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ON MINIMALITY OF NONAUTONOMOUS DYNAMICAL SYSTEMS*

ПРО МІНІМАЛЬНІСТЬ НЕАВТОНОМНИХ ДИНАМІЧНИХ СИСТЕМ

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The minimality of a nonautonomous dynamical system given by a compact Hausdorff space X and a sequence of continuous selfmaps of X is studied. A sufficient condition for nonminimality of such a system is formulated. A special attention is paid to the particular case when X is a real compact interval I. A sequence of continuous selfmaps of I forming a minimal nonautonomous system may uniformly converge. For instance, the limit may be any topologically transitive map. But if all the maps in the sequence are surjective then the limit is necessarily monotone. An example is given when the limit is the identity. As an application, in a simple way we construct a triangular map in the square I^2 with the property that every point except of those in the leftmost fibre has an orbit whose ω-limit set coincides with the leftmost fibre.

Вивчається мінімальність неавтономної динамічної системи, що задається компактним хаусдорфовим простором X та послідовністю неперервних відображень на ньому. Сформульовано достатню умову для немінімальності таких систем. Особливу увагу приділено випадку, коли X є відрізком прямої I. Послідовність неперервних відображень на I, що формує мінімальну неавтономну динамічну систему, може рівномірно збігатись. Наприклад, границею може бути будь-яке транзитивне відображення. Але якщо всі відображення з цієї послідовності є сюр'єктивними, тоді границею є необхідно монотонне відображення. Наведено приклад, коли границею є тотожне відображення. Як деяку аплікацію наведено просту конструкцію трикутного відображення в квадраті I^2 з властивістю, що довільна точка, за винятком точок із крайнього лівого вертикального шару, має орбіту, ω-гранична множина якої збігається з цим шаром.

1. Introduction. Minimality is a central topic in topological dynamics (see, e.g., [1–3]). A dynamical system (X, f) where X is a topological space and $f : X \to X$ continuous, is called (topologically) *minimal* if there is no proper subset $M \subseteq X$ which is nonempty, closed and f-invariant

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(i.e., $f(M) \subseteq M$). In such a case we also say that the map f itself is minimal. Clearly, the system (X, f) is minimal if and only if the (forward) orbit of every point $x \in X$ is dense in X. Recently the authors proved (see [4]) that the minimal maps send open sets to sets with nonempty interiors (so called feebly open or almost open maps) and, moreover, they are almost one-to-one (a typical point has just one preimage). To show this, among others a simple but useful sufficient condition for nonminimality of a map (see Lemma 2.1 below) was used.

We will call a space X *minimal* if it admits a minimal map $f : X \to X$. An important and old question is which compact Hausdorff spaces (compact metric spaces, continua, etc.) admit minimal maps. This problem is investigated for instance in [5] and [6]. In particular, from [5] we know that if we are interested in whether a space admits a minimal homeomorphism or not and whether it admits a minimal nonivertible map or not, then all four possibilities can occur (examples are interval (no, no), circle (yes, no), torus (yes, yes), pinched torus (no, no)). In [6] it is proved that the only 2-manifolds (compact or not) which admit minimal maps are either finite unions of tori or finite unions of Klein bottles.

In this paper we will study a *nonautonomous* discrete dynamical system $(X; f_{1,\infty})$ given by a compact Hausdorff topological space X and a sequence $f_{1,\infty} := (f_i)_{i=1}^{\infty}$ of continuous selfmaps of X. The trajectory of a point x is defined to be the sequence $x, f_1(x), f_2(f_1(x)), \ldots$ and its orbit is the set of values of this sequence. The system is minimal if every orbit is dense in X. Easy examples (see, e.g., the sequence of constant maps from the example in Remark after Lemma 2.3) show that from minimality of such a system one cannot deduce the feeble openness of f_n or $f_n \circ \cdots \circ f_2 \circ f_1$. Neither one can deduce that these maps are almost one-to-one. This indicates that there is a larger variety of nonautonomous minimal systems than autonomous ones. (One of our aims is to illustrate this observation by showing less degenerate examples.)

