UDC 517.9

ASYMPTOTICALLY PERIODIC SOLUTIONS TO NONLOCAL CAUCHY PROBLEMS GOVERNED BY COMPACT EVOLUTION FAMILIES*

АСИМПТОТИЧНО ПЕРІОДИЧНІ РОЗВ'ЯЗКИ НЕЛОКАЛЬНИХ ЗАДАЧ КОШІ, ЩО РЕГУЛЮЮТЬСЯ КОМПАКТНИМИ ЕВОЛЮЦІЙНИМИ СІМ'ЯМИ

R.-N. Wang, Q.-M. Xiang

Nanchang Univ.
Nanchang, Jiangxi 330031, P. R. China
e-mail: rnwang@mail.ustc.edu.cn
qmxiangncu@163.com

Y. Zhou

Xiangtan Univ.
Xiangtan, Hunan 411105, P.R. China
e-mail: yzhou@xtu.edu.cn

This paper is devoted to a study of a class of abstract Cauchy problems for semilinear nonautonomous evolution equations involving nonlocal initial conditions. Combining the theory of evolution families and the fixed point theorem due to Krasnoselskii, as well as a decomposition technique, we prove the existence of the asymptotically periodic mild solutions to such problems. Our results generalize and improve some previous results since the (locally) Lipschitz continuity on the nonlinearity is not required. A partial differential equation is also presented as an application.

Статтю присвячено вивченню класу абстрактних задач Коші для напівлінійних неавтономних еволюційних рівнянь з нелокальними початковими умовами. З використанням теорії еволюційних сімей та теореми Красносельського про нерухому точку, а також техніки розкладу доведено існування асимптотично періодичних м'яких розв'язків таких задач. Наведені результати узагальнюють та покращують попередні результати, оскільки не вимагається, щоб нелінійність задовольняла (локальну) умову Ліпшиця. Як приклад наведено диференціальне рівняння з частинними похідними.

1. Introduction. The study of asymptotically periodic solutions is one of the most attracting topics in the qualitative theory of differential or integral equations. The motivation for this study lies in both its mathematical interest and the applications in physics, mathematical biology, control theory, and so forth. Some recent contributions have been made. For instance, de Andrade and Cuevas [1] studied the existence and uniqueness of asymptotically ω-periodic solutions to an abstract differential equation with linear part dominated by a Hille – Yosida operator with non-dense domain, Pierri [2] established some conditions under which an S-asymptotically ω-periodic function is asymptotically ω-periodic and discussed the existence of

* Research supported by the Natural Science Foundations of China (No. 11271309, 11101202, 11101203), the Youth Foundation of Tianyuan Mathematics of China (11126278), the Specialized Research Fund for the Doctoral Program of Higher Education (20114301110001) and Hunan Provincial Natural Science Foundation of China (12JJ2001).

asymptotically ω-periodic solutions to an abstract integral equation, and Agarwal et al. [3] examined the asymptotically ω-periodic solutions to an abstract neutral integro-differential equation with infinite delay. However, we note that in the papers mentioned above, most results were done under the (locally) Lipschitz continuity on the nonlinearity. It is also noted that Henriquez et al. [4] gave a relationship between S-asymptotically ω-periodic function and the class of asymptotically ω-periodic functions.

When dealing with parabolic evolution equation, it is usually assumed that the partial differential operator in the linear part (possibly unbounded) depends on time (i.e., it is the case of equations being non-autonomous), stimulated by the fact that this class of operators appears very often in the applications (see, e.g., [5, 6]). In the present work we deal with the asymptotically periodic solutions for the Cauchy problem consisting in the standard nonautonomous parabolic evolution equation supplied with a nonlocal initial condition in the Banach space X with norm $\|\cdot\|$. The precise problem is

$$\begin{aligned} u'(t) &= A(t)u(t) + F(t, u(t)), \quad t > 0, \\ u(0) &= H(u), \end{aligned} \tag{1.1}$$

where F, H are given nonlinear functions to be specified later. As can be seen, H constitutes a nonlocal condition. Throughout, it is assumed that $A(t)$ (usually unbounded) for each t, having domain $D(A(t))$, is a closed and densely defined linear operator on X satisfying the Acquistapace and Terreni conditions (AT_1) and (AT_2):

(AT_1) There are constants $\lambda_0 \geq 0, \theta \in \left(\frac{\pi}{2}, \pi\right)$ and $K_1 \geq 0$ such that $\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0)$ and for all $\lambda \in \Sigma_\theta \cup \{0\}$ and $t \in \mathbb{R}$,

$$\|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K_1}{1 + |\lambda|}.$$

(AT_2) There are constants $K_2 \geq 0$ and $\alpha, \beta \in (0, 1]$ with $\alpha + \beta > 1$ such that for all $\lambda \in \Sigma_\theta$ and $t, s \in \mathbb{R}$

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \frac{K_2|t - s|^\alpha}{|\lambda|^\beta}.$$

Here $\Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\}; |\lambda| \leq \theta\}$. Conditions (AT_1) and (AT_2), which are initiated by Acquistapace and Terreni [7, 8] for $\lambda_0 = 0$, are well understood and widely used in the literature.

As indicated in [9–15] and the related references given there, another motivation for the study to the problem (1.1) is due to the fact that the nonlocal initial condition $u(0) = H(u)$ models many interesting nature phenomena, with which the normal initial condition $u(0) = u_0$ may not fit in. It is mentioned that in some previous work such as Byszewski [16] (see also Deng [17], Lin and Liu [18]), the function $H(u)$ is given by $H(u) = g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) - u_0$ (here $0 < t_1 < \ldots < t_p < +\infty$ and g is a given function), which allows the measurements at $t = 0, t_1, \ldots, t_p$, rather than just at $t = 0$. So more information is available. Nonautonomous problems of type (1.1) on a finite interval have been investigated in many papers; see, e.g., [19] for the existence of mild solutions and [20] for the existence of classical solutions.

