UDC 517.9

STABILITY OF SYNCHRONIZED AND CLUSTERED STATES IN COUPLED PIECEWISE LINEAR MAPS

СТІЙКІСТЬ СИНХРОНІЗОВАНИХ ТА КЛАСТЕРНИХ СТАНІВ У СИСТЕМІ ЗВ'ЯЗАНИХ КУСКОВО-ЛІНІЙНИХ ВІДОБРАЖЕНЬ

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Parameter regions for different types of stability of synchronized and clustered states are obtained for two interacting ensembles of globally coupled one-dimensional piecewise linear maps. We analyze strong (asymptotic) and weak (Milnor) stability of the synchronized state, as well as its instability. We found that the stability and instability regions in the phase space depend only on parameters of the individual skew tent map, and do not depend on the ensembles size. In the simplest nontrivial case of four coupled chaotic maps we obtain stability regions for coherent and two-cluster states. The regions appear to be large enough to provide an effective control of coherent and clustered chaotic regimes. Transition from desynchronization to synchronization is identified to be qualitatively different in smooth and piecewise linear models.

Знайдено параметричні області для різних типів стійкості синхронізованих та кластерних станів для двох взаємодіючих ансамблів глобально зв'язаних одновимірних кусково-лінійних відображень. Досліджено сильну (асимптотичну) та слабку (за Мілнором) стійкість та нестійкість синхронізованого стану системи. Визначено, що області стійкості та нестійкості у просторі параметрів залежать лише від коефіцієнтів кусково-лінійного відображення і не залежать від розміру ансамблів. Для найпростішого нетривіального випадку чотирьох зв'язаних відображень отримано області стійкості для когерентного та двокластерних станів. Досить великі розміри областей стійкості у просторі параметрів дають можливість проводити ефективне керування когерентним та кластерними режимами у системі. Крім цього, виявлено якісно різні способи десинхронізації у системах кусково-лінійних та гладких відображень.

1. Introduction. Network organizations appear in a wide variety of phenomena in physics, engineering, biology, medicine and other fields. That is why investigation of the network formations and dependence of their properties on the network structure has become an important applied problem. There are a lot of different mathematical models describing netwok organizations, developed to understand the collective behavior of their elements. Examples of network formations can be presented as ensembles of globally (mean field) coupled oscillators

$$x_i^{t+1} = (1-\varepsilon)f(x_i^t) + \frac{\varepsilon}{N}\sum_{j=1}^{N} f(x_j^t), \tag{1}$$

which have been first suggested and intensively studied by Kaneko [1, 2]. If not all oscillators in the ensemble interact, we get a so-called 'fractally' coupled system,

$$x_i^{t+1} = (1-\varepsilon)f(x_i^t) + \frac{\varepsilon}{A_i}\sum_{j\in \operatorname{con} n} f(x_j^t), \tag{2}$$

where A_i is the number of connections at the ith element, and j runs over all elements that are connected to the element i. An interesting example of such networks is given by 'small-world' systems, introduced by Watts and Strogatz [3, 4] and interpreted as mathematical models of many biologic and social networks. Another example is given by neural network, where groups of peripheral oscillators interact with only one leading oscillator [5].

Investigation of such systems is typically concerned with studying their dynamics properties such as full and partial synchronization [6].

In the present paper we consider a $2N$-dimensional system of coupled maps which models the situation when there are two globally coupled Kaneko systems of the mean field type (1), and their elements are pairwise coupled with each others,

$$\begin{aligned} x_i^{t+1} &= \left(1-\varepsilon-\frac{\delta}{2}\right) f(x_i^t) + \frac{\varepsilon}{N}\sum_{j=1}^{N} f(x_j^t) + \frac{\delta}{2} f(y_i^t), \\ y_i^{t+1} &= \left(1-\varepsilon-\frac{\delta}{2}\right) f(y_i^t) + \frac{\varepsilon}{N}\sum_{j=1}^{N} f(y_j^t) + \frac{\delta}{2} f(x_i^t), \quad i=1,\dots,N. \end{aligned} \tag{3}$$

Here $\{x_i^t\}_{i=1}^N$ and $\{y_i^t\}_{i=1}^N$ are N-dimensional state vectors; $t = 0, 1, \dots$ is a discrete time index, $f : \mathbb{R} \to \mathbb{R}$ is a one-dimensional map.

