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SWEPT VOLUME DINAMICAL SYSTEMS AND THEIR KINETIC MODELS

ДИНАМІЧНІ СИСТЕМИ, ПОРОДЖЕНІ ОРБІТАМИ ТОЧОК ДЕЯКОЇ ЧАСТИНИ ФАЗОВОГО ПРОСТОРУ, ТА ЇХНІ КІНЕТИЧНІ МОДЕЛІ

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We study swept-volume dynamical systems for which several hydrodynamical models are formulated. The properties of those hydrodynamical models are studied by means of the algebraic Kostant – Symes technique. A differential-geometric description of swept volumes dynamical systems are devised based on the Cartan Movina frame approach.

Для динамічних систем, породжених орбітами точок деякої частини фазового простору, наведено декілька гідродинамічних моделей, властивості яких досліджуються за допомогою методу Костанта – Сімза. В рамках підходу Картана дано диференціально-геометричний опис вказаних систем.

1. It is well known [1, 2] that motion planning, numerically controlled machining and robotics are just a few of the many areas of manufacturing automation in which the analysis and representation of swept volumes plays a crucial role. Swept volume modeling is also an important part of task-oriented robot motion planning. In these problems a robot carrying some objects is moved through a domain in space containing obstacles (phase space constraints) for the purpose of reaching a desired goal. A typical motion planning problem consists of moving a collection of solid objects around obstacles from an initial to a final position. This may include, in particular, solving collision detection problems and obtaining optimal solution paths.

When an object undergoes a rigid motion, the totality of points through which it passes constitutes a region in space called the swept volume generated by the (rigid) sweep. In order to have a mathematical framework for these sweeps in real 3-space $\mathbf{R}^3$, we introduce some definitions that will prove useful in the sequel.

An Euclidean motion in $(n+1)$-dimensional Euclidean space E^{n+1} is a mapping $\sigma : E^{n+1} \to \to E^{n+1}$ such that $\|\sigma(x) - \sigma(y)\| = \|x - y\|$ for all $x, y \in E^{n+1}$, where $\|\cdot\|$ is the standard Euclidean norm in E^{n+1}. The simplest cases of these sweeps or motions are translations and rotations of the form

$$\alpha(x) := x + a, \quad \beta(x) := gx, \tag{1}$$

where $x, a \in E^{n+1}$ $(= R^{n+1})$ and $g \in O(n+1)$. The following result characterizes Euclidean motions.

Theorem 1. *Let σ be an Euclidean motion in E^{n+1}. Then there exist a unique orthogonal mapping $\beta \in O(n+1)$ and a unique translation α such that $\sigma = \alpha \circ \beta$.*

Sketch of Proof. Let $a := \sigma(0)$ and α be as in (1). Define $\beta := \alpha^{-1} \circ \sigma$. It is easy to show that β is orthogonal. Indeed, β is a motion since

$$\|\beta(x) - \beta(y)\| = \|\sigma(x) - \sigma(y)\| = \|x - y\|, \tag{2}$$

and $\|\beta\| = 1$ because

$$\|\beta\| = \|\beta(x) - \beta(0)\| = \|x - 0\| = \|x\|$$

for all $x, y \in E^{n+1}$. We need to show that β is linear. Note first that the standard inner product $\langle \cdot, \cdot \rangle$ in E^{n+1} is preserved by the mapping $\beta : E^{n+1} \to E^{n+1}$, viz.

$$\langle \beta(x), \beta(y) \rangle = \frac{1}{2}\left(\|\beta(x)\|^2 + \|\beta(y)\|^2 - \|\beta(x) - \beta(y)\|^2\right) =$$

$$= \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x - y\|^2\right) = \langle x, y \rangle \tag{3}$$

for all $x, y \in E^{n+1}$. To prove the linearity of β, it suffices to show that

$$\beta(k_1 x + k_2 y) = k_1 \beta(x) + k_2 \beta(y), \tag{4}$$

or equivalently that the form

$$E(x, y) := \beta(k_1 x + k_2 y) - k_1 \beta(x) - k_2 \beta(y) \equiv 0 \tag{5}$$

for all $x, y \in E^{n+1}, k_1, k_2 \in R$. To prove (5) it is necessary that $E(x, y)$ in (5) be orthogonal to each vector of some basis in E^{n+1}. If $\{e_1, ..., e_{n+1}\}$ is an orthonormal basis for E^{n+1}, obviously so is $\{\beta(e_1), ..., \beta(e_{n+1})\}$ since β preserves the scalar product in view of (3). Therefore we find that for all $1 \le j \le n+1$,