As we all know, the study of the existence of asymptotically ω-periodic solutions for nonautonomous problems of type (1.1) is a topic not yet considered in the literature. In this paper we aim at extending the early investigations to the nonautonomous Cauchy problem (1.1). New existence results of asymptotically ω-periodic mild solutions are established, without imposing (locally) Lipschitz condition on the nonlinearity F with respect to the second variable. On the way towards our main results, we make use of the theory of evolution families generated by $A(t)$ (including the compactness of evolution families) and the fixed point theorem due to Krasnoselskii as well as a decomposition technique. The results obtained in this paper are generalizations of related results. Moreover, even for the Cauchy problems of abstract evolution equations without nonlocal initial conditions, the results are also new.

We end this section with the following remark.

Remark 1.1. We mention that much less is known about the existence of asymptotically ω-periodic solutions to the problem (1.1) (including the autonomous case, i.e., $A(t) \equiv A$) when the nonlinearity F as a whole loses the Lipschitz continuity with respect to the second variable. As the reader will see, in our results (see Theorem 3.1) the nonlinearity F does not satisfy (locally) Lipschitz continuity with respect to the second variable.

2. Preliminaries. We begin this section, by recalling some definitions and fixing some notations. Throughout this paper, $\mathcal{L}(X)$ denotes the space of all bounded linear operators from X to X.

Definition 2.1. *A two parameter family of bounded linear operators $\{U(t,s)\}_{t\geq s}$ on X is called an evolution family if*

(1) $U(t,r)U(r,s) = U(t,s)$ and $U(t,t) = I$ for all $t \geq r \geq s$ and $t,r,s \in \mathbb{R}$,

(2) the map $(t,s) \mapsto U(t,s)x$ is continuous for all $x \in X$, $t \geq s$ and $t,s \in \mathbb{R}$.

It is worth pointing out that "evolution family" as a basic concept in the theory of nonautonomous evolution equations is also called evolution system, evolution operator, evolution process, propagator, or fundamental solution. More details can be found in, e.g., [6, 21, 22].

Definition 2.2. *An evolution family $\{U(t,s)\}_{t\geq s}$ is said to be compact if for all $t > s$, $U(t,s)$ is continuous and maps bounded subsets of X into precompact subsets of X.*

Remark 2.1. Let us note that if for each $t \in \mathbb{R}^+$ and some $\lambda \in \rho(A(t))$ (the resolvent set of $A(t)$), the resolvent $R(\lambda, A(t))$ is a compact operator, then $U(t,s)$ is a compact operator whenever $t > s$ (see [23], Proposition 2.1).

Similar to one-parameter semigroups, $\{U(t,s)\}_{t\geq s}$ verifies the following property.

Lemma 2.1. *Let $\{U(t,s)\}_{t\geq s}$ be a compact evolution family on X. Then for each $s \in \mathbb{R}$, the function $t \mapsto U(t,s)$ is continuous on $(s,+\infty)$ in the uniform operator topology.*

By an obvious rescaling from [7] (Theorem 2.3) and [24] (Theorem 2.1) (see also [8, 25]), the Acquistapace and Terreni conditions (AT_1) and (AT_2) ensures that there exists a unique evolution family $\{U(t,s)\}_{t\geq s}$ on X such that

(I) $U(\cdot,s) \in C^1((s,\infty),\mathcal{L}(X))$, $\dfrac{\partial U(t,s)}{\partial t} = A(t)U(t,s)$ for $t > s$, and

$$\|A(t)^k U(t,s)\| \leq C(t-s)^{-k}$$

for $0 < t-s \leq 1$, $k = 0,1$;

(II) $\dfrac{\partial^+ U(t,s)x}{\partial s} = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

In this case we say that $(A(t))_{t\in\mathbb{R}}$ generate the evolution family $\{U(t,s)\}_{t\geq s}$. It should be mentioned that when $(A(t))_{t\in\mathbb{R}}$ have a constant domains $D(A(t))$, (AT_2) can be replaced with the following condition: there exist constants $K_2 > 0$, $0 < \mu \leq 1$ such that

$$\|(A(t) - A(s))R(\lambda_0, A(r))\| \leq K_2|t-s|^{\mu}$$

for all $s, t, r \in \mathbb{R}$ (see, e.g., [5, 6]). Throughout this paper, we further suppose that

(H_1) the evolution family $\{U(t,s)\}_{t\geq s}$ is exponentially stable, i.e., there exist constants $M > 0$, $\delta > 0$ such that

$$\|U(t,s)\| \leq Me^{-\delta(t-s)}$$

for all $t \geq s$ and $t, s \in \mathbb{R}$, and

(H_2) $U(t+T, s+T) = U(t,s)$ for all $t \geq s$ and $U(t,s)$ is compact for all $t > s$ and $t, s \in \mathbb{R}$.

Below, $C_b(\mathbb{R}^+; X)$ stands for the Banach space of all bounded, continuous functions u from $\mathbb{R}^+$ to X equipped with the sup norm

$$\|u\|_0 = \sup\{\|u(t)\|; t \in \mathbb{R}^+\}.$$

Let $C_0(\mathbb{R}^+; X)$ and $C_\omega(\mathbb{R}; X)$ be the spaces of functions

$$C_0(\mathbb{R}^+; X) := \{x \in C_b(\mathbb{R}^+; X); \lim_{t\to+\infty} \|x(t)\| = 0\},$$

$$C_\omega(\mathbb{R}; X) := \{x \in C(\mathbb{R}; X); x \text{ is } \omega\text{-periodic}\}.$$

It is easy to see that $C_0(\mathbb{R}^+; X)$ and $C_\omega(\mathbb{R}; X)$, endowed with the norms $\|\cdot\|_0$ and $\|\cdot\|'_\infty := \sup_{t\in\mathbb{R}} \|\cdot(t)\|$, are Banach space, respectively. We abbreviate $C_0(\mathbb{R}^+; X)$ to $C_0(\mathbb{R}^+)$ when $X = \mathbb{R}^+$. Write

$$\Omega_r := \{x \in C_0(\mathbb{R}^+; X); \|x\|_0 \leq r\}$$

for some $r > 0$.

