t,\tau)(F(\tau,v(\tau),v_\tau) - F(\tau,v(\tau),w_\tau))\|\,d\tau +$$

$$+ \int\limits_0^t \|U(t,\tau)(F(\tau,v(\tau),w_\tau) - F(\tau,w(\tau),w_\tau))\|\,d\tau +$$

$$+ \sum_{0<t_i<t} \|U(t,t_i)I_i(v(t_i)-w(t_i))\| \le$$

$$\le M_0\|\phi_1-\phi_2\|_0 + M_1\int\limits_0^t \|v_\tau - w_\tau\|_{PC}d\tau + \sum_{0<t_i<t} C_i\|v(t_i)-w(t_i)\|,$$

where M_1 and C_i, $i = 1,2,\ldots$, are some constants, which implies that for all $t \in [0,T]$,

$$\|v_t - w_t\|_{PC} \le M_0\|\phi_1-\phi_2\|_0 + M_1\int\limits_0^t \|v_\tau - w_\tau\|_{PC}d\tau + \sum_{0<t_i<t} C_i\|v_{t_i} - w_{t_i}\|_{PC}.$$

So, an application of [16] (Lemma 2.5) yields that

$$\|v_t - w_t\|_{PC} \le M_2\|\phi_1-\phi_2\|_0, \quad t \in [0,T],$$

where M_2 is a positive constant. This proves that

$$\|P(\phi_1) - P(\phi_2)\|_0 \le M_2\|\phi_1-\phi_2\|_0,$$

which implies that P is a continuous operator.

(b) One can easily show, along the same lines as in the proof of [16] (Theorem 2.10), that if the solutions of Eqs. (1.1) are ultimately bounded, then they are also bounded. Hence, it suffices to show that when the solutions of Eqs. (1.1) are bounded, P is a compact operator.

Let G be a bounded subset of $C([-r,0];X)$. In view of the fact that the solutions of Eqs. (1.1) are bounded we see that $P(G) \subset C([-r,0];X)$ is bounded.

For $s \in [-r,0]$, we have

$$(P\phi)(s) = U(T+s,0)\phi(0) + \int_0^{T+s} U(T+s,\tau)F(\tau,u(\tau),u_\tau)d\tau +$$

$$+ \sum_{0<t_i<T+s} U(T+s,t_i)I_i(u(t_i)) := (P_1\phi)(s) + (P_2\phi)(s),$$

where

$$(P_1\phi)(s) = \int_0^{T+s} U(T+s,\tau)F(\tau,u(\tau),u_\tau)d\tau, \quad s \in [-r,0],$$

$$(P_2\phi)(s) = U(T+s,0)\phi(0) + \sum_{0<t_i<T+s} U(T+s,t_i)I_i(u(t_i)), \quad s \in [-r,0].$$

For any $\epsilon > 0$ with $\epsilon < T-r$ we define a mapping P_ϵ as

$$(P_\epsilon\phi)(s) = \int_0^{T+s-\epsilon} U(T+s,\tau)F(\tau,u(\tau),u_\tau)d\tau =$$

$$= U(T+s,T+s-\epsilon) \int_0^{T+s-\epsilon} U(T+s-\epsilon,\tau)F(\tau,u(\tau),u_\tau)d\tau, \quad \phi \in G.$$

From (H_2) it follows that $U(T+s,T+s-\epsilon)$ is compact for any $0 < \epsilon < T-r$ and $s \in [-r,0]$. Also, as assumed in (H_3), F maps a bounded set into a bounded set. This, together with the fact that the solutions of Eqs. (1.1) are bounded, implies that the set $\{F(\tau,u(\tau),u_\tau) : \tau \in [0,T], \phi \in G\}$ is bounded. Hence, we get that the set $\{(P_\epsilon\phi)(s) : \phi \in G\}$ is precompact for each $s \in [-r,0]$. At the same time, we note that

$$\|(P_1\phi)(s) - (P_\epsilon\phi)(s)\| \le \int_{T+s-\epsilon}^{T+s} \|U(T+s,\tau)F(\tau,u(\tau),u_\tau)\|d\tau \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Accordingly, the set $\{(P_1\phi)(s) : \phi \in G\}$ is precompact for each $s \in [-r,0]$.