The paper is organized as follows. We first generalize the above mentioned sufficient condition for nonminimality to the case of nonautonomous dynamical systems on a compact Hausdorff space (see Section 2). A special attention is paid to the particular case when X is a real compact interval I (see Section 3). We show that the interval admits a nonautonomous dynamical system given by a sequence of continuous *surjective* maps. This sequence may even uniformly converge. For instance, the limit may be any topologically transitive map and we discuss the question whether the minimality of the sequence of maps always implies the topological transitivity of its uniform limit. We show that, rather surprisingly, this is not necessarily the case. If all the maps in the sequence are surjective then the uniform limit of such a sequence is definitely *not* transitive — in fact, it is necessarily monotone (see Proposition 3.1). An example is given when the limit is the identity (see Theorem 3.1). As an application, we construct a triangular map in the square I^2 with the property that every point except of those in the leftmost fibre has an orbit whose ω-limit set coincides with the leftmost fibre. To show this property for our triangular map is much simpler than in [7] where this is proved for a map from [8].

2. Sufficient conditions for nonminimality of a system. Let X be a Hausdorff topological space and $f : X \to X$ continuous (written $f \in C(X)$). If $Y \subseteq X$ is nonempty, closed and f-invariant then Y is called a *minimal set* of the system $(X; f)$ if the system $(Y; f|_Y)$ is minimal.

Recall also that the system $(X; f)$ or the map f itself is called (topologically) *transitive* if for every pair of nonempty open sets U and V in X, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$. Clearly, minimality implies transitivity. If f is transitive then $f(X)$ is obviously dense in X and since we additionally assume X to be compact, $f(X)$ is also compact. Hence $f(X) = X$.

If X has an isolated point and f is transitive then X is just a periodic orbit of f. Hence, if X admits a transitive map then either it has no isolated point or otherwise it is finite. If X is a compact metric space without isolated points, then the above definition of transitivity is equivalent to the existence of a dense orbit (for a survey of results on transitivity see, e.g., [9]).

Lemma 2.1 [4]. *Let X be a compact Hausdorff space and $f \in C(X)$. Then the following two conditions are equivalent and each of them is sufficient for f not to be minimal:*
1) there is a closed set $A \neq X$ in X with $f(A) = f(X)$;
2) there is an open set $B \neq \emptyset$ in X with $f(B) \subseteq f(X \setminus B)$.

Nonautonomous dynamical systems (in fact their topological entropy) have been studied in [10] and [11]. Let us recall the definitions and some notations.

Let $f_{1,\infty} := (f_i)_{i=1}^{\infty}$ be a sequence of continuous selfmaps of X. For any positive integers i, n set $f_i^n = f_{i+(n-1)} \circ \dots \circ f_{i+1} \circ f_i$ and additionally $f_i^0 = \mathrm{id}_X$. We will call $(X; f_{1,\infty})$ a *nonautonomous discrete dynamical system*. The *trajectory* and the *orbit* of a point $x \in X$ will be the sequence $(f_1^n(x))_{n=0}^{\infty}$ and the set $\{f_1^n(x) : n = 0, 1, 2, \dots\}$, respectively. If k is a positive integer, the sequence $(f_1^n(x))_{n=0}^{k-1}$ is said to be a finite trajectory of x (with length k).

We define the minimality of a nonautonomous system as the density of all orbits. Again, this is equivalent to the nonexistence of any proper subset $M \subseteq X$ which is nonempty, closed and $f_{1,\infty}$-invariant (i.e., $f_1^n(M) \subseteq M$ for $n = 1, 2, \dots$). Note that for autonomous systems (when $f_n = f$ for every n) this agrees with the usual definition.

In the previous lemma we worked with compact Hausdorff spaces. In the proof of the next lemma we need that a sequence of points in the space have a convergent subsequence. Therefore we additionally assume metrizability (in fact to add 1st countability instead of metrizability would be sufficient, see [12, p. 229, 217]).

Recall also that for autonomous systems the next lemma is equivalent to the well known fact that f is minimal if and only if for any open set B there is a k such that every point visits the set B not later than in time k.

Lemma 2.2. *Let (X, ρ) be a compact metric space and let $(X; f_{1,\infty})$ be a nonautonomous dynamical system. Then the following are equivalent:*
1) $(X; f_{1,\infty})$ is not minimal;
2) there is a nonempty open set $B \subseteq X$ such that $(X; f_{1,\infty})$ has arbitrarily long finite trajectories disjoint with B.