To continue, we establish without proof the following compact criterion.

Lemma 2.2. *A set $D \subset C_0(\mathbb{R}^+; X)$ is relatively compact if*
(1) D is equicontinuous;
(2) $\lim_{t\to+\infty} u(t) = 0$ uniformly for $u \in D$;
(3) the set $D(t) := \{u(t); u \in D\}$ is relatively compact in X for every $t \geq 0$.

Definition 2.3. *A function $u \in C_b(\mathbb{R}^+; X)$ is said to be asymptotically ω-periodic if it can be decomposed as*

$$u = u_1 + u_2,$$

where $u_1 \in C_\omega(\mathbb{R}; X)$ and $u_2 \in C_0(\mathbb{R}^+; X)$. The set of such functions is denoted by $AP_\omega(\mathbb{R}^+; X)$.

$AP_\omega(\mathbb{R}^+; X)$ turns out to be a Banach space with the supremum norm $\|\cdot\|_0$. Write

$$S_r := \{u \in AP_\omega(\mathbb{R}^+; X); \|u\|_0 \leq r\}$$

for some $r > 0$.

Remark 2.2. Take $u \in AP_\omega(\mathbb{R}^+; X)$. Let us note that the decomposition of u is unique. Indeed, if there exist $u_1, u_1' \in C_\omega(\mathbb{R}; X)$ and $u_2, u_2' \in C_0(\mathbb{R}^+; X)$ such that

$$u(t) = u_1(t) + u_2(t) = u_1'(t) + u_2'(t), \quad t \in \mathbb{R}^+,$$

then one can find that for fixed $t \in \mathbb{R}$,

$$u_1(t) - u_1'(t) = u_2'(t + n\omega) - u_2(t + n\omega), \quad n \in \mathbb{N} \quad \text{with} \quad n\omega \geq -t$$

in view of $u_1(t) = u_1(t + n\omega)$ and $u_1'(t) = u_1'(t + n\omega)$. Taking the limit as $n \to +\infty$, one has that $u_1(t) = u_1'(t)$, $t \in \mathbb{R}$, as required.

A continuous function f from $\mathbb{R} \times X$ to X is said to be ω-periodic if $f(t + \omega, x) = f(t, x)$ for all $t \in \mathbb{R}$ and $x \in X$. The set of such functions will be denoted by $C_\omega(\mathbb{R} \times X; X)$. Let the notation $C_0(\mathbb{R}^+ \times X; X)$ stand for the set of functions

$$C_0(\mathbb{R} \times X; X) = \Big\{ f \in C(\mathbb{R} \times X; X); \lim_{t \to +\infty} \|f(t, x)\| = 0 \quad \text{uniformly for } x \text{ in any bounded subset of } X \Big\}.$$

Definition 2.4. *A function $f : \mathbb{R}^+ \times X \to X$ is said to be asymptotically ω-periodic if it can be decomposed as*

$$f = f_1 + f_2,$$

where $f_1 \in C_\omega(\mathbb{R} \times X; X)$ and $f_2 \in C_0(\mathbb{R}^+ \times X; X)$.

Definition 2.5. *An asymptotically ω-periodic function $f : \mathbb{R}^+ \times X \to X$ is said to be semi-Lipschitz continuous with the Lipschitz constant L if writing $f = f_1 + f_2$ with $f_1 \in C_\omega(\mathbb{R} \times X; X)$ and $f_2 \in C_0(\mathbb{R}^+ \times X; X)$, there exists a constant $L_f > 0$ such that*

$$\|f_1(t, x) - f_1(t, y)\| \leq L_f \|x - y\|$$

for all $t \in \mathbb{R}$ and $x, y \in X$.

The following fixed point theorem plays a key role in the proofs of our main results, which can be found in many books.

Lemma 2.3 (Krasnoselskii). *Let E be a Banach space and B be a bounded closed and convex subset of E, and let J_1, J_2 be maps of B into E such that $J_1 x + J_2 y \in B$ for every pair $x, y \in B$. If J_1 is a contraction and J_2 is completely continuous, then the equation $J_1 x + J_2 x = x$ has a solution on B.*

3. Main results and their proofs. This section is devoted to the study of the existence of asymptotically ω-periodic mild solutions to the Cauchy problem (1.1).

Let us introduce the following assumptions:

(H_3) $F = F_1 + F_2 : \mathbb{R}^+ \times X \to X$ is asymptotically ω-periodic and semi-Lipschitz continuous with the Lipschitz constant L_F, where $F_1 \in C_\omega(\mathbb{R} \times X; X)$ and $F_2 \in C_0(\mathbb{R}^+ \times X; X)$.

Moreover, there exists a function $\theta \in C_0(\mathbb{R}^+)$ and a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in \mathbb{R}^+$ and $x \in X$ satisfying $\|x\| \leq r$,

$$\|F_2(t,x)\| \leq \theta(t)\varphi(r), \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\varphi(r)}{r} = \sigma_1. \tag{3.1}$$

(H_4) $H : AP_\omega(\mathbb{R}^+; X) \to X$ is Lipschitz continuous with Lipschitz constant L_H, there exists a nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $u \in S_r$,

$$\|H(u)\| \leq \phi(r), \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\phi(r)}{r} = \sigma_2. \tag{3.2}$$

Let θ be the function involved in assumption (H_3). It is not difficult to see that

$$\int_0^{\cdot} e^{-\delta(\cdot - s)}\theta(s)\,ds \in C_0(\mathbb{R}^+). \tag{3.3}$$

Put

$$\sigma_3 := \sup_{t\in\mathbb{R}^+} \int_0^t e^{-\delta(t-s)}\theta(s)\,ds.$$

Before stating the existence theorem, we first prove the following lemma.