Set $F^* := \max\{F(\tau,u(\tau),u_\tau) : \tau \in [0,T], \phi \in G\}$. Let $\delta > 0$ be small enough such that

$\delta < T - r$. For $-r \le s_1 < s_2 \le 0$, we have

$$\|(P_1\phi)(s_2) - (P_1\phi)(s_1)\| \le \int\limits_{T+s_1-\delta}^{T+s_1} \|(U(T+s_2,\tau) - U(T+s_1,\tau))F(\tau, u(\tau), u_\tau)\|\, d\tau +$$

$$+ \int\limits_{0}^{T+s_1-\delta} \|(U(T+s_2,\tau) - U(T+s_1,\tau))F(\tau, u(\tau), u_\tau)\|\, d\tau +$$

$$+ \int\limits_{T+s_1}^{T+s_2} \|U(T+s_2,\tau)F(\tau, u(\tau), u_\tau)\|\, d\tau \le$$

$$\le MF^* \int\limits_{T+s_1-\delta}^{T+s_1} \left(e^{-\omega(T+s_2-\tau)} + e^{-\omega(T+s_1-\tau)}\right) d\tau +$$

$$+ F^* \sup_{\tau\in[0,T+s_1-\delta]} \|(U(T+s_2,\tau) - U(T+s_1,\tau)\| \int\limits_{0}^{T+s_1-\delta} d\tau +$$

$$+ MF^* \int\limits_{T+s_1}^{T+s_2} e^{-\omega(T+s_2-\tau)} d\tau \to 0 \quad \text{as} \quad s_2 - s_1 \to 0, \quad \delta \to 0.$$

Accordingly, the set $\{(P_1\phi)(\cdot) : \phi \in G\}$ is equicontinuous. Hence, an application of Arzela–Ascoli theorem yields that P_1 is a compact operator.

Now, we treat P_2. According to (H_4) we have $t_p < T - r$ and $T < t_{p+1}$. This gives

$$(P_2\phi)(s) = U(T+s,0)\phi(0) + \sum_{1\le i\le p} U(T+s,t_i)I_i(u(t_i)), \quad s \in [-r, 0].$$

From (H_4) we note that I_i maps a bounded set into a bounded set for each $1 \le i \le p$. This, together with the fact that the solutions of Eqs. (1.1) are bounded, implies that the set $\{I_i(u(t_i)) : \phi \in G\}$ is bounded for each $1 \le i \le p$. Hence, by the compactness of $U(t,\tau)$ for $t > \tau$, we conclude that the set $\{U(T+s,t_i)I_i(u(t_i)) : \phi \in G\}$ is precompact for each $s \in [-r, 0]$ and $1 \le i \le p$. Also, one finds that the set $\{U(T+s,0)\phi(0) : \phi \in G\}$ is precompact for each $s \in [-r, 0]$. We thus obtain that the set $\{(P_2\phi)(s) : \phi \in G\}$ is precompact for each $s \in [-r, 0]$.

As proved above, there exists a $k_i > 0$ such that $\|I_i(u(t_i))\| \le k_i$ for each $1 \le i \le p$. Since $0 < t_i < T - r \le T + s \le T$ for each $s \in [-r, 0]$ and $1 \le i \le p$, we obtain that for $-r \le s_1 < s_2 \le 0$, $1 \le i \le p$,

$$\|U(T+s_2,t_i)I_i(u(t_i)) - U(T+s_1,t_i)I_i(u(t_i))\| \le k_i\|U(T+s_2,t_i) - U(T+s_1,t_i)\| \to$$

$$\to 0 \quad \text{as} \quad s_2 - s_1 \to 0,$$

which implies that $\{U(T+\cdot,t_i)I_i(u(t_i)) : \phi \in G\}$ is equicontinuous for each $1 \leq i \leq p$. Along the same lines one can show that $\{U(T+\cdot,0)\phi(0) : \phi \in G\}$ is equicontinuous. Thus, $\{(P_2\phi)(\cdot) : \phi \in G\}$ is equicontinuous. Again using Arzela–Ascoli theorem we get the compactness of P_2.

Lemma 3.1 is proved.

Now we are in a position to prove our existence result of periodic solutions.