A model of the form (3), in the case of smooth, logistic map $f(x) = ax(1-x)$ was studied in [7]. In the present work, f is chosen as a piecewise linear, skew tent map of the form

$$f(x) = f_{l,p}(x) = \begin{cases} lx + 1 - l - \dfrac{l}{p}, & x \le 1 + \dfrac{1}{p}; \\ px - p, & x > 1 + \dfrac{1}{p}. \end{cases} \tag{4}$$

Piecewise linear maps systems are widely spread in a variety of technical, engineering, and electronic applications. Depending on parameters, such systems are characterized by regular or complex chaotic dynamics [8, 9]. In many cases piecewise linear dynamics appear to be rather different from smooth ones, especially as for different types of bifurcations and their [10–12].

The behavior of $2N$-dimensional system (3) is controlled by four parameters, $l > 0$ and $p < -1$, which are the coefficients of linear parts of the individual skew tent map $f = f_{l,p}$, and coupling parameters $\varepsilon,\ \delta$.

Model (3) can be interpreted as two groups of interacting oscillators (i. e. neurons), as well as the simplest example of a deterministic complex 'small-world' network having two groups of elements with a strong coupling inside and weaker coupling between the groups.

In the present paper we obtain parameter regions for strong and weak stability and instability of the synchronizing chaotic set on the main diagonal of the $2N$-dimensional phase space of system (3) in terms of the coefficients of the individual skew tent map $f_{l,p}$. In the parameter (δ, ε)-plane we delineate schematically stability regions and identify variation of their border with change of l and p. As the simplest nontrivial example we consider a system of four coupled skew tent maps and delineate parameter regions for stability of synchronized (coherent) and two-cluster states.

2. Stability of chaotic synchronizing set. Consider a $2N$-dimensional map F defined by the system (3), where N is a number of elements in each globally coupled ensemble. The Jacobian matrix of this system is

$$DF\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} G(X) & L(Y) \\ L(X) & G(Y) \end{pmatrix},$$

where

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix},$$

$$G(Z) = \begin{pmatrix} \left(1 - \dfrac{(N-1)\varepsilon}{N} - \dfrac{\delta}{2}\right) f'(z_1) & \cdots & \dfrac{\varepsilon}{N} f'(z_N) \\ \vdots & \ddots & \vdots \\ \dfrac{\varepsilon}{N} f'(z_1) & \cdots & \left(1 - \dfrac{(N-1)\varepsilon}{N} - \dfrac{\delta}{2}\right) f'(z_N) \end{pmatrix}$$

and

$$L(Z) = \begin{pmatrix} \dfrac{\delta}{2} f'(z_1) & 0 & \cdots & 0 \\ 0 & \dfrac{\delta}{2} f'(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{\delta}{2} f'(z_N) \end{pmatrix}.$$

We denote by $D_{2N} = \{\mathbf{z} \in \mathbb{R}^{2N} | z_1 = \ldots = z_{2N}\}$ the main diagonal of the $2N$-dimensional phase space. It is not difficult to obtain that the Jacobian matrix $DF(\mathbf{x})$, taken in the point $\mathbf{x} = (x, \ldots, x) \in D_{2N}$, has only four different eigenvalues,

$$\begin{gathered} \nu_{\|}(x) = f'(x), \quad \nu_{\perp,1}(x) = f'(x)(1-\varepsilon), \\ \nu_{\perp,2}(x) = f'(x)(1-\delta), \quad \nu_{\perp,3}(x) = f'(x)(1-\varepsilon-\delta). \end{gathered} \tag{5}$$

Note that $\nu_{\perp,2}(x)$ and $\nu_{\perp,3}(x)$ have multiplicity $N-1$.

The eigenvector $v_{\|}$ corresponding to the eigenvalue $\nu_{\|}$ is directed along the main diagonal D_{2N}, hence this eigenvalue is responsible for 'dispersion' of the trajectories along D_{2N}. Three other eigenvectors are transverse to D_{2N}. Hence, the corresponding eigenvalues ($\nu_{\perp,1}$, $\nu_{\perp,2}$, and $\nu_{\perp,3}$), named usually transverse, control the attracting (repelling) strength for the trajectories in the corresponding transverse directions.

The following consideration is concerned with the next definitions.