$$\langle E(x,y), \beta(e_j)\rangle =$$

$$= \langle \beta(k_1 x + k_2 y), \beta(e_j)\rangle - k_1 \langle \beta(x), e_j\rangle - k_2 \langle \beta(x), \beta(e_j)\rangle =$$

$$= k_1 \langle x, e_j\rangle + k_2 \langle y, e_j\rangle - k_1 \langle x, e_j\rangle - k_2 \langle x, e_j\rangle = 0. \tag{6}$$

This proves the linearity of β, and the uniqueness of α and β are easily verified.

The following result is a direct consequence of the above theorem:

Corollary 1. *Let σ be an Euclidean motion in E^{n+1}. Then:*
(i) σ is a smooth mapping;
(ii) σ maps E^{n+1} orthogonally onto itself; and
(iii) $\langle \sigma'(x), \sigma'(y)\rangle = \langle x, y\rangle$ for all $x, y \in E_p^{n+1} \cong T_p(E^{n+1}), p \in E^{n+1}$.
Indeed, for each point $(p, x) \in E_p^{n+1}, p \in E^{n+1}$ we have

$$\sigma'(p)x = \frac{d}{dt}\sigma(p+tx)\mid_{t=0} = \frac{d}{dt}\alpha \circ \beta(p+tx)\mid_{t=0} =$$

$$= \frac{d}{dt}\alpha(\beta(p) + t\beta(x))\mid_{t=0} = \frac{d}{dt}(\beta(p) + t\beta(x) + \sigma(0))\mid_{t=0} = \beta(x). \tag{7}$$

Whence for $(p, x), (p, y) \in E_p^{n+1}$, we obtain

$$\left\langle \sigma'(p)x, \sigma'(p)y\right\rangle = \langle \beta(x), \beta(y)\rangle = \langle x, y\rangle, \tag{8}$$

from which the required properties follow.

2. Let us consider a simply-connected solid body V embedded in E^3 having boundary surface $S = \partial V$ parametrized as follows:

$$S = \bigcup_{\tau \in [0,h]} \left\{ x(s,\tau) \in \mathbf{R}^3 : x(s+2\pi, \tau) = x(s,\tau), s \in \frac{\mathbf{R}}{2\pi \mathbf{Z}} \right\}. \tag{9}$$

Here $\tau \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}}$ is the usual parametrization via the identity $\langle dx, dx\rangle = ds^2$ of curve $x(\cdot,\tau), \tau \in$ $\in [0,h]$, obtained by cutting V straight across its diameter through a point $\tau \in [0,h]$. This means that the diameter of V is parametrized by τ and the set of curves $\{x(\cdot,\tau) : \tau \in [0,h]\}$ covers S in a unique fashion.

To proceed further in our description of the motion of a solid body V in E^3, we reformulate it in terms of manifolds swept out by the surface S in a time interval $[0, t_0]$. We denote these surfaces by $S_{t_0}(\tau)$; they can be represented by

$$S_{t_0}(\tau) := \bigcup_{t \in [0,t_0]} \{x(t,\tau)\}. \tag{10}$$

Therefore, the swept volume manifold $S_{t_0}(V)$ of the solid body defined in [2] can be written as

$$S_{t_0}(V) = \bigcup_{\tau \in [0,h]} S_{t_0}(\tau). \tag{11}$$

It is obvious that $S_{t_0}(V)$ is equal to the compact three-dimensional submanifold with boundary comprised of points swept by the set of curves $\{x(\cdot,\tau)\} \subset S$ corresponding to the diameter points $\tau \in [0,h]$. This leads naturally to the problem of constructing special dynamical systems – called swept volume dynamical systems – intimately associated with the Euclidean motion of a solid body in space and studying their differential-geometric and differential-topological properties [1] which are useful for applications in manufacturing automation.