Theorem 3.1. *Let the hypotheses (H_1)–(H_4) hold. Suppose in addition that the solutions of Eqs. (1.1) are ultimately bounded. Then Eqs. (1.1) has a T-periodic solution.*

Proof. Let $u(t;\phi)$ be a solution of Eqs. (1.1) with $u_0 = \phi \in C([-r,0];X)$ and $y(t) := u(t+T;\phi)$.

To prove the theorem, we first show that $u(t;\phi)$ is a T-periodic solution of Eqs. (1.1) if and only if ϕ is a fixed point of P. Note that

$$y(t) = U(t+T,0)\phi(0) + \int_0^{t+T} U(t+T,\tau)F(\tau,u(\tau),u_\tau)d\tau +$$

$$+ \sum_{0<t_i<t+T} U(t+T,t_i)I_i(u(t_i)) := J_1(t) + J_2(t).$$

From (H_2) we have

$$J_1(t) = U(t+T,0)\phi(0) + \int_0^T U(t+T,\tau)F(\tau,u(\tau),u_\tau)d\tau + \int_T^{t+T} U(t+T,\tau)F(\tau,u(\tau),u_\tau)d\tau =$$

$$= U(t+T,T)U(T,0)\phi(0) + \int_0^T U(t+T,T)U(T,\tau)F(\tau,u(\tau),u_\tau)d\tau +$$

$$+ \int_0^t U(t+T,z+T)F(z+T,u(z+T),u_{z+T})dz =$$

$$= U(t,0)U(T,0)\phi(0) + U(t,0)\int_0^T U(T,\tau)F(\tau,u(\tau),u_\tau)d\tau + \int_0^t U(t,\tau)F(\tau,y(\tau),y_\tau)d\tau.$$

Also, for $t_i > T$, it is clear that $i = p+k$ and $t_i = t_{p+k} = t_k + T$ and $I_i = I_{p+k} = I_k$. Therefore, we get

$$J_2(t) = \sum_{0<t_i<T} U(t+T,t_i)I_i(u(t_i)) + \sum_{T<t_i<t+T} U(t+T,t_i)I_i(u(t_i)) =$$

$$= \sum_{0<t_i<T} U(t+T,T)U(T,t_i)I_i(u(t_i)) + \sum_{0<t_k<t} U(t+T,t_k+T)I_{p+k}(u(t_k+T)) =$$

$$= U(t,0)\sum_{0<t_i<T} U(T,t_i)I_i(u(t_i)) + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)).$$

Hence, we obtain

$$y(t) = U(t,0)U(T,0)\phi(0) + U(t,0)\int_0^T U(T,\tau)F(\tau,u(\tau),u_\tau)d\tau + \int_0^t U(t,\tau)F(\tau,y(\tau),y_\tau)d\tau +$$

$$+ U(t,0)\sum_{0<t_i<T} U(T,t_i)I_i(u(t_i)) + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)) =$$

$$= U(t,0)\left(U(T,0)\phi(0) + \int_0^T U(T,\tau)F(\tau,u(\tau),u_\tau)d\tau + \sum_{0<t_i<T} U(T,t_i)I_i(u(t_i))\right) +$$

$$+ \int_0^t U(t,\tau)F(\tau,y(\tau),y_\tau)d\tau + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)) =$$

$$= U(t,0)u(T) + \int_0^t U(t,\tau)F(\tau,y(\tau),y_\tau)d\tau + \sum_{0<t_i<t} U(t,t_i)I_i(y(t_i)),$$

which yields that $y(t) = u(t+T;\phi)$ is a solution of Eqs. (1.1) with $y_0 = u_T(\phi) = P\phi$. Since solutions of Eqs. (1.1) are unique, we have $u(t+T;\phi) = u(t;P\phi)$. Noticing this, a similar argument as in the proof of [16] (Lemma 2.7) shows that $u_{nT}(\phi) = P^n\phi$, $n \geq 1$.

Now, if P has a fixed point ϕ, i.e., $P\phi = \phi$, then one has $u(t+T;\phi) = u(t;\phi)$ due to $u(t+ +T;\phi) = u(t;P\phi)$, which implies that $u(t;\phi)$ is a T-periodic solution of Eqs. (1.1). Conversely, if $u(t;\phi)$ is a T-periodic solution of Eqs. (1.1), i.e., $u_T(\phi) = u_0(\phi)$, then $P\phi = u_T(\phi) = u_0(\phi) = = \phi$. This proves that ϕ is a fixed point of P.