Lemma 3.1. *Let the hypotheses (H_1) be satisfied. Given $u_0 \in X$, $v \in C_\omega(\mathbb{R}; X)$ and $w \in C_0(\mathbb{R}^+; X)$. Write*

$$W(t) := U(t,0)u_0 - \int_{-\infty}^0 U(t,s)v(s)\,ds + \int_0^t U(t,s)w(s)\,ds, \quad t \in \mathbb{R}^+.$$

Then W belongs to $C_0(\mathbb{R}^+; X)$.

Proof. Given $\epsilon > 0$. One can choose $N > 0$ such that $\|w(t)\| < \epsilon$ for all $t \geq N$, since $w \in C_0(\mathbb{R}^+; X)$. From (H_1) we note that

$$\left\| U(t,0)u_0 - \int_{-\infty}^0 U(t,0)v(s)\,ds \right\| \leq M(\|u_0\| + \delta^{-1}\|v\|'_\infty)e^{-\delta t} \to 0 \text{ as } t \to +\infty.$$

Also, we derive, by a similar reasoning,

$$\left\| \int_0^N U(2t,s)w(s)\,ds \right\| \leq M\delta^{-1}e^{-\delta(2t-N)}\|w\|_0 \to 0 \text{ as } t \to +\infty.$$

Moreover, it follows readily that for all $t \geq N$,

$$\left\| \int_N^{2t} U(2t,s)w(s)\,ds \right\| \leq M\delta^{-1}\epsilon.$$

We thus gain from the arguments above that

$$\left\| \int\limits_0^{2t} U(2t,s)w(s)\,ds \right\| \leq \left\| \int\limits_0^{N} U(2t,s)w(s)\,ds \right\| + \left\| \int\limits_N^{2t} U(2t,s)w(s)\,ds \right\| \to 0 \text{ as } t \to +\infty.$$

Accordingly, W belongs to $C_0(\mathbb{R}^+; X)$.

Lemma 3.1 is proved.

Definition 3.1. *By a mild solution of the Cauchy problem (1.1), we mean a function* $u \in C(\mathbb{R}^+; X)$ *satisfying the integral equation*

$$u(t) = U(t,0)H(u) + \int\limits_0^t U(t,s)F(s,u(s))\,ds, \quad t \geq 0.$$

Now we are in a position to present our first existence result.

Theorem 3.1. *Assume that the hypotheses* $(H_1)-(H_4)$ *hold. Then the Cauchy problem (1.1) admits at least an asymptotically* ω*-periodic mild solution provided that*

$$M\max\{L_H, \sigma_2\} + ML_F\delta^{-1} + M\sigma_1\sigma_3 < 1. \tag{3.4}$$

Proof. As assumed in (H_3), F is asymptotically ω-periodic and is split into $F_1 + F_2$ with $F_1 \in C_\omega(\mathbb{R} \times X; X)$ and $F_2 \in C_0(\mathbb{R}^+ \times X; X)$. From this it follows at once that $F_1(\cdot, v(\cdot)) \in C_\omega(\mathbb{R}; X)$ for every $v \in C_\omega(\mathbb{R}; X)$.

Let the mapping Γ^1 be defined by

$$(\Gamma^1 v)(t) = \int\limits_{-\infty}^t U(t,s)F_1(s,v(s))\,ds, \quad v \in C_\omega(\mathbb{R}; X).$$

By (H_1) we obtain

$$\left\| \int\limits_{-\infty}^t U(t,s)F_1(s,v(s))\,ds \right\| \leq M\delta^{-1}\|F_1(\cdot, v(\cdot))\|'_\infty,$$

which implies that Γ^1 is well defined and continuous on $\mathbb{R}$. Moreover, one easily calculates, by (H_2),

$$(\Gamma^1 v)(t+\omega) = \int\limits_{-\infty}^{t+\omega} U(t+\omega,s)F_1(s,v(s))\,ds = \int\limits_{-\infty}^{t} U(t+\omega,s+\omega)F_1(s+\omega,v(s+\omega))\,ds = (\Gamma^1 v)(t).$$

Accordingly, Γ^1 maps $C_\omega(\mathbb{R}; X)$ into itself.

At the same time, for any $v_1, v_2 \in C_\omega(\mathbb{R}; X)$, we obtain, thanks to the semi-Lipschitz continuity of F,

$$\|(\Gamma^1 v_1)(t) - (\Gamma^1 v_2)(t)\| \le ML_F \int\limits_{-\infty}^{t} e^{-\delta(t-s)} \|v_1(s) - v_2(s)\|\, ds \le ML_F \delta^{-1} \|v_1 - v_2\|'_\infty.$$

As a result, we have

$$\|\Gamma^1 v_1 - \Gamma^1 v_2\|'_\infty \le ML_F \delta^{-1} \|v_1 - v_2\|'_\infty.$$

This, together with (3.4), allows us to conclude, using the Banach contraction principle, that Γ^1 has a unique fixed point $v \in C_\omega(\mathbb{R}; X)$.

Now, on $C_0(\mathbb{R}^+; X)$ we consider an integral equations in the form

$$w(t) = U(t,0)H(v|_{\mathbb{R}^+} + w) + \int\limits_0^t U(t,s)F(s, v(s) + w(s))\, ds -$$

$$- \int\limits_{-\infty}^{t} U(t,s)F_1(s, v(s))\, ds, \quad t \in \mathbb{R}^+. \tag{3.5}$$

Here, our objective is to show that the integral equation (3.5) admits at least a solution in $C_0(\mathbb{R}^+; X)$.

Let us define a mapping $\Gamma^2 = \Gamma_{F_1} + \Gamma_{F_2}$ as

$$(\Gamma_{F_1} w)(t) := U(t,0)H(v|_{\mathbb{R}^+} + w) + \int\limits_0^t U(t,s)[F_1(s, v(s) + w(s)) -$$

$$- F_1(s, v(s))]\, ds, \quad w \in C_0(\mathbb{R}^+; X),$$

$$(\Gamma_{F_2} w)(t) := - \int\limits_{-\infty}^{0} U(t,s)F_1(s, v(s))\, ds + \int\limits_0^t U(t,s)F_2(s, v(s) + w(s))\, ds, \quad w \in C_0(\mathbb{R}^+; X).$$

It is clear that the result follows if we can show that the mapping $\Gamma^2 : C_0(\mathbb{R}^+; X) \to C_0(\mathbb{R}^+; X)$ has a fixed point. The proof will be divided into four steps.