Let $A_D \subset D_{2N}$ be an invariant set for the system (3). We denote by $B(A_D)$ the *basin of attraction* of A_D. Recall that $B(A_D) = \{\mathbf{z} \in \mathbb{R}^{2N} | \omega(\mathbf{z}, F) \subset A_D\}$, where $\omega(\mathbf{z}, F) = \bigcap_{n \in \mathbb{N}} \overline{\{F^k(\mathbf{z}) : k > n\}}$.

The set A_D is called *strongly or asymptotically stable* if for any its neighborhood $U(A_D) \subset \mathbb{R}^{2N}$ there exists another neighborhood $V(A_D) \subset \mathbb{R}^{2N}$ such that for any $\mathbf{z} \in V(A_D)$:

1) $F^n(\mathbf{z}) \in U(A_D)$ for all $n \in \mathbb{N}$;

2) $\rho(F^n(\mathbf{z}), A_D) \to 0$ for $n \to \infty$, $\rho(\cdot, \cdot)$ being the distance between a point and a set in $\mathbb{R}^{2N}$.

Let $\operatorname{Pre}(A_D) = \{\mathbf{z} \in \mathbb{R}^{2N} | \exists n \in \mathbb{N} : F^n(\mathbf{z}) \in A_D\}$ be a set of preimages for A_D. Obviously $\operatorname{Pre}(A_D) \subseteq B(A_D)$. The set A_D is called *strongly or asymptotically unstable* if $B(A_D) = \operatorname{Pre}(A_D)$, which means that A_D attracts only its preimages.

The other type of stability considered in the present paper was introduced by Milnor [13]. Typically, it is referred to as weak stability. The set A_D is called *weakly or Milnor stable* if its basin of attraction $B(A_D)$ has a positive Lebesgue measure in $\mathbb{R}^{2N}$. Alternatively, A_D is called *weakly or Milnor unstable* in the opposite case, i. e., if $\operatorname{mes} B(A_D) = 0$. Note that strong stability implies weak stability, but not vice versa.

Let $A_D \subset D_{2N}$ be a chaotic synchronizing set for the system (3). Depending on the coupling parameters values, it can be characterized by the defined above types of stability or instability. Being strongly stable for certain parameter values, when ε and δ change, the set A_D typically loses its strong stability through a *riddling bifurcation* [14]. After the riddling bifurcation, the synchronizing set A_D can still be weakly stable.

Further change of the coupling parameters can cause a loss of weak stability of A_D through *blowout bifurcation* [15]. After blowout bifurcation there still can exist trajectories attracted by the synchronizing set A_D, but the Lebesgue measure of their initial points is equal to zero.

In the present paper such a sequence of parameter regions for strong and weak stability, as well as instability, of the chaotic synchronizing set A_D for the system (3) are obtained analytically.

As shown above, the eigenvalues (5) do not depend on the system size $2N$. So the parameter regions for different types of stability for system (3) do not depend on the number of oscillators, too.

Denote

$$b_1^{(1)} = 1 - \left(\frac{1}{l^{k-1}|p|}\right)^{1/k}, \quad b_1^{(2)} = 1 + \left(\frac{1}{l^{k-1}|p|}\right)^{1/k},$$

$$b_2^{(1)} = 1 - \frac{1}{|p|}, \quad b_2^{(2)} = 1 + \frac{1}{|p|},$$

and $k = \left[2 - \dfrac{\ln l + p(l-1)}{\ln l}\right]$, $[\cdot]$ being the integer part of a real number.

Lemma 1. *For the skew tent map $f_{l,p}$ (4) the most unstable among all existing cycles are:*
the chaotic cycle of the maximal period k, if $l > |p|$;
the fixed point of the map $f_{l,p}$, if $l < |p|$.

Proof. (1) Let $l > |p|$. Suppose for $f_{l,p}$ there exists a cycle

$$\gamma_k = \{(x_1, \dots, x_1); \dots ; (x_k, \dots, x_k) \mid \quad x_{i+1} = f_{l,p}(x_i),\ i = 1, \dots, k-1;\ k > 1; \\ x_1 = f_{l,p}(x_k), \}$$

that belongs to the chaotic synchronizing set A_D. The multiplier of γ_k is

$$\nu^{(k)} = \nu(x_1) \dots \nu(x_k) = |f'_{l,p}(x_1)| \dots |f'_{l,p}(x_k)| = l^m |p|^{k-m},$$