Proof. 2) follows from 1) trivially. So let 2) hold. Then there is an increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ and a sequence of points $(x_k)_{k=1}^{\infty}$ such that for every k, the $f_{1,\infty}$-trajectory of x_k of length n_k is disjoint with B. Without loss of generality we may assume that the sequence x_k converges. Denote its limit by y. Obviously, $y \notin B$. Due to continuity, for any positive integer r, $f_1^r(y)$ is the limit of $f_1^r(x_k)$ when k tends to infinity. For all sufficiently large k we have $n_k > r$ and so $f_1^r(x_k) \notin B$. Hence $f_1^r(y) \notin B$. Thus the $f_{1,\infty}$-trajectory of y is disjoint with B and so $(X; f_{1,\infty})$ is not minimal.

Corollary 2.1. *Let (X, ρ) be a compact metric space with no isolated points and let $(X; f_{1,\infty})$ be a nonautonomous dynamical system. Suppose that there is a nonempty open set $B \subseteq X$ and $n_0 \in \mathbb{N}$ such that:*
1) $f_1^{n_0 - 1}$ is onto;
2) $(X; f_{n_0,\infty})$ has arbitrarily long finite trajectories disjoint with B.
Then $(X; f_{1,\infty})$ is not minimal.

Proof. By Lemma 2.2, there is a point $y \in X$ whose $f_{n_0,\infty}$-trajectory is not dense in X, i.e., does not intersect some nonempty open set in X (in fact the set B as it can be seen from the proof of Lemma 2.2). By 1) there is a point z with $f_1^{n_0-1}(z) = y$. Then the $f_{1,\infty}$-trajectory of z contains at most $n_0 - 1$ points from the set B and so it cannot be dense in B (note that X has no isolated point). Thus $(X; f_{1,\infty})$ is not minimal.

Remark 2.1. The assumption 1) in Corollary 2.1 cannot be removed. To see it, take $X = I = [0,1]$. For $n \geq 2$ let f_n be the standard tent map and let $f_1(x) = d$, $x \in I$, where d is a point whose trajectory under the tent map is dense in I. Then 2) is fulfilled with, say, $n_0 = 2$ and $B = \left(0, \frac{1}{2}\right)$ but $(X; f_{1,\infty})$ is minimal. (One can construct an analogous example with any other topologically transitive map in place of the tent map.)

Lemma 2.3. *Let (X, ρ) be a compact metric space with no isolated points and let $(X; f_{1,\infty})$ be a nonautonomous dynamical system. Suppose that there is a nonempty open set $B \subseteq X$ and $n_0 \in \mathbb{N}$ such that:*

1) $f_1^{n_0-1}$ as well as the maps $f_n, n \geq n_0$, are onto;

2) for every $n \geq n_0$, $f_n(B) \subseteq f_n(X \setminus B)$.

Then $(X; f_{1,\infty})$ is not minimal.

Proof. In view of Corollary 2.1 it is sufficient to find arbitrarily long finite trajectories of the system $(X; f_{n_0,\infty})$ disjoint with B. To this end, fix $k \in \mathbb{N}$ and $p \in X \setminus B$. For any $n \geq n_0$ we have $f_n(X \setminus B) = f_n(X) = X$ and so any point has an f_n-preimage in $X \setminus B$. Thus there are points $x_k, x_{k-1}, \ldots, x_0 \in X \setminus B$ such that $f_{n_0+k}(x_k) = p, f_{n_0+(k-1)}(x_{k-1}) = x_k, \ldots, f_{n_0}(x_0) = x_1$. Then $\{x_0, x_1, \ldots, x_k, p\}$ is an $f_{n_0,\infty}$-trajectory of x_0 of length $k+2$ disjoint with $X \setminus B$.

Remark 2.2. The assumption 1) in Lemma 2.3 cannot be removed. To see this, let $X = I = [0,1]$ and let $\{q_n : n \in \mathbb{N}\}$ be an enumeration of the rationals from $[0,1]$. Put $f_n(x) = q_n$ for all $x \in I$ and $n \in \mathbb{N}$ and take $B = \left(0, \frac{1}{2}\right)$. Then $(I; f_{1,\infty})$ is minimal though all the assumptions of Lemma 2.3 except for 1) are satisfied.