Next, to obtain the existence of T-periodic solution of Eqs. (1.1) it suffices to show that P has a fixed point ϕ. Assume that $B > 0$ is the bound in the definition of ultimate boundedness. As shown in Lemma 3.1, if the solutions of Eqs. (1.1) are ultimately bounded, then they are also bounded. From this, we see that there is a constant $B_1 > B$ such that

$$\|u(t;\phi)\| < \frac{1}{2}B_1 \quad \text{for all} \quad t \geq 0$$

when $\|\phi\|_0 \leq B$. Similarly, there is a constant $B_1' > B_1$ such that

$$\|u(t;\phi)\| < B_1' \quad \text{for all} \quad t \geq 0$$

when $\|\phi\|_0 < B_1$. Also, from the ultimate boundedness it is clear that there exists a positive integer m such that

$$\|u(t;\phi)\| < B \quad \text{for all} \quad t \geq (m-2)T$$

when $\|\phi\|_0 \leq B_1$. Hence, in view of $u_{nT}(\phi) = P^n\phi$, $n \geq 1$, we see that

$$\|P^{j-1}(\phi)\| = \|u((j-1)T + \cdot;\phi)\| < B_1' \quad \text{for} \quad 1 \leq j \leq m-1 \quad \text{and} \quad \|\phi\|_0 \leq B_1,$$
$$\|P^{j-1}(\phi)\| = \|u((j-1)T + \cdot;\phi)\| < B \quad \text{for} \quad j \geq m \quad \text{and} \quad \|\phi\|_0 \leq B_1. \tag{3.1}$$

Denote

$$A := \{\phi \in C([-r,0];X) : \|\phi\|_0 < B\}, \quad E_0 := \overline{\text{cov.}(P(A))},$$
$$C := \{\phi \in C([-r,0];X) : \|\phi\|_0 < B_1'\}, \quad E_2 := \overline{\text{cov.}(P(C))}, \tag{3.2}$$
$$D := \{\phi \in C([-r,0];X) : \|\phi\|_0 < B_1\}, \quad E_1 := D \cap E_2.$$

Then from the compactness of P and the fact that a convex hull of a precompact set is also precompact, we obtain that $E_0 \subset E_1 \subset E_2$ are all convex subsets of $C([-r,0];X)$, E_0 and E_2 are compact subsets of $C([-r,0];X)$, and E_1 is open relative to E_2. Therefore, according to (3.1) and (3.2) we deduce that

$$P^j(E_1) \subset E_2, \quad 1 \leq j \leq m-1, \quad P^j(E_1) \subset E_0, \quad m \leq j \leq 2m-1.$$

Thus, by Lemma 2.1 P has a fixed point, i.e., there exits $\phi \in C([-r,0];X)$ such that $P\phi = \phi$. As proved above, we know that the solution $u(t;\phi)$ of Eqs. (1.1) corresponding to the initial value $u_0 = \phi$ is just T-periodic. Therefore $u(t;\phi)$ is a T-periodic solution of Eqs. (1.1).

Theorem 3.1 is proved.

4. An example. To illustrate our abstract results, let us consider the partial differential equation in the form

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + d(t)u(t,x) = \sin t \cos u(t,x) \cos u(t+\theta,x), \quad t > 0, \quad t \neq t_i, \quad x \in [0,\pi],$$
$$u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}^+,$$
$$u(\theta,x) = \phi(\theta,x), \quad \theta \in \left[-\frac{\pi}{4}, 0\right], \quad x \in [0,\pi], \tag{4.1}$$
$$\triangle u(t_i,x) = (-1)^i e^{\sin(u(t_i,x))}, \quad t_i = \frac{2i-1}{2}\pi, \quad i = 1,2,\ldots,q, \quad x \in [0,\pi],$$

where q is a given positive integer, $d : \mathbb{R}^+ \to \mathbb{R}$ is continuously differentiable, $d(t+2\pi) = d(t)$ for all $t \in \mathbb{R}^+$, and

$$d_{\min} := \min_{t \in \mathbb{R}^+} d(t) > -1.$$

Take $X = L^2([0,\pi])$ with norm $\|\cdot\|_2$ and inner product $(\cdot,\cdot)_2$. Define

$$D(A(t)) := D(B), \quad t \in \mathbb{R}^+,$$

$$A(t)\xi := B\xi - d(t)\xi, \quad \xi \in D(A(t)),$$

where the operator $B : D(B) \subset X \to X$ is given by the following form:

$$B\xi = \frac{\partial^2 \xi}{\partial x^2}, \quad \xi \in D(B),$$

$$D(B) := \{\xi \in X : \xi, \xi' \text{ are absolutely continuous, } \xi'' \in X, \text{ and } \xi(0) = \xi(\pi) = 0\}.$$