where $0 < m \le k$. The most unstable cycle is the first to lose its stability that is why the value of $\nu^{(k)}$ for this cycle is maximal among multipliers of all existing cycles. $\nu^{(k)}$ reaches its maximal value if the product includes as more multipliers l as possible. A cycle with the kneading sequence $LL \dots L$ does not exist, that is why $\nu^{(k)}$ reaches its maximal value at $\nu^{(k)} = l^{k-1}|p|$ in the case of a cycle of maximal period k, the kneading of which is $L^{k-1}R$. This means that the cycle of period higher than k does not exist. The cycle of period $k+1$ exists when $p \le \le -\dfrac{1 - l^{k-1}}{l^{k-2}(1-l)}$. From this condition we get the value of the k-maximal period of the existing cycle in terms of the coefficients l and p of the map $f_{l,p}$,

$$k = \left[2 - \frac{\ln l + p(l-1)}{\ln l} \right], \tag{6}$$

$[\cdot]$ being the integer part of a real number.

(2) Let $l < |p|$. In this case the value of $\nu^{(k)}$ is maximal if the product includes as more multipliers $|p|$ as possible. This corresponds to the fixed point $\nu^{(k)} = |p|^k$ with the kneading sequence $(R)^k$.

Lemma 2. *For the skew tent map $f_{l,p}$ (4) the most stable among all existing cycles are:*
the fixed point of the map $f_{l,p}$, if $l > |p|$;
the chaotic cycle of maximal period k, if $l < |p|$.

Proof. By analogy to the proof of Lemma 1, the most stable cycle is the last to lose its stability, its multiplier is less than multipliers of the other cycles.

In the case $l > |p|$, $\nu^{(k)}$ reaches its minimal value if the product includes as more multipliers $|p|$ as possible, $\nu^{(k)} = |p|^k$. This means that the most stable is the fixed point with the kneading $(R)^k$.

If $l < |p|$, the multiplier of γ_k is minimal at $\nu^{(k)} = l^{k-1}|p|$ and the cycle of the maximal period k $(L^{k-1}R)$ is the most stable. k can be found, as in Lemma 1, from the condition that the cycle of higher period $k+1$ does not exist.

Theorem 1. *In the system (3), where the individual map $f = f_{l,p}$ has form (4), the chaotic synchronizing set $A_D \subset D_{2N}$ is strongly stable, if*

$\varepsilon,\ \delta,\ (\varepsilon + \delta) \in (b_1^{(1)}, b_1^{(2)})$ *for $l > |p|$;*

$\varepsilon,\ \delta,\ (\varepsilon + \delta) \in (b_2^{(1)}, b_2^{(2)})$ *for $l < |p|$,*

and strongly unstable if

$\varepsilon, \delta, (\varepsilon + \delta) \in \left(-\infty; b_2^{(1)}\right) \bigcup \left(b_2^{(2)}, +\infty\right)$ *for $l > |p|$;*

$\varepsilon, \delta, (\varepsilon + \delta) \in \left(-\infty; b_1^{(1)}\right) \bigcup \left(b_1^{(2)}, +\infty\right)$ *for $l < |p|$.*

Proof. The synchronizing chaotic set A_D loses its strong stability at the moment when the most unstable periodic cycle loses its stability in the transverse direction. We will consider two cases, $l > |p|$ and $l < |p|$.

1) $l > |p|$.

Lemma 1 shows that the most unstable is the chaotic cycle with the maximal period k. It is transversely stable if its transverse multiplier

$$\nu_{\perp,1} = |l^{k-1}p(1-\varepsilon)^k| < 1,$$
$$1 - \left(\frac{1}{l^{k-1}|p|}\right)^{1/k} < \varepsilon < 1 + \left(\frac{1}{l^{k-1}|p|}\right)^{1/k}. \tag{7}$$

By analogy, for the synchronizing chaotic set A_D to be strongly stable in two other transverse directions, the inequalities

$$\nu_{\perp,2} = |l^{k-1}p(1-\delta)^k| < 1, \quad \nu_{\perp,3} = |l^{k-1}p(1-\varepsilon-\delta)^k| < 1,$$

should be satisfied. From the inequalities we get that

$$\delta,\ (\varepsilon+\delta) \in (b_1^{(1)},\ b_1^{(2)}). \tag{8}$$

The region of strong stability of A_D is the intersection of the regions in the parameter plane, defined by (8) and (9).

2) $l < |p|$.