Let us assume that a sweep of a solid body V in 3-space is generated by a family of Euclidean motions $\sigma(t), t \in [0,t_0]$, giving rise to a swept volume such that each of the curves $x(\cdot,\tau)$ maintains its planarity and arc-length for all $t \in [0,t_0]$. This means, in particular, that the Gaussian curvature of each curve $x(\cdot,\tau), \tau \in [0,h]$, is time independent while its torsion ξ is zero for all $t \in [0,h]$. The invariance of planarity of the family of Euclidean motions $\sigma(t), t \in [0,t_0]$, completely characterizes the motion of V in E^3. To the above properties one needs only to add length invariance, which can be expressed as

$$\left\langle \frac{dx}{ds}, K'(t,x)\frac{dx}{ds} \right\rangle = 0 \tag{12}$$

for all $t \in [0,t_0]$. Here we have postulated the evolution of curves $x(\cdot,\tau,t)$ due to the Euclidean motion as follows:

$$\frac{dx}{dt} = K(t,x), \tag{13}$$

where $K(t,\cdot) : E^3 \to T(E^3)$ is a parametric family of vector fields on E^3. Vector fields satisfying (12) are called Killing vector fields. A rather complete description of such fields associated with infinitesimal rigid body motions can be found in [3, 4]. Our next goal is to obtain an analogous theory for swept volumes using modern differential-geometric and algebraic-topological tools.

Now suppose we are given a swept volume manifold $S_{t_0}(V)$ as defined in (11). Over this manifold we can define a connection Γ [5] together with the following representation of parallel transport with respect to local spatial parameters $\tau \in [0,h], s \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}}$, and the temporal parameter $t \in [0,t_0]$ via the covariant derivative

$$\nabla_{\frac{\partial}{\partial y^j}} f := \frac{\partial f}{\partial y^j} + \Gamma_j f, \tag{14}$$

where $y = (y^1,y^2,y^3)^T := (s,\tau,t)^T \in S_{t_0}(V)$ and $\Gamma_j, 1 \le j \le 3$, are Christoffel matrices acting in the adjoint vector bundle over the swept volume manifold. The Christoffel matrices can be determined uniquely by requiring that $\nabla_{\frac{\partial}{\partial y^j}} f^* = 0$ for all horizontal [4, 5] vector fields $f^* \in$ $\in T(S_{t_0}(V))$. This means that parallel transport along the manifold $S_{t_0}(V)$ must be generated by some Euclidean motion $\sigma(t) : E^3 \to E^3, t \in [0,t_0]$. We first consider Cartan's main structure equations in the differential-geometric setting [6]:

$$d\theta = -\frac{1}{2}[\omega,\theta] + \Theta, \quad d\omega = -\frac{1}{2}[\omega,\omega] + \Omega, \tag{15}$$

where $\omega : P\left(S_{t_0}(V); GL(3;\mathbf{R})\right) \to gl(3;\mathbf{R})$ is the connection form,

$\theta : P\left(S_{t_0}(V); GL(3;\mathbf{R})\right) \to E^3$

is the canonical affine form, Ω is the corresponding curvature form and Θ is the torsion form, all on the principal fiber bundle $P\left(S_{t_0}(V); GL(3;\mathbf{R})\right)$ of frames over the manifold $S_{t_0}(V)$. In component form the structure equations (15) are as follows:

$$d\theta^j = -\omega^i_j \wedge \theta^j + \Theta^j, \quad d\omega^i = -\omega^i_k \wedge \omega^k_j + \Omega^i_j, \tag{16}$$

where

$$\theta = \sum_{i=1}^{3} \theta^i e_i, \;\; \omega = \sum_{i,j=1}^{3} \omega^i_j A^j_i, \;\; \Theta = \sum_{i=1}^{3} \Theta^i e_i, \;\; \Omega = \sum_{i,j=1}^{3} \Omega^i_j A^j_i, \;\; \{e_i : 1 \le i \le 3\}$$

is a basis for E^3 and $\{A^i_j \in gl(3;\mathbf{R}) : 1 \le i, j \le 3\}$ are in the Lie algebra $gl(3;\mathbf{R})$ and satisfy the condition $A^i_j e_k = e_j \delta^i_k$ for all $i, j, k = 1, 2, 3$. The structure equations (16) completely describe the swept volume dynamical system generated by a rigid sweep $\sigma(t) : E^3 \to E^3$ of a solid body in 3-space. The motion $\sigma(t)$ considered as an affine motion in $\mathbf{R}^3$ must satisfy the main defining conditions on the canonical and connection one-forms:

$$Ra^*\omega = Ad_{a^{-1}}\omega, \quad Ra^*\theta = a^{-1}\theta$$

for all $a \in GL(3;\mathbf{R})$. These conditions are obviously satisfied if the following canonical conditions hold:

$$\theta = X^{-1}dy, \quad \omega = X^{-1}\left(dX + \langle dy, \Gamma(y)\rangle X\right), \tag{17}$$

where $y = (s, \tau, t)^T \in S_{t_0}(V)$ and $X = [X_1, X_2, X_3] \in GL(3;\mathbf{R})$ is an arbitrary basis for the tangent space $T_y(S_{t_0})$. To determine the Riemannian connection matrix $\Gamma(y)$, $y \in S_{t_0}$, we write the first fundamental form as follows:

$$dl^2 := \langle dx, dx\rangle = \sum_{i,j=1}^{3} g_{ij}(y) dy^i dy^j. \tag{18}$$

If the embedding $x : S_{t_0}(V) \to E^3$ is generated by a sweep, the above Riemannian connection matrices must have the form

$$\Gamma^j_{k,i} = \frac{1}{2}\sum_{s=1}^{3} g^{is}\left(\frac{\partial g_{is}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^i} - \frac{\partial g_{ki}}{\partial y^s}\right), \tag{19}$$

where $\sum\limits_{s=1}^{3} g^{is} g_{sj} = \delta^j_i$, $i, j, k = 1, 2, 3$; and as is well known [6], element (19) is uniquely determined by the condition $\nabla_{\frac{\partial}{\partial y^j}} g_{ks} = 0$ for all $1 \le j, k, s \le 3$. From (18) we see that the above swept volume manifold (with boundary) is not Euclidean but actually a Riemannian space with nontrivial curvature and torsion. Thus we have arrived at the following important classification problem: to describe an effective differential-geometric procedure for determining the manifolds with boundary that are generated by a Euclidean sweep of a solid object. We can simplify this problem by employing the geometric theory of Cartan. For the case under consideration we assume that a Lie group G acts on $S_{t_0}(V)$ in a manner described by

$$dy^j + \sum_{i=1}^{n} \xi^j_i(y)\bar{\omega}^i(a, da) = 0, \tag{20}$$

where $y \in S_{t_0}(V)$, $\bar{\omega}^i(a, da), 1 \leq i \leq n = \dim G$, are the Maurer – Cartan left-invariant forms of the Lie group G, $a \in G$ is an arbitrary element and the $\xi^i_j(y)$ are characteristic functions on the manifold $S_{t_0}(V)$. The following theorem of Cartan is useful in describing a geometric object that is invariant with respect to the group action $G \times S_{t_0}(V) \to S_{t_0}(V)$.

Theorem 2. *The differential system (20), with characteristic left-invariant one-forms* $\bar{\omega}^i(a, da)$, $1 \leq i \leq n = \dim G$, *on the Lie group* G, *is tantamount to invariance of the group action on* $S_{t_0}(V)$ *if and only if the following conditions hold:*

(i) The coefficients ξ^i_j *are analytic functions of* $y \in S_{t_0}(V)$.

(ii) The system (20) is completely integrable in the Frobenius – Cartan sense.

We intend to investigate the above geometric aspects of the problem more thoroughly in a paper that is now in preparation. Here we are just going to formulate several hydrodynamic models for swept volume dynamical systems in E^3 having very rich symmetry groups and discuss some of their interesting and useful properties.

3. If $S_{t_0}(V)$ is a swept volume manifold generated by a Euclidean motion $\sigma(t)$, $t \in [0, t_0]$, then there exists a set of tangent vector fields

$$\frac{ds}{d\alpha} = u(s, \tau, t), \qquad \frac{d\tau}{d\alpha} = v(s, \tau, t), \qquad \frac{dt}{d\alpha} = w(s, \tau, t), \tag{21}$$

where $(s, \tau, t) \in S_{t_0}(V)$, $\alpha \in \mathbf{R}$ is an evolution parameter and (u, v, w) is a smooth vector field on $S_{t_0}(V)$. To more effectively describe the vector field (21), let us assume that the surface $S = \partial V$ of the solid body satisfies the invariant equation

$$\bar{\gamma}(\bar{x}) = 0, \tag{22}$$

where $\bar{\gamma} : \mathbf{R}^3 \to \mathbf{R}$ is a smooth function. This means that (22) is identically satisfied for the parametrized surface $\bar{x} = \bar{x}(s, \tau)$ for all $\tau \in [0, h]$ and $s \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}}$:

$$\bar{\gamma}\left(\bar{x}(s, \tau)\right) \equiv 0. \tag{23}$$

When $t \neq 0$, Theorem 1 implies that the Euclidean motion can be written in the form

$$x(s, \tau, t) = \xi(t) + a(t)\bar{x}(s, \tau) \tag{24}$$

for all $(s, \tau, t)^T \in S_{t_0}(V)$.