Lemma 2.4. *Let (X, ρ) be a compact metric space with no isolated points and let $(X; f_{1,\infty})$ be a nonautonomous dynamical system. Suppose that the sequence $(f_n)_{n=1}^{\infty}$ uniformly converges to a map f. If f is not onto then $(X; f_{1,\infty})$ is not minimal (even no $f_{1,\infty}$-trajectory is dense).*

Proof. Since f is not onto, there is a nonempty open set B with $\rho(B, f(X)) > 0$. Then for all sufficiently large n, $f_n(X) \cap B = \emptyset$. This implies that every trajectory of the system $(X; f_{1,\infty})$ has only finitely many points in B and so is not dense in X (note that X has no isolated points).

3. Minimality of nonautonomous systems on the interval and an application. In $C(I)$ where I is a real compact interval, say $I = [0,1]$, there are no minimal maps. Nevertheless, a sequence of maps $f_n \in C(I)$ may be minimal, see for instance the remark after Lemma 2.3. One can even find an example with surjective maps. Let $\{Q_n : n \in \mathbb{N}\}$ be an enumeration of open intervals with rational endpoints in $[0,1]$. For every n choose a point $q_n \in Q_n$. Let f_n be any continuous selfmap of I such that $f_n(I \setminus Q_n) = \{q_n\}$ and $f_n(Q_n) = I$. For every $x \in I$ and every $n \in \mathbb{N}$ we have that $x \in Q_n$ or $f_n(x) \in Q_n$. Hence the system $(I; f_{1,\infty})$ is minimal.

But what happens if f_n uniformly converges to a map f ? Since f cannot be minimal, does this mean that the sequence $(f_n)_{n=1}^{\infty}$ itself cannot be minimal ? The answer is negative as we saw in the remark after Corollary 2.1. But notice that in this example f_1 was not onto. If all the maps f_n are onto then still $(f_n)_{n=1}^{\infty}$ may be minimal and convergent (see Theorem 3.1 below) but the limit function must be monotone. This is rather paradoxical: a minimal sequence of onto maps (i.e., a complicated sequence in a sense) may uniformly converge but only to a monotone map (i.e., to a very simple map which is even far from being topologically transitive).

Proposition 3.1. *Let $(I; f_{1,\infty})$ be a minimal nonautonomous dynamical system such that the sequence $(f_n)_{n=1}^{\infty}$ uniformly converges to a map f and all the maps f_n are onto. Then f is (not necessarily strictly) monotone.*

Proof. Suppose on the contrary that f is not monotone. Then there are points $a < b < c$ in I with $f(a) < f(b) > f(c)$ or $f(a) > f(b) < f(c)$. In either case, one can find a nondegenerate open interval $B \subseteq (a, b)$ such that for all sufficiently large n, $f_n(B) \subseteq f_n(I \setminus B)$. Hence by Lemma 2.3 the system is not minimal, a contradiction.

In what follows, by a *u-homeomorphism* or a *d-homeomorphism* we will mean any increasing homeomorphism $f : I \to I$ such that for all $x \in (0,1)$, $f(x) > x$ or $f(x) < x$, respectively. It is not difficult to see that one can define a minimal sequence $(f_n)_{n=1}^{\infty}$ uniformly converging to the identity, of the form

$$f_1, U_1, D_1, \ldots, U_n, D_n, \ldots$$

where $f_1(x) = \dfrac{1}{2}$ and U_i or D_i is a block of u-homeomorphisms or d-homeomorphisms, respectively. On the other hand, $(f_n)_{n=1}^{\infty}$ cannot be minimal if f_n, $n \geq n_0$, are homeomorphisms and $f_1^{n_0-1}$ is onto (for there is a point x with $f_1^{n_0-1}(x) \in \{0,1\}$ and $g(\{0,1\}) = \{0,1\}$ whenever g is a homeomorphisms). Nevertheless, we have the following theorem.