Step 1. As above, $F_1(\cdot, v(\cdot)) \in C_\omega(\mathbb{R}; X)$ for every $v \in C_\omega(\mathbb{R}; X)$. Also, from the semi-Lipschitz continuity of F we observe that for all $t \in \mathbb{R}^+$, $x \in X$,

$$\|F_1(t, v(t) + x) - F_1(t, v(t))\| \le L_F \|x\|,$$

which implies that $F_1(\cdot, v(\cdot) + w(\cdot)) - F_1(\cdot, v(\cdot) \in C_0(\mathbb{R}^+; X)$ for every $w \in C_0(\mathbb{R}^+; X)$. Moreover, we infer that $F_2(\cdot, v(\cdot) + w(\cdot)) \in C_0(\mathbb{R}^+; X)$ for every $w \in C_0(\mathbb{R}^+; X)$, since $F_2 \in$ $\in C_0(\mathbb{R}^+ \times X; X)$. Hence, an application of Lemma 3.1 yields that Γ^2 is well defined and maps $C_0(\mathbb{R}^+; X)$ into itself.

Next, we claim that there exists a $r_0 > 0$ such that $\Gamma_{F_1} w_1 + \Gamma_{F_2} w_2 \in \Omega_{r_0}$ for every pair $w_1, w_2 \in \Omega_{r_0}$. Indeed, from (3.1), (3.2) and (3.4) it follows readily that there exists a $r_0 > 0$ such that

$$M\phi(r_0 + \|v\|'_\infty) + ML_F \delta^{-1} r_0 + M\delta^{-1} \sup_{t\in\mathbb{R}} \|F_1(t, v(t))\| + M\varphi(r_0 + \|v\|'_\infty)\sigma_3 \le r_0.$$

Noticing this and the representation of σ_3, we obtain that for every pair $w_1, w_2 \in \Omega_{r_0}$,

$$\begin{aligned}\|(\Gamma_{F_1} w_1)(t) + (\Gamma_{F_2} w_2)(t)\| &\le M\|H(v|_{\mathbb{R}^+} + w_1)\| + M \int_{-\infty}^{0} e^{-\delta(t-s)} \|F_1(s, v(s))\|\, ds + \\ &+ M \int_0^t e^{-\delta(t-s)} \|F_1(s, v(s) + w_1(s)) - F_1(s, v(s))\|\, ds + \\ &+ M \int_0^t e^{-\delta(t-s)} \|F_2(s, v(s) + w_2(s))\|\, ds \le \\ &\le M\phi(r_0 + \|v\|'_\infty) + ML_F\delta^{-1} r_0 + M\delta^{-1} \sup_{t\in\mathbb{R}} \|F_1(t, v(t))\| + \\ &+ M\varphi(r_0 + \|v\|'_\infty) \int_0^t e^{-\delta(t-s)} \theta(s)\, ds \le r_0.\end{aligned}$$

Accordingly, the claim follows.

Step 2. Taking $w_1, w_2 \in \Omega_{r_0}$, from the Lipschitz continuity of H and the semi-Lipschitz continuity of F we obtain

$$\begin{aligned}\|(\Gamma_{F_1} w_1)(t) - (\Gamma_{F_1} w_2)(t)\| &\le M\|H(v|_{\mathbb{R}^+} + w_1) - H(v|_{\mathbb{R}^+} + w_2)\| + \\ &+ M \int_0^t e^{-\delta(t-s)} \|F_1(s, v(s) + w_1(s)) - F_1(s, v(s) + w_2(s))\|\, ds \le \\ &\le ML_H \|w_1(t) - w_2(t)\|_0 + ML_F \int_0^t e^{-\delta(t-s)} \|w_1(s) - w_2(s)\|\, ds \le \\ &\le M(L_H + L_F\delta^{-1})\|w_1 - w_2\|_0,\end{aligned}$$

from which together with (3.4) we see that Γ_{F_1} is a strict contraction on Ω_{r_0}.

Step 3. We show that Γ_{F_2} is completely continuous on Ω_{r_0}.

Given $\epsilon > 0$. Let $\{w_k\}_{k=1}^{+\infty} \subset \Omega_{r_0}$ with $w_k \to w_0$ in $C_0(\mathbb{R}^+; X)$ as $k \to +\infty$.

Since $\theta \in C_0(\mathbb{R}^+)$, one may choose a $t_1 > 0$ big enough such that

$$3M\varphi(r_0 + \|v\|'_\infty)\theta(t) < \delta\epsilon$$

whenever $t \geq t_1$. Also, in view of (H_3) we have $F_2(s, v(s) + w_k(s)) \to F_2(s, v(s) + w(s))$ for all $s \in [0, t_1]$ as $k \to \infty$ and

$$\|F_2(\cdot, v(\cdot) + w_k(\cdot)) - F_2(\cdot, v(\cdot) + w(\cdot))\| \leq 2\varphi(r_0 + \|v\|'_\infty)\theta(\cdot) \in L^1(0, t_1).$$

Hence, by the Lebesgue dominated convergence theorem we deduce that there exists an $N > 0$ such that for any $t \in \mathbb{R}^+$,

$$\|(\Gamma_{F_2} w_k)(t) - (\Gamma_{F_2} w)(t)\| \leq M \int_0^{t_1} \|F_2(s, v(s) + w_k(s)) - F_2(s, v(s) + w(s))\|\, ds +$$

$$+ 2M\varphi(r_0 + \|v\|'_\infty) \int_{t_1}^{\max\{t,t_1\}} e^{-\delta(t-s)}\theta(s)\, ds < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$$

whenever $k \geq N$. Accordingly, Γ_{F_2} is continuous on Ω_{r_0}.