From Lemma 1 we get that the most unstable cycle is the fixed point of the individual map $f_{l,p}$. Its multiplier in the first transverse direction is

$$\nu_{\perp,1} = |p^k(1-\varepsilon)^k| < 1,$$
$$1 - \frac{1}{|p|} < \varepsilon < 1 + \frac{1}{|p|}.$$

From the condition of stability in two other transverse directions, we find parameter regions for strong stability of the synchronizing chaotic set A_D,

$$\delta,\ (\varepsilon+\delta) \in (b_2^{(1)},\ b_2^{(2)}).$$

Transition to strong instability of the chaotic synchronizing set A_D is the moment when the most stable cycle which belongs to A_D loses its stability in the transverse direction.

By analogy to the case of strong stability, using results of Lemma 2 we get that A_D is strongly unstable if

$$\varepsilon,\ \delta,\ (\varepsilon+\delta) \in \left(-\infty, 1 - \left(\frac{1}{l^{k-1}|p|}\right)^{1/k}\right] \bigcup \left[1 + \left(\frac{1}{l^{k-1}|p|}\right)^{1/k},\ +\infty\right)$$

for $l < |p|$. We note that a k can be found from the condition that a cycle of a period bigger than k does not exist.

In the case $l > |p|$, the synchronizing chaotic set A_D is strongly unstable if

$$\varepsilon,\ \delta,\ (\varepsilon+\delta) \in \left(-\infty, 1-\frac{1}{|p|}\right] \bigcup \left[1+\frac{1}{|p|}, +\infty\right).$$

The theorem is proved.

Consider case where the skew tent map $f(x) = f_{l,p}(x)$ has a chaotic interval $I = [0,1]$. According to the theorem of Lasota and Yorke [16], for the map $f_{l,p}$ there exists a unique probability invariant measure $\mu = \mu_{l,p}$, absolutely continuous with respect to the Lebesgue measure. Denote $m = \mu_{l,p}(\{x \in [1+1/p; 1]\})$, i. e., $m = \int_{1+1/p}^{1} \rho(x)dx$, where $\rho(x)$ is a density function of the invariant measure.

Theorem 2. *For the system (3), where the individual map $f = f_{l,p}$ has form (4), the chaotic synchronizing set $A_D \subset D_{2N}$ is weakly stable if*

$$\varepsilon, \delta, (\varepsilon+\delta) \in \left(1-\frac{1}{l^{1-m}|p|^m},\ 1+\frac{1}{l^{1-m}|p|^m}\right),$$

and weakly unstable if

$$\varepsilon, \delta, (\varepsilon+\delta) \in \left(-\infty, 1-\frac{1}{l^{1-m}|p|^m}\right) \bigcup \left(1+\frac{1}{l^{1-m}|p|^m}, +\infty\right).$$

Proof. Transition from weak (Milnor) stability to instability occurs in the moment when the Lyapunov exponent of the typical trajectory of the map $f_{l,p}$ becomes equal to zero.

Let $\{f^n_{l,p}(x)\}$ be a typical trajectory of the piecewise linear map $f_{l,p}(x)$. Since m is a measure of the interval $\left[1+\frac{1}{p}, 1\right]$, $(1-m)$ is the measure of the interval $\left[0, 1+\frac{1}{p}\right]$. Let m be known, then

$$\lambda^A_{\perp,1} = \int_A \ln|f'(x)(1-\varepsilon)|d\mu(x) = \ln|(l(1-\varepsilon))^{1-m}(p(1-\varepsilon))^m|.$$

For the synchronizing set to be Milnor stable the condition $\lambda^A_{\perp,1} < 0$ must be satisfied. It means that

$$|(l(1-\varepsilon))^{1-m}(p(1-\varepsilon))^m| < 1,$$

$$|1-\varepsilon| < \frac{1}{l^{1-m}|p|^m}, \tag{9}$$

$$1-\frac{1}{l^{1-m}|p|^m} < \varepsilon < 1+\frac{1}{l^{1-m}|p|^m}.$$

From analogous conditions for the transverse directions $v_{\perp,2}$ and $v_{\perp,3}$ we get identical stability regions for parameter values δ and $(\varepsilon+\delta)$.