It is well-known that B has a discrete spectrum and its eigenvalues are $-n^2$, $n \in \mathbb{N}^+$ with the corresponding normalized eigenvectors $y_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Clearly, $(A(t))_{t\in\mathbb{R}^+}$ satisfies the conditions (AT_1) and (AT_2). Hence, $A(t)$ generater an evolution family $U = \{U(t,s) : t \geq s, t, s \in \mathbb{R}^+\}$ and

$$U(t,s)\xi = \sum_{n=1}^{\infty} e^{-\left(\int_s^t d(\tau)d\tau + n^2(t-\tau)\right)} (\xi, y_n)_2 y_n \quad \text{for all} \quad t \geq s, \quad \xi \in X.$$

A direct calculation gives the following estimate:

$$\|U(t,s)\| \leq e^{-(1+d_{\min})(t-s)} \quad \text{for all} \quad t \geq s.$$

Note also that for each $t > s$, $s \in \mathbb{R}^+$, the operator $U(t,s)$ is a nuclear operator, which yields the compactness of $U(t,s)$ for $t > s$.

Define

$$u(t)(x) = u(t,x),$$

$$\phi(t)(x) = \phi(t,x),$$

$$F(t, u(t), u_t(\theta))(x) = \sin t \cos u(t,x) \cos u(t+\theta, x),$$

$$I_i(u(t_i))(x) = (-1)^i e^{\sin(u(t_i,x))}.$$

Then it is clear that $t_{2+i} = t_i + 2\pi$, $I_{2+i} = I_i$ and

$$F(t+2\pi, \xi_1, \xi_2) = F(t, \xi_1, \xi_2) \quad \text{for all} \quad t \in \mathbb{R}^+, \quad \xi_1, \xi_2 \in X,$$

$$\|I_i(\xi_1) - I_i(\xi_2)\|_2 \leq e\|\xi_1 - \xi_2\|_2, \quad i = 1, 2, \ldots, \quad q, \xi_1, \xi_2 \in X,$$

$$\|I_i(\xi)\|_2 \leq e\sqrt{\pi}, \quad i = 1, 2, \ldots, \quad q, \xi \in X.$$

Moreover, it is easy to verify that the hypotheses (H_1)–(H_4) are satisfied.

Note that the partial differential equation (4.1) can be reformulated as the abstract Eqs. (1.1). Let $\phi \in C\left(\left[-\frac{\pi}{4}, 0\right]; L^2([0,\pi])\right)$ and $B > 0$ be a constant with $\|\phi\|_0 \leq B$. If $u(t;\phi)$ is a solution of Eq. (1.1), then we have that for $t > 0$,

$$\|u(t)\|_2 \leq \|U(t,0)\phi(0)\|_2 + \int_0^t \|U(t,\tau)\sin\tau\cos u(\tau)\cos u_\tau\|_2 d\tau + \sum_{0<t_i<t} \|U(t,t_i)I_i(u(t_i))\|_2 \leq$$

$$\leq BMe^{-(1+d_{\min})t} + M\sqrt{\pi}\int_0^t e^{-(1+d_{\min})(t-\tau)}d\tau + Me\sqrt{\pi}\sum_{0<t_i<t} e^{-(1+d_{\min})(t-t_i)} =$$

$$= BMe^{-(1+d_{\min})t} + \frac{M\sqrt{\pi}}{(1+d_{\min})}(1-e^{-(1+d_{\min})t}) + Me\sqrt{\pi}\sum_{0<t_i<t} e^{-(1+d_{\min})(t-t_i)}.$$

Taking $K(B) > t_q$, one can find a constant $M' > 0$ such that

$$BMe^{-(1+d_{\min})K(B)} + \frac{M\sqrt{\pi}}{(1+d_{\min})} + Me\sqrt{\pi}\sum_{0<t_i<K(B)} e^{-(1+d_{\min})(K(B)-t_i)} \leq M'.$$

Accordingly, when $\|\phi\|_0 \leq B$, it follows that $\|u(t;\phi)\|_2 < M'$ for all $t \geq K(B)$, which implies that $u(t;\phi)$ is ultimately bounded and hence a 2π-periodic solution of (4.1) due to Theorem 3.1.