Theorem 3.1. *There exists a minimal nonautonomous dynamical system $(I; f_{1,\infty})$ such that the sequence $(f_n)_{n=1}^{\infty}$ uniformly converges to the identity and all the maps f_n are onto. Moreover, all f_n can be chosen piecewise linear with nonzero slopes and at most three pieces of linearity, and even for every n the system $(I; f_{n,\infty})$ is minimal.*

Proof. First introduce some notation and terminology. For $\varepsilon > 0$ let $\mathcal{F}_\varepsilon$ denote the set of maps which are ε-close to the identity, onto, piecewise linear with nonzero slopes and at most three pieces of linearity.

If $S = (f_1, \ldots, f_n)$ is a finite sequence of selfmaps of I and $x \in I$, we will say that the trajectory of x under S is ε-dense in I if for every $y \in I$ there is $i \in \{0, 1, \ldots, n\}$ with $|f_1^i(x) - y| \leq \varepsilon$.

Finally, if f is a selfmap of I and A_i, B_i, $i = 1, \ldots, r$, are subsets of I with $f(A_i) = B_i$, $i = 1, \ldots, r$, we will write $f\langle A_1, \ldots, A_r\rangle = \langle B_1, \ldots, B_r\rangle$.

Now we are ready to prove the theorem. Suppose for a moment that for any small $\varepsilon > 0$ we are able to define a finite sequence S_ε of maps from $\mathcal{F}_\varepsilon$ such that the trajectory under S_ε of any point from I is ε-dense in I. Then, whenever $\varepsilon(n) \to 0$, the infinite sequence

$$S_{\varepsilon(1)}, S_{\varepsilon(2)}, \ldots, S_{\varepsilon(n)}, \ldots$$

fulfills all the properties required in the theorem.

Thus it remains only to show how one can construct S_ε. So fix a small $\varepsilon > 0$ $\left(\text{say, } \varepsilon < \dfrac{1}{4}\right)$ and put $A = \left[0, \dfrac{\varepsilon}{2}\right]$, $B = \left[\dfrac{\varepsilon}{2}, 1 - \dfrac{\varepsilon}{2}\right]$, $C = \left[1 - \dfrac{\varepsilon}{2}, 1\right]$, $A^+ = \left[\dfrac{\varepsilon}{2}, \varepsilon\right]$ and $C^- = \left[1 - \varepsilon, 1 - \dfrac{\varepsilon}{2}\right]$. It is clear that there are increasing homeomorphisms $f_\varepsilon, g_\varepsilon, h_\varepsilon \in \mathcal{F}_\varepsilon$ and positive integers $n_1(\varepsilon)$, $n_2(\varepsilon), n_3(\varepsilon)$ such that

$$(f_\varepsilon)^{n_1(\varepsilon)}\langle A, B, C\rangle = \langle A, A^+, I \setminus (A \cup A^+)\rangle \,,$$

$$(g_\varepsilon)^{n_2(\varepsilon)}\langle A, A^+, I \setminus (A \cup A^+)\rangle = \langle I \setminus (C^- \cup C), C^-, C\rangle \,,$$

$$(h_\varepsilon)^{n_3(\varepsilon)}\langle I \setminus (C^- \cup C), C^-, C\rangle = \langle A, B, C\rangle$$

(f_ε is the identity on A and a d-homeomorphism on $I \setminus \mathrm{int}(A)$, g_ε is a u-homeomorphism and h_ε is the identity on C and a d-homeomorphism on $I \setminus \mathrm{int}(C)$). Consider the finite sequence

$$T_\varepsilon = \underbrace{f_\varepsilon, \ldots, f_\varepsilon}_{n_1(\varepsilon)}, \underbrace{g_\varepsilon, \ldots, g_\varepsilon}_{n_2(\varepsilon)}, \underbrace{h_\varepsilon, \ldots, h_\varepsilon}_{n_3(\varepsilon)}.$$

It is easy to check that the trajectory under T_ε of any point $x \in I$ ends in the interval A, B or C in which it started and, moreover, if $x \in B$ then the trajectory of x is ε-dense in I. Nevertheless, the T_ε-trajectory of a point from $A \cup C$ may not be ε-dense in I. Therefore take a piecewise linear map φ_ε having three pieces of linearity such that $\varphi_\varepsilon(0) = \varepsilon$, $\varphi_\varepsilon(\varepsilon) = 0$, $\varphi_\varepsilon(1-\varepsilon) = 1$, $\varphi_\varepsilon(1) = 1 - \varepsilon$ (notice that $\varphi_\varepsilon \in \mathcal{F}_\varepsilon$). Since $\varphi_\varepsilon(A \cup C) \subseteq B$, the trajectory of any point from $A \cup C$ under the finite sequence $\varphi_\varepsilon, T_\varepsilon$ is ε-dense in I. Hence the sequence

$$S_\varepsilon = T_\varepsilon, \varphi_\varepsilon, T_\varepsilon$$

has all the desired properties.