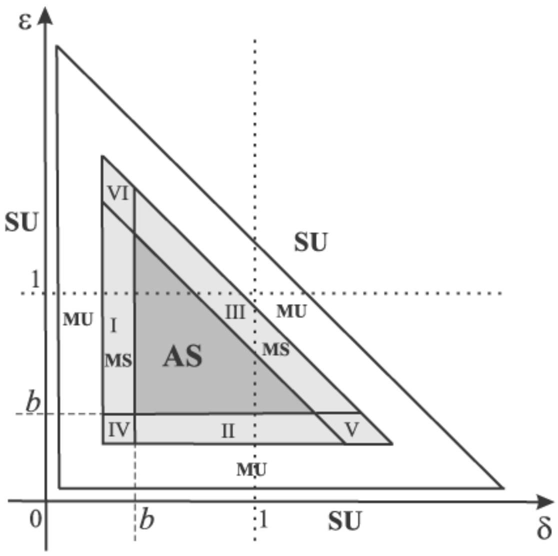

Fig. 1. Parameter regions for different types of stability of the set A_D. In the labeled regions, A_D is: asymptotically stable (AS), Milnor (weakly) stable (MS), Milnor unstable (MU), strongly unstable (SU).

The synchronizing set is Milnor unstable in every transverse direction when $\lambda^A_{\perp,1} > 0$, $\lambda^A_{\perp,2} > 0$, and $\lambda^A_{\perp,3} > 0$. From these conditions we get

$$\varepsilon,\ \delta,\ (\varepsilon+\delta) \in \left(-\infty, 1-\frac{1}{l^{1-m}|p|^m}\right) \bigcup \left(1+\frac{1}{l^{1-m}|p|^m}, +\infty\right). \tag{10}$$

The theorem is proved.

The invariant measure μ can be found in an implicit form only in special cases, in particular when the trajectory of the extreme point $x_0 = 1-1/p$ becomes, eventually, an unstable periodic orbit.

Suppose that the parameters l and p are such that the map $f_{l,p}$ is chaotic and its invariant measure μ is found, being distributed in the interval $[0,1]$. Then one can find borders of the parameter region of weak stability for the synchronizing chaotic set A_D using results of Theorem 2, as well as regions of strong stability and instability from results of Theorem 1.

All stability regions are shown in Fig. 1. Triangle colored with dark gray and labeled AS is the region of strong (asymptotic) stability of A_D. The region colored with light gray and labeled MS is the region of Milnor stability. Lines which bound the AS region, divide the MS region into six parts, marked on the figure with numbers from I to VI. In these parameter regions, the diagonal still attracts trajectories from a set of initial points of positive Lebesque measure.

The synchronizing chaotic set A_D on the main diagonal D_{2N} can lose its stability in three transverse directions which correspond to crossing each of three sides of the stability triangle AS in Fig. 1.

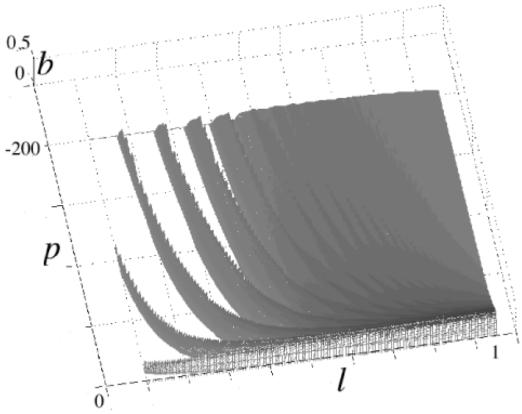

Fig. 2. A dependence of the values of b: the border of strong stability region of the chaotic synchronizing set A_D as the coefficients l and p of the single skew tent map $f_{l,p}$ change.

After the loss of the stability, a transition to a *cluster* state takes place. *Clustering* in the system of coupled oscillators occurs when the population of oscillators splits into subgroups (clusters) with different dynamics, but such that all oscillators within each cluster asymptotically move in synchrony.

When A_D loses stability entering region I, a transition to the two-cluster state $C_2^{(1)} = \{x_1 = \ldots = x_N; y_1 = \ldots = y_N\}$ takes place. When entering region II, we get transition to N-cluster state $C_N = \{x_1 = y_1; \ldots ; x_N = y_N\}$. Finally, when the loss of stability happens through the border of region III, then for even $N = 2n$ it results in the two-cluster state $C_2^{(3)} = \{x_1 = \ldots = x_n = y_{n+1} = \ldots = y_N;\ x_{n+1} = \ldots = x_N = y_1 = \ldots = y_n\}$. Coexistence of different clusters is possible in the parameter regions IV, V and VI.