Solving (24) for $\bar{x}$ and substituting this in (23), we find that

$$\bar{\gamma}\left(a^*(t)(x - \xi(t)\right) := \gamma(t, x) = 0 \tag{25}$$

for $t \in [0, h]$ and $x \in S_{t_0}(V)$. Recalling now the general form (21) of a vector field on the manifold $S_{t_0}(V)$, it follows directly from (25) that

$$w\frac{\partial \gamma}{\partial t} + \left\langle \mathrm{grad}\gamma, x' \cdot (u, v, w)^T \right\rangle = 0, \tag{26}$$

where the prime denotes the Fréchet derivative of the mapping (24). Thus (26) gives a necessary condition for the set of vector fields (21) to belong to $T\left(S_{t_0}(V)\right)$.

If (21) satisfies (26) for the parameter $\alpha := t \in [0, t_0]$, then we obtain

$$\frac{ds}{dt} = u(s, \tau, t), \quad \frac{d\tau}{dt} = v(s, \tau, t), \tag{27}$$

$w(s, \tau, t) \equiv 1$ for all $(s, \tau, t)^T \in S_{t_0}(V)$. Using once again the representation (24), a simple but tedious calculation shows that equations (27) admit the prolongation

$$\begin{aligned} \frac{Du}{Dt} &= -\frac{\partial p}{\partial s}, \\ \frac{Dv}{Dt} &= -\frac{\partial p}{\partial \tau}, \end{aligned} \tag{28}$$

where $\frac{D}{Dt}$ is the Eulerian total or material derivative [7] and $p(s, \tau, t) := q(s, t) + r(\tau, t)$. Moreover, if we use the equation $\frac{Dv}{Dt} = -\frac{\partial r}{\partial \tau}$, the dynamical system (28) takes the form of the well known Navier – Stokes equations for the virtual flow of an ideal incompressible two-dimensional fluid under external pressure $p(s, \tau)$, namely

$$\begin{aligned} \frac{Du}{Dt} &= -\rho^{-1}\frac{\partial p}{\partial s}, \\ \frac{Dv}{Dt} &= -\rho^{-1}\frac{\partial p}{\partial \tau}, \\ \frac{Dp}{Dt} &= -\rho\left(\frac{\partial u}{\partial s} + \frac{\partial v}{\partial \tau}\right), \end{aligned} \tag{29}$$

where $(u, v)^T$ is the velocity vector in the s, τ-plane and $\rho > 0$ is the density of the liquid.

As the liquid is incompressible, i.e., $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{u\partial \rho}{\partial s} + \frac{v\partial \rho}{\partial \tau} \equiv 0$, we easily establish the condition $\frac{\partial u}{\partial s} + \frac{\partial v}{\partial \tau} = 0$ for all $s \in \frac{\mathbf{R}}{2\pi\mathbf{Z}}$ and $\tau \in [0, h]$. We may assume that the density is normalized to unity, i.e., $\rho \equiv 1$.

Consider the Navier – Stokes equations (29) with a free surface given by the equation $\tau = h(s, t)$, where $h(\cdot, t)$, $t \in [0, t_0]$, is the height of the fluid above the bottom (the s-axis) at time t. Then (29) reduces to the system

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dq}{dt} \\ \frac{dh}{dt} \end{pmatrix} = K[u, q, h] := \begin{pmatrix} -uu_s - u_\tau \int\limits_0^\tau u_s d\tau - q_s \\ -h_s \\ -\frac{\partial}{\partial s}\int\limits_0^h u d\tau \end{pmatrix}, \tag{30}$$

which is similar to formula (109) in [8] subject to the condition $u_\tau = 0$. Here we have also assumed that $\frac{Dh}{Dt} = v \mid_{\tau=h}$, which ensures that the virtual fluid does not pass through the free surface $\tau = h(s, t)$, and $\frac{dq}{dt} = -\frac{\partial h}{\partial s}$, which stems from a wind pressure of unity along the s-axis.