In the sequel, we consider the compactness of Γ_{F_2}. Since that the function

$$t \mapsto \int_{-\infty}^{0} U(t, s)F_1(s, v(s))\, ds$$

belongs to $C_0(\mathbb{R}^+; X)$ due to Lemma 3.1 and is independent of w, it suffices to show that the mapping Γ'_{F_2} given by

$$(\Gamma'_{F_2} w)(t) := \int_0^t U(t, s)F_2(s, v(s) + w(s))\, ds, \quad w \in C_0(\mathbb{R}^+; X),$$

is compact.

Let $t > 0$ be fixed. For any $\varepsilon \in (0, t)$, note that $U(t, t - \varepsilon/2) \in \mathcal{L}(X)$ and $U(t - \varepsilon/2, t - \varepsilon)$ is compact in X due to (H_2). Therefore, for every $w \in \Omega_{r_0}$, as

$$(\Gamma'_{F_2} w)(t) = \int_{t-\varepsilon}^{t} U(t, s)F_2(s, v(s) + w(s))\, ds + U\left(t, t - \frac{\varepsilon}{2}\right) U\left(t - \frac{\varepsilon}{2}, t - \varepsilon\right) \times$$

$$\times \int_0^{t-\varepsilon} U(t - \varepsilon, s)F_2(s, v(s) + w(s))\, ds,$$

and

$$\left\| \int\limits_{t-\varepsilon}^{t} U(t,s)F_2(s, v(s)+w(s))\,ds \right\| \le M\varphi(r_0 + \|v\|'_\infty) \int\limits_{t-\varepsilon}^{t} e^{-\delta(t-s)}\theta(s)\,ds \to 0 \ \text{ as } \ \varepsilon \to 0$$

due to (H_1) and (3.1), we conclude, using the total boundedness, that for each $t > 0$, the set $\{(\Gamma'_{F_2}w)(t); w \in \Omega_{r_0}\}$ is relatively compact in X.

Next, we verify the equicontinuity of the set $\{\Gamma'_{F_2}w; w \in \Omega_{r_0}\}$. Let $\kappa > 0$ be small enough and $s_1, s_2 \in \mathbb{R}^+$, $w \in \Omega_{r_0}$. Then by (H_1) and (3.1) we have that for the case when $s_1 < s_2$,

$$\begin{aligned}
\|(\Gamma'_{F_2}w)(s_2) - (\Gamma'_{F_2}w)(s_1)\| \le & \int\limits_{s_1}^{s_2} \|U(s_2,s)F_2(s,v(s)+w(s))\|\,ds + \\
& + \int\limits_{0}^{s_1-\kappa} \|(U(s_2,s)-U(s_1,s))F_2(s,v(s)+w(s))\|\,ds + \\
& + \int\limits_{s_1-\kappa}^{s_1} \|(U(s_2,s)-U(s_1,s))F_2(s,v(s)+w(s))\|\,ds \le \\
& \le M\varphi(r_0+\|v\|'_\infty) \int\limits_{s_1}^{s_2} e^{-\delta(s_2-s)}\theta(s)\,ds + \\
& + \varphi(r_0+\|v\|'_\infty) \sup_{s\in[0,s_1-\kappa]} \|(U(s_2,s)-U(s_1,s))\| \int\limits_{0}^{s_1-\kappa} \theta(s)\,ds + \\
& + M\varphi(r_0+\|v\|'_\infty) \int\limits_{s_1-\kappa}^{s_1} \left(e^{-\delta(s_2-s)} + e^{-\delta(s_1-s)}\right)\theta(s)\,ds \to 0 \\
& \text{as} \quad s_2 - s_1 \to 0, \quad \kappa \to 0,
\end{aligned}$$

and for the case when $0 = s_1 < s_2$,

$$\|(\Gamma'_{F_2}w)(s_2) - (\Gamma'_{F_2}w)(s_1)\| \le M\varphi(r_0+\|v\|'_\infty) \int\limits_{0}^{s_2} e^{-\delta(s_2-s)}\theta(s)\,ds \to 0 \ \text{ as } \ s_2 \to 0,$$

which verifies that the result follows.

Finally, as

$$\|(\Gamma'_{F_2}w)(t)\| \le M\varphi(r_0+\|v\|'_\infty) \int\limits_{0}^{t} e^{-\delta(t-s)}\theta(s)\,ds \to 0 \ \text{ as } \ t \to \infty,$$

uniformly for $w \in \Omega_{r_0}$ in view of (3.3), we conclude that $\Gamma'_{F_2} w$ vanishes at infinity uniformly for $w \in \Omega_{r_0}$.

Now an application of Lemma 2.2 justifies the compactness of Γ'_{F_2}, which together with the representation of Γ_{F_2} implies that Γ_{F_2} is compact.

Step 4. As shown in Step 2 and in Step 3, respectively, Γ_{F_1} is a strict contraction and Γ_{F_2} is completely continuous. Accordingly, we deduce, thanks to Lemma 2.3, that the integral equation (3.5) admits at least a solution $w \in C_0(\mathbb{R}^+; X)$.

Noting that $w \in C_0(\mathbb{R}^+; X)$ satisfies the integral equation (3.5) and $v \in C_\omega(\mathbb{R}; X)$ satisfies the integral equation

$$v(t) = \int_{-\infty}^{t} U(t,s) F_1(s, v(s))\, ds, \quad v \in C_\omega(\mathbb{R}; X), \quad t \in \mathbb{R},$$

we deduce that $v+w \in AP_\omega(\mathbb{R}^+; X)$ is an asymptotically ω-periodic mild solution to the Cauchy problem (1.1).

Theorem 3.1 is proved.

The following is a direct consequence of Theorem 3.1.

Corollary 3.1. *Assume that the hypotheses $(H_1)-(H_3)$ hold, $H(u) \equiv u_0 \in X$ and $ML_F\delta^{-1} + M\sigma_1\sigma_3 < 1$. Then there exists an asymptotically ω-periodic mild solution to the Cauchy problem (1.1).*

Below, we will establish the existence result of asymptotically ω-periodic mild solutions to the Cauchy problem (1.1) for the case of H being completely continuous.