The set A_D is Milnor unstable in the blank region labeled MU. For the parameter values outside this region MU, A_D is strongly unstable (SU).

To obtain an analytically asymptotic stability region AS, it is enough to know the value $b = \begin{cases} b_1^{(1)}, & l > |p|; \\ b_2^{(1)}, & l < |p|, \end{cases}$ which is the left corner point of AS in Fig. 1. Indeed, as it follows from the Theorem 1, the strong stability region is bounded by the straight lines $\varepsilon = b$, $\delta = b$, and $\varepsilon + \delta = 2 - b$. The value of b varies from 0 to $0,5$, with the change of coefficients of the single map $f_{l,p}$, $b = 0$ corresponds to the strong stability of the synchronizing set A_D for all $0 \leq \varepsilon,\ \delta \leq 1$. Figure 2 shows a plot of values b versus l and p. Blank regions in the (l, p)-plane correspond to periodic windows of the piecewise linear map $f_{l,p}$. For coefficient pairs (l, p) belonging to these regions, the diagonal is strongly stable in the whole unit square $[0, 1] \times [0, 1]$ of the (δ, ε)-parameter plane. Figure 3 shows the variation of b versus l for four fixed values of $p = -10;\ -50;\ -100;\ -200$.

3. An example: a system of four coupled maps. As the simplest nontrivial example, consider system (3) with $2N = 4$ coupled skew tent maps. The coupling parameters are supposed to be $0 \leq \varepsilon,\ \delta \leq 1$.

A partial case of system (3) with four coupled maps is an example of much interest. Its structure is similar to the nearest-neighboring systems, where every element is coupled only

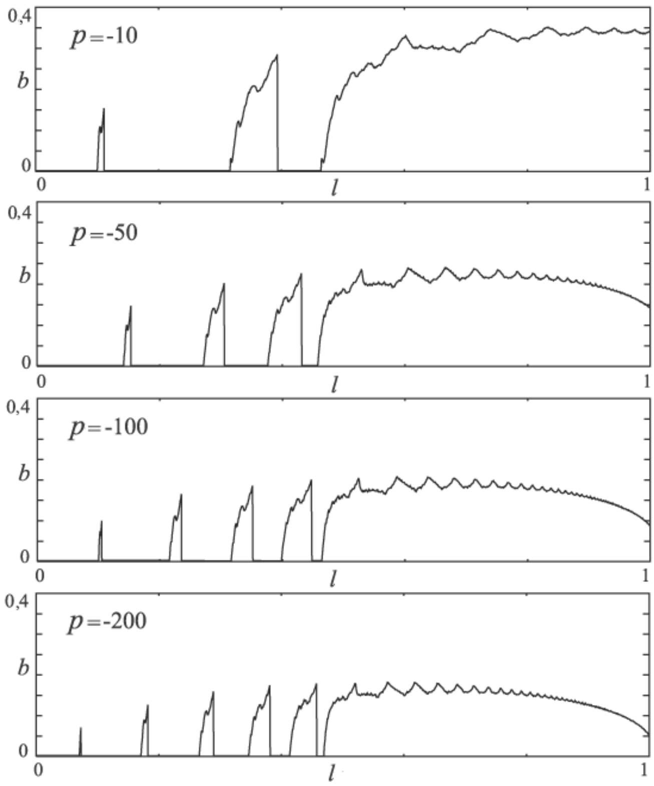

Fig. 3. A dependence of b for fixed values of the parameter p of a single skew tent map. In the periodic windows of the skew tent map, b is equal to zero, which means that the region of strong stabilitycovers the whole unit square $[0,1] \times [0,1]$ of the parameter plane.

with two neighboring elements. In the present system we have two couplings with strength ε and two with strength δ. Such an ensemble of four coupled maps can be interpreted as a description of quadrupeds legs locomotion. This problem was studied in [17–20].

Figure 4(a) shows a stability diagram for the system of four coupled skew tent maps $f_{l,p}$ with the coefficients $l = 0,6$, $p = -10$. The region of strong synchronization (coherence) is shown in gray. Stability regions of two-cluster states $C_2^{(1)} = \{x_1 = x_2;\ y_1 = y_2\}$, $C_2^{(2)} = \{x_1 = y_1;\ x_2 = y_2\}$ and $C_2^{(3)} = \{x_1 = y_2;\ x_2 = y_1\}$ are hatched. The coherent state can lose its stability in three transverse directions, which correspond to the transition to a two-cluster state. Stable $C_2^{(1)}$ and $C_2^{(2)}$ clusters coexist in the doubly hatched region near the left lower corner of the coherence region.