The system (30) generates a nonlinear integro-differential dynamical system on an infinite-dimensional functional manifold $M_{(u,q,h)} \subset C^{\infty}(\mathbf{R}^2; \mathbf{R} \times \mathbf{R}_+^2)$. In order to better understand this system, we will further investigate its Lax integrability, i.e., we study the existence of an infinite hierarchy of involutive conservation laws (with respect to some Poisson bracket) and a special operator representation of Lax type. In addition, we shall show that (30) has a natural connection with the nonlinear kinetic Boltzmann – Vlasov equation for a one-dimensional particle flow with a pointwise interaction potential between particles. This property of (30) enables us to establish a physical analogy between turbulence in kinetic multiparticle systems connected with stochastization of particle trajectories and instability and shocks in the flow of an ideal incompressible fluid flowing over a horizontal bottom and having a free boundary.

4. The Boltzmann – Vlasov equation can be obtained from (30) by use of the representation

$$\tau = \int\limits_{-\infty}^{u(s,\tau)} dp f(s,p;t),$$

where $\tau \in [0,h]$ and $f \in C^2(\mathbf{R}^2; \mathbf{R}_+)$ is the Boltzmann distribution function for the kinetics of a one-dimensional system of particles. The equation of the free boundary in (30) is determined by the compatibility condition for the distribution function:

$$h(s) = \int\limits_{-\infty}^{u(s,h)} dp f(s,p;t), \quad s \in \frac{\mathbf{R}}{2\pi\mathbf{Z}}.$$

Note also that the above transformation of the dynamical system (30) is canonical, i.e., the Boltzmann – Vlasov equation obtained is Hamiltonian and has a special symplectic structure on the functional manifold $M_{(f)} \subset C^2(\mathbf{R}^2; \mathbf{R}_+)$, and the same is true of (30).

In order to better understand the dynamical system (30), we introduce the moment functionals

$$a_n(s) := \int\limits_0^{h(s)} d\tau u^n(s,\tau),$$

for all $s \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}}, n \in \mathbf{Z}_+$.

Then by direct calculation we find that (30) is equivalent to the following infinite-dimensional system of moment equations on the functional manifold

$$M_{(\mathbf{Z}_+)} := \left\{ a_n \in C^2(\mathbf{R}; \mathbf{R}) : n \in \mathbf{Z}_+, \sup_n n^k \, |a_n| < \infty, \ k \in \mathbf{Z}_+ \right\} : \tag{31}$$

$$\begin{pmatrix} \dfrac{da_n}{dt} \\ \dfrac{dq}{dt} \end{pmatrix} = K[a,q] := \begin{pmatrix} -na_{n-1}q_x - a_{n+1,x} \\ -a_{0,x} \end{pmatrix}.$$

We first establish the complete integrability of the dynamical system (31) on $M_{(\mathbf{Z}_+)}$. For this purpose we consider the Lie algebra G_0 of symbols

$$l(\xi) := \sum_{j \gg \infty} a_j(s)\xi^{-(j+1)},$$

where $\{a_j(s)\} \in M_{(\mathbf{Z}_+)}$, with bracket defined by the formula [8]:

$$[l_1(\xi), l_2(\xi)]_0 = \frac{\partial l_1}{\partial \xi}\frac{\partial l_2}{\partial x} - \frac{\partial l_2}{\partial \xi}\frac{\partial l_1}{\partial x}.$$

As shown in [9], the bracket $[\cdot,\cdot]_0$ is a natural hydrodynamic limit of the standard bracket $[\cdot,\cdot]$ on the Lie algebra G (see formula (56) in [8]) of symbols of pseudodifferential operators on R.

The Lie algebra G_0 admits a natural direct sum decomposition: $G_0 = G_{0+} \oplus G_{0-}$. Moreover, the identifications $G_{0+}^* \cong G_{0-}$, $G_{0-}^* \cong G_{0+}$ hold for the space G_0^* dual to G_0 with respect to the standard invariant inner product having the form $(l_1, l_2) := Tr\,(l_1 \circ l_2)$, $l_1, l_2, \in G_0$, where

$$\operatorname{Tr} l(\xi) := \int\limits_{\mathbf{R}} ds \operatorname{res} l(\xi), \quad l \in \mathcal{G}_0.$$