Theorem 3.2. *Let the hypotheses $(H_1)-(H_3)$ hold. Assume in addition that*

(H'_4) $H : AP_\omega(\mathbb{R}^+; X) \to X$ is completely continuous, there exists a nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $u \in S_r$,

$$\|H(u)\| \le \phi(r), \quad \text{and} \quad \liminf_{r \to +\infty} \frac{\phi(r)}{r} = \sigma_2.$$

Then the Cauchy problem (1.1) admits at least an asymptotically ω-periodic mild solution provided that

$$M\sigma_2 + ML_F\delta^{-1} + M\sigma_1\sigma_3 < 1. \tag{3.6}$$

Proof. Assume that the operator Γ^1 is defined the same as in Theorem 3.1 and $v \in C_\omega(\mathbb{R}; X)$, coming from Theorem 3.1, is a unique fixed point of Γ^1.

Consider a mapping $Q = Q^1 + Q^2$ defined by

$$(Q^1 w)(t) := \int_0^t U(t,s)\left[F_1(s, v(s) + w(s)) - F_1(s, v(s))\right] ds, \quad w \in C_0(\mathbb{R}^+; X),$$

$$(Q^2w)(t) := U(t,0)H(v|_{\mathbb{R}^+} + w) - \int\limits_{-\infty}^{0} U(t,s)F_1(s,v(s))\,ds +$$

$$+ \int\limits_{0}^{t} U(t,s)F_2(s,v(s)+w(s))\,ds, \quad w \in C_0(\mathbb{R}^+;X).$$

From our assumptions it follows that Q is well defined and maps $C_0(\mathbb{R}^+;X)$ into itself and there exists a $r_0 > 0$ such that $Q^1w_1 + Q^2w_2 \in \Omega_{r_0}$ for every pair $w_1, w_2 \in \Omega_{r_0}$ (see the Step 1 in the proof of Theorem 3.1 for more details). Thus, to be able to apply Lemma 2.3 to obtain a fixed point of Q, we need to prove that Q^1 is a strict contraction and Q^2 is completely continuous on Ω_{r_0}.

From (3.6) and the Step 2 in the proof of Theorem 3.1 we see that Q^1 is a strict contraction. Also, since $H : AP_\omega(\mathbb{R}^+;X) \to X$ is completely continuous, it follows from the Step 3 in the proof of Theorem 3.1 that Q^2 is completely continuous. Now, applying Lemma 2.3 we obtain that Q has a fixed point $w \in C_0(\mathbb{R}^+;X)$, which gives rise to an asymptotically ω-periodic mild solution $v + w$.

Theorem 3.2 is proved.

As an application, let us consider a partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial condition

$$\frac{\partial u(t,\xi)}{\partial t} - \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + d(t)u(t,\xi) = \sin\frac{2\pi t}{\omega}\,\sin u(t,\xi) + e^{-t}u(t,\xi)\cos u^3(t,\xi), \;\; t > 0, \;\; \xi \in [0,\pi],$$

$$u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}^+, \tag{3.7}$$

$$u(0,\xi) = u_0(\xi) + \sum_{i=1}^{p} C_i u(s_i,\xi), \quad \xi \in [0,\pi],$$

where $0 < s_1 < \ldots < s_p < +\infty$ and C_i, $i = 1,\ldots,p$, are given constants, $u_0 \in L^2[0,\pi]$, $d : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $d(t+\omega) = d(t)$ for all $t \in \mathbb{R}$, and

$$d_{\min} := \min_{t\in\mathbb{R}} d(t) > -1.$$

Take $X = L^2[0,\pi]$ with the norm $\|\cdot\|_{L^2[0,\pi]}$ and inner product $(\cdot,\cdot)_2$. Define

$$D(A(t)) := D(B), \quad t \in \mathbb{R},$$

$$A(t)x := Bu - d(t)x, \quad x \in D(A(t)),$$

where the operator $B : D(B) \subset X \to X$ is given by

$$Bx = \frac{\partial^2 x}{\partial \xi^2}, \quad x \in D(B),$$

$$D(B) = \{x \in X; x, x' \text{ are absolutely continuous}, x'' \in X, \text{ and } x(0) = x(\pi) = 0\}.$$

It is well-known that B has a discrete spectrum and its eigenvalues are $-n^2, n \in \mathbb{N}^+$, with the corresponding normalized eigenvectors $y_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$. More details about these facts can be seen from the monograph [6] of Pazy.

It is clear that $(A(t))_{t\in\mathbb{R}}$ satisfies the conditions (AT_1) and (AT_2), which ensure that it generates an evolution family $\{U(t,s)\}_{t\geq s}$ and

$$U(t,s)x = \sum_{n=1}^{\infty} e^{-\left(\int_s^t d(\tau)\, ds\tau + n^2(t-s)\right)}(x, y_n)_2 y_n \quad \text{for all} \quad t \geq s, x \in X. \tag{3.8}$$

A direct calculation gives

$$\|U(t,s)\| \leq e^{-(1+d_{\min})(t-s)} \quad \text{for all} \quad t \geq s.$$

Note also that for each $t > s$, the operator $U(t,s)$ is a nuclear operator, which yields the compactness of $U(t,s)$ for $t > s$.