The theorem is proved.

A continuous map F of the space $X \times Y$ into itself is called *skew product* or *triangular* (see e.g. [8, 13]) if it is of the form $F(x, y) = (f(x), g(x, y))$. The continuous map $f : X \to X$ is called the *basis map* of F. Instead of $g(x, y)$ we also write $g_x(y)$. Then we shortly write $F = (f, g_x)$. Here g_x, $x \in X$, is a family of continuous maps $Y \to Y$ depending continuously on x. The maps g_x are called *fibre maps*, the sets $Y_x = \{x\} \times Y$ are called *fibres* (Y_x is the fibre over x). Since $\pi_X \circ F = f \circ \pi_X$, where $\pi_X : (x, y) \mapsto x$ is the X-projection, the map F is an extension of f. The Y-projection $\pi_Y : (x, y) \mapsto y$ is also called the 2*nd projection*. We denote by $C_\triangle(X \times Y)$ the set of all continuous triangular maps from $X \times Y$ into itself.

Consider a map $F \in C_\triangle(I^2)$. It can happen as well that an orbit of F is not dense in $I \times I$ (it may even be nowhere dense) and yet its 2nd projection *is* dense in I. It is not possible that *all* orbits of F have dense 2nd projections because the square has the fixed point property. Nevertheless, as an application of Theorem 3.1 we prove the following theorem.

Theorem 3.2. *There is a triangular map $F = (f, g_x) \in C_\triangle(I^2)$ such that:*

1) all points of the form $(0, y)$ are fixed;

2) $\lim_{n\to\infty} f^n(x) = 0$ for every x;

3) every point from $I \times I$ which is not of the form $(0, y)$ has the orbit whose 2nd projection is dense in I (hence, in view of (2), its ω-limit set equals $\{0\} \times I$);

4) the map F has zero topological entropy.

Proof. First we define the basis map f on $I = [0, 1]$. Put $f(0) = 0$ and $f(1) = 1/2$. Further, for any $n = 1, 2, \ldots$ let $f(1/2^n) = f(1/2^n + 1/2^{n+1}) = 1/2^{n+1}$. To finish the definition of

f let f be linear on each closed interval J having the property that f has been defined at the endpoints of J but has not been defined in the interior points of J. Notice that for every $x \in I$, $\lim_{n\to\infty} f^n(x) = 0$.

Now, for each $x \in I$, we need to define the fibre map $g_x(y)$, $y \in I$. We use the system $(I; f_{1,\infty})$ from the previous theorem. Put $g_0(y) = y$ and for every $n \in \mathbb{N} \cup \{0\}$ let $g_{1/2^n} = f_{n+1}$. So far we have defined a map $F_0 \in C_\triangle(\{1/2^n : n = 0, 1, \dots\} \times I)$. Now use Extension lemma from [14] (cf. [15]) to extend F_0 to a map $F \in C_\triangle(I^2)$.

It remains to show that F satisfies 3) and 4). The map f is piecewise linear with infinitely many pieces, each of them having the slope zero or one. For every $x \in I \setminus \{0\}$ there is a nonnegative integer k (depending on x) such that $f^k(x)$ belongs to the set $\{1/2^n : n = 0, 1, \dots\}$. From this and the fact that the system $(I; f_{k,\infty})$ is minimal for every k, it easily follows that every point from $I \times I$ which is not of the form $(0, y)$ has the orbit whose 2nd projection is dense in I. Since the set $\Omega(F)$ of nonwandering points of F is the fibre I_0 where F is the identity, for the entropy $h(F)$ of F we get (see [16]) $h(F) = h(F_{|\Omega(F)}) = 0$.

The theorem is proved.

Let us remark that the existence of such maps as in Theorem 3.2 has been established in [7] (by using a map from [8]), but the proof for our map is simpler.

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