1. *Acquistapace P.* Evolution operators and strong solution of abstract parabolic equations // Different. Integr. Equat. — 1988. — **1**. — P. 433–457.
2. *Acquistapace P., Terreni B.* A unified approach to abstract linear nonautonomous parabolic equations // Rend. Semin. mat. Univ. Padova. — 1987. — **78**. — P. 47–107.
3. *Ahmed N. U.* Measure solutions for impulsive evolution equations with measurable vector fields // J. Math. Anal. and Appl. — 2006. — **319**. — P. 74–93.
4. *Benchohra M., Henderson J., Ntouyas S.* Impulsive differential equations and inclusions. — New York: Hindawi Publ. Corp., 2006.
5. *Boichuk A. A., Samoilenko A. M.* Generalized inverse operators and Fredholm boundary value problems. — De Gruyter, 2004.
6. *Burton T.* Stability and periodic solutions of ordinary differential equations and functional differential equations. — Acad. Press, Inc., 1985.
7. *Burton T. A., Zhang B.* Periodic solutions of abstract differential equations with infinite delay // J. Different. Equat. — 1991. — **90**. — P. 357–396.
8. *Diekmann O., van Gils S., Lunel S., Walther H. O.* Delay equations. Functional-, complex-, and nonlinear analysis // Appl. Math. Sci. — Berlin: Springer-Verlag, 1995. — **110**.
9. *Engel K. J., Nagel R.* One-parameter semigroups for linear evolution equations // GTM. — Berlin: Springer, 2000. — **194**.
10. *Ezzinbi K., Liu J., Minh N.* Periodic solutions of impulsive evolution equations // Int. J. Evol. Equat. — 2009. — **4**. — P. 103–111.
11. *Grimmer R.* Resolvent operators for integral equations in Banach space // Trans. Amer. Math. Soc. — 1982. — **273**. — P. 333–349.
12. *Guo D., Liu X.* Extremal solutions of nonlinear impulsive integrodifferential equations in Banach spaces // J. Math. Anal. and Appl. — 1993. — **177**. — P. 538–552.
13. *Lakshmikantham V., Bainov D. D., Simeonov P. S.* Theory of impulsive differential equations. — Singapore; London: World Sci., 1989.

14. *Li Y. X.* Existence and asymptotic stability of periodic solution for evolution equations with delays // J. Funct. Anal. — 2011. — **261**. — P. 1309 – 1324.

15. *Liang J., Liu J. H., Xiao T. J.* Nonlocal impulsive problems for nonlinear differential equations in Banach spaces // Math. Comput. Modelling. — 2009. — **49**. — P. 798 – 804.

16. *Liang J., Liu J. H., Xiao T. J.* Periodic solutions of delay impulsive differential equations // Nonlinear Anal. T.M.A. — 2011. — **74**. — P. 6835 – 6842.

17. *Liu J. H.* Bounded and periodic solutions of finite delay evolution equations // Nonlinear Anal. T.M.A. — 1998. — **34**. — P. 101 – 111.

18. *Liu J. H.* Periodic solutions of infinite delay evolution equations // J. Math. Anal. and Appl. — 2000. — **247**. — P. 627 – 644.

19. *Liu J. H.* Bounded and periodic solutions of infinite delay evolution equations // J. Math. Anal. and Appl. — 2003. — **286**. — P. 705 – 712.

20. *Samoilenko A. M., Perestyuk N. A.* Impulsive differential equations. — Kiev: Vyshcha Shkola, 1987 (in Russian).

21. *Xiang X., Ahmed N.* Existence of periodic solutions of semilinear evolution equations with time lags // Nonlinear Anal. T.M.A. — 1992. — **11**. — P. 1063 – 1070.

22. *Zhu J. M., Liu Y. C., Li Z. X.* The existence and attractivity of time periodic solutions for evolution equations with delays // Nonlinear Anal. R.W.A. — 2008. — **9**. — P. 842 – 851.

Received 11.04.13,
after revision — 19.03.14