Now, we define

$$F_1(t, x(\xi)) := \sin\frac{2\pi t}{\omega}\sin x(\xi), \quad t \in \mathbb{R}, \quad x \in X,$$

$$F_2(t, x(\xi)) := e^{-t}x(\xi)\cos x^3(\xi), \quad t \in \mathbb{R}^+, \quad x \in X,$$

$$H(u(t,\xi)) := u_0(\xi) + \sum_{i=1}^{p} C_i u(t_i, \xi), \quad u \in AP_\omega(\mathbb{R}^+; X).$$

It is easy to verify that $F_1 : \mathbb{R} \times X \to X$ and $F_2 : \mathbb{R}^+ \times X \to X$ are continuous, $F_1(t+\omega, x) = F_1(t,x)$ for all $t \in \mathbb{R}$ and $x \in X$, and

$$\|F_1(t,x) - F_1(t,y)\| \leq \|x - y\| \quad \text{for all} \quad t \in \mathbb{R}^+, \quad x, y \in X,$$

$$\|F_2(t,x)\| \leq e^{-t}\|x\| \quad \text{for all} \quad t \in \mathbb{R}^+, \quad x \in X,$$

$$\|H(u) - H(v)\| \leq \sum_{i=1}^{p} |C_i| \|u - v\|_0 \quad \text{for all} \quad u, v \in AP_\omega(\mathbb{R}^+; X).$$

Moreover, the assumptions $(H_1)-(H_4)$ hold with

$$L_H = \sigma_2 = \sum_{i=1}^{p} |C_i|, \quad L_F = 1, \quad \varphi(r) = r, \quad \phi(r) = \|u_0\| + r\sum_{i=1}^{p}|C_i|,$$

$$\theta(t) = e^{-t}, \quad \sigma_1 = 1, \quad \sigma_3 \leq \frac{1}{1 + d_{\min}}, \quad M = 1, \quad \delta = 1 + d_{\min}.$$

Therefore, (3.7) can be reformulated as the abstract Cauchy problem (1.1). Hence, when $\sum_{i=1}^{p}|C_i| + \dfrac{2}{1 + d_{\min}} < 1$ such that condition (3.4) is satisfied, it follows from Theorem 3.1 that the partial differential equation (3.7) at least has one asymptotically ω-periodic mild solution.

1. *de Andrade B., Cuevas C.* S-asymptotically ω-periodic and asymptotically ω-periodic solutions to semi-linear Cauchy problems with non-dense domain // Nonlinear Anal. — 2010. — **72**. — P. 3190 – 3208.
2. *Pierri M.* On S-asymptotically ω-periodic functions and applications // Nonlinear Anal. — 2012. — **75**. — P. 651 – 661.
3. *Agarwal R. P., Cuevas C., Soto H., El-Gebeily M.* Asymptotic periodicity for some evolution equations in Banach spaces // Nonlinear Anal. — 2011. — **74**. — P. 1769 – 1798.
4. *Henriquez H., Pierri M., Táboas P.* On S-asymptotically ω-periodic functions on Banach spaces and applications // J. Math. Anal. and Appl. — 2008. — **343 (2)**. — P. 1119 – 1130.
5. *Amann H.* Linear and quasilinear parabolic problems. — Berlin: Birkhäuser, 1995.
6. *Pazy A.* Semigroups of linear operators and applications to partial differential equations // Appl. Math. Sci. — 1983. — **44**.
7. *Acquistapace P.* Evolution operators and strong solution of abstract parabolic equations // Different. Integr. Equat. — 1988. — **1**. — P. 433 – 457.
8. *Acquistapace P., Terreni B.* A unified approach to abstract linear nonautonomous parabolic equations // Rend. Semin. mat. Univ. Padova. — 1987. — **78**. — P. 47 – 107.
9. *Balachandran K., Trujillo J. J.* The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces // Nonlinear Anal. — 2010. — **72**. — P. 4587 – 4593.
10. *Byszewski L., Akca H.* On a mild solution of a semilinear functional-differential evolution nonlocal problem // J. Appl. Math. Stochast. Anal. — 1997. — **10**. — P. 265 – 271.
11. *Liang J., Liu J. H., Xiao T. J.* Nonlocal Cauchy problems governed by compact operator families // Nonlinear Anal. — 2004. — **57**. — P. 183 – 189.
12. *Liang J., Liu J. H., Xiao T. J.* Nonlocal impulsive problems for nonlinear differential equations in Banach spaces // Math. Comput. Modelling. — 2009. — **49**. — P. 798 – 804.
13. *Wang R. N., Xiao T. J., Liang J.* A note on the fractional Cauchy problems with nonlocal initial conditions // Appl. Math. Lett. — 2011. — **24**. — P. 1435 – 1442.
14. *Wang J. R., Feckan M., Zhou Y.* On the new concept of solutions and existence results for impulsive fractional evolution equations // Dynam. Part. Different. Equat. — 2011. — **8**. — P. 345 – 361.
15. *Zhou Y., Jiao F.* Nonlocal Cauchy problem for fractional evolution equations // Nonlinear Anal. R.W.A. — 2010. — **11**. — P. 4465 – 4475.
16. *Byszewski L.* Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem // J. Math. Anal. and Appl. — 1991. — **162**. — P. 494 – 505.
17. *Deng K.* Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions // J. Math. Anal. and Appl. — 1993. — **179**. — P. 630 – 637.
18. *Lin Y., Liu J. H.* Semilinear integrodifferential equations with nonlocal Cauchy problem // Nonlinear Anal. — 1996. — **26**. — P. 1023 – 1033.
19. *Liang J., Liu J. H., Xiao T. J.* Nonlocal Cauchy problems for nonautonomous evolution equations // Communs. Pure and Appl. Anal. — 2006. — **5**. — P. 529 – 535.
20. *Xiao T. J., Liang J.* Existence of classical solutions to nonautonomous nonlocal parabolic problems // Nonlinear Anal. — 2005. — **63**. — P. 225 – 232.
21. *Engel K. J., Nagel R.* One-parameter semigroups for linear evolution equations // GTM. — Berlin; New York: Springer, 2000. — **194**.
22. *Yosida K.* Functional analysis. — 6th ed. — Berlin: Springer-Verlag, 1980.
23. *Fitzgibbon W. E.* Semilinear functional equations in Banach space // J. Different. Equat. — 1978. — **29**. — P. 1 – 14.
24. *Yagi A.* Abstract quasilinear evolution equations of parabolic type in Banach spaces // Boll. Unione mat. ital. B. — 1991. — **5(7)**. — P. 341 – 368.
25. *Yagi A.* Parabolic equations in which the coefficients are generators of infinitely differentiable semigroups II // Funkc. ekvacioj. — 1990. — **33**. — P. 139 – 150.

Received 23.12.12