Figure 4(b) shows another example; here $l = 0,85$, $p = -50$. Note that the stability region for the coherent state is wider in comparison with the previous case, and for the two-cluster states, the stability regions are smaller. Moreover, there is no coexistence of two-cluster states $C_2^{(1)}$ and $C_2^{(2)}$.

From the obtained stability diagrams we conclude that for small values of the coupling parameters ε and δ, in the blank regions in Fig. 4(a) and 4(b) there is no clustering in the system (3). Then, increasing and varying the parameters ε and δ, one or even two cluster states $C_2^{(1)}$ or

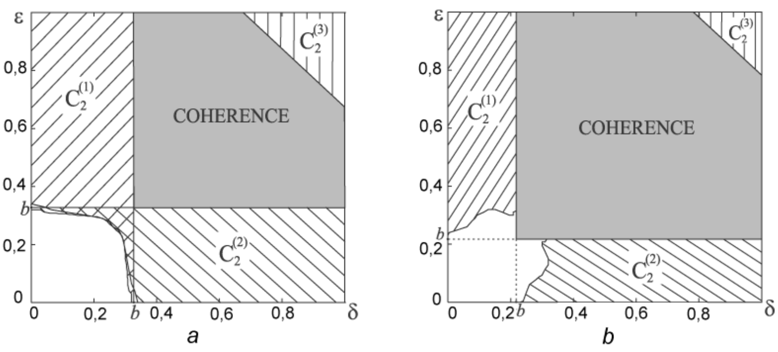

Fig. 4. (a) Stability diagram for the system (3) with $2N = 4, l = 0{,}6, p = -10$. The region of stability of the coherent state is shown in gray; the regions of stability of two-cluster states are denoted by $C_2^{(i)},\ i = 1, 2, 3$. There is a doubly hatched region of coexistence of $C_2^{(1)}$ and $C_2^{(2)}$. (b) A stability diagram for the system of four coupled maps with $l = 0,85,\ p = -50$.

$C_2^{(2)}$ can stabilize, followed by stabilization of the coherent state. Moreover, for some values of the coefficients l and p, the coherent state can be the first to stabilize as the coupling parameters grow.

4. Conclusions. To investigate stability properties of the chaotic synchronizing set $A_D \subset D_{2N}$ for $2N$-dimensional system (3) we found analytically parameter regions for strong and weak stability of A_D as well as both types of instability. Borders of stability regions are obtained in terms of the coefficients l and p of individual skew tent map $f_{l,p}$ and do not depend on the ensemble size $2N$.

Supposing the coefficients l and p to be fixed, we have delineated schematically stability regions in the (δ, ε)-parameter plane and got a sequence of 'embedded' triangles. In the case when the pair of coefficients (l, p) corresponds to a periodic window of the individual skew tent map $f_{l,p}$, the synchronizing chaotic set A_D is strongly stable in the unit square $[0, 1] \times [0, 1]$ of the coupling parameter plane.

The parameter b that defines borders of the strong stability region of A_D has been introduced, and a dependence of b on the coefficients l and p was analyzed.

As an example, results of numerical experiments for the system (3) with $2N = 4$ oscillators were shown. In the unit square $0 \le \varepsilon, \delta \le 1$ we have delineated regions of stability of strongly synchronized (coherent) and partially synchronized (clustered) states.

The type of the synchronization transition in the case of coupled piecewise linear maps, where for small values of the coupling parameters, as they increase, a stabilization of the two-cluster states $C_2^{(1)}$ or $C_2^{(2)}$ takes place, is essentially different from the case of the coupled smooth maps $f_a(x) = ax(1 - x)$ [7]. In the smooth case, with an increase of small values of the coupling parameters ε and δ, the two-cluster state $C_2^{(3)}$ is the first to stabilize, only after we get a stabilization of $C_2^{(1)}$ or $C_2^{(2)}$, followed by a transition to coherence.

Moreover, from the obtained stability diagrams it is obvious that parameter regions of stability for synchronized and clustered states are rather large, which makes it possible to provide an effective control of the system dynamics varying the coupling coefficients.

Acknowledgements. The author is grateful to Yu. L. Maistrenko and S. I. Popovych for a number of illuminating discussions.

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Received 20.02.2004