Let the gradient $\tau\gamma(l)G_0$ for given $\gamma \in D(G_0^*)$ and all $m \in G_0^*$ is defined as

$$(\Delta\gamma(l), m) := \frac{d}{d\varepsilon}\gamma(l + \varepsilon m)|_{\varepsilon=0}$$

and let $R = P_+ - P_-$, where P_+, P_- are projection operators of G_0 on G_{0+}, G_{0-} respectively. Consider the vector field K on G_0^* defined by

$$\frac{dl(\xi)}{dt} = K[l(\xi)] := ad^*_{\mathcal{R}\nabla\gamma(l)} l(\xi), \tag{32}$$

which is a coadjoint action of $R\nabla\gamma(l) \in G_0$ on G_0^* , where $\gamma \in D(G_0^*)$ a Casimir functional. The isomorphism $G_0^* \cong G_0$ obtained from the inner product on G_0 implies that (31) can be represented on G_0^* in the form

$$\frac{dl}{dt} = K[l] := [l, \mathcal{R}\nabla\gamma(l)]. \tag{33}$$

Let us show that (33) is Hamiltonian with respect to the standard symplectic Lie – Poisson structure on G_0^*. Indeed, the Lie – Poisson bracket

$$\{\gamma, \mu\}_{\mathcal{L}}(l) := (l, [\nabla\gamma(l), \nabla\mu(l)]_0)$$

is defined naturally on G_0^*. Then it clearly follows from the properties of the scalar product $(\cdot,\cdot)$ that (33) is equivalent to the Hamiltonian system

$$dl/dt = \{\gamma, l\}_\theta,$$

where $\theta := LR + R^*L$. Define

$$l(a(\xi)) = \frac{\xi^2}{2} + q + A(\xi), \tag{34}$$

where

$$A(\xi) := \sum_{j\in\mathbf{Z}_+} a_j(s)\xi^{-(j+1)} \in \mathcal{G}_{0-}$$

is constructed from the values of the moment functions on $M_{(\mathbf{Z}_+)}$. According to the Kostant – Symes theorem [10, 11], all of the functionals $\gamma_j = Tr\, l^{j/2}$ are Casimir and in involution with respect to the Lie – Poisson bracket $\{\cdot,\cdot\}_\theta$ on G_0^*. Consequently, (33) with $\gamma = H = Tr\, l^2 \in \in D(M_{(\mathbf{Z}_+)})$ is a completely integrable Hamiltonian flow on $M_{(\mathbf{Z}_+)} \cong M_{(u,q,h)}$, where the Lie – Poisson bracket is given by

$$\{\gamma,\mu\}_\theta := \int\limits_{\mathbf{R}} ds \left\langle \mathrm{grad}\gamma, \theta(a,q)\mathrm{grad}\mu \right\rangle. \tag{35}$$

Here $\langle\cdot,\cdot\rangle$ is the standard scalar product on the space of sequences $l_2(R)$ and

$$\begin{aligned} &\theta(a,q) := \|\theta_{mn}(a)\| \otimes \theta(q),\ m,n \in \mathbf{Z}_+, \\ &\theta_{mn}(a) := m a_{m+n-1}\frac{d}{ds} + n\frac{d}{ds}a_{m+n-1}, \\ &\theta(q) := \frac{d}{ds}. \end{aligned} \tag{36}$$

With the above we have proved the following result.

Theorem 3. *The moment dynamical system (31) on the functional manifold $M_{(\mathbf{Z}_+)}$ is a completely Lax integrable Hamiltonian flow with respect to the Lie – Poisson bracket (35), (36); the Hamiltonian functional is $H := Tr\, l^2$ and the Lax representation has the form*

$$\frac{dl}{dt} = [l, \mathcal{R}\nabla H(l)], \tag{37}$$

where $l \in G_0^*$ is given by (34).

We now prove that the above mappings of $M_{(\mathbf{Z}_+)}$ onto $M_{(u,q,h)}$ and $M_{(f,q)}$ are canonical. In the first case it is easy to verify that the mapping

$$a_n(s) = \int\limits_0^{h(s)} d\tau u^n(s,\tau) \in M_{(\mathbf{Z}_+)},$$

where $n \in \mathbf{Z}_+, s \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}}$, transforms the Hamiltonian structure $\{\cdot,\cdot\}_{\theta(a,q)}$ into $\{\cdot,\cdot\}_{\theta(u,q,h)}$. Moreover

$$\frac{d(u,q,h)^T}{dt} = -\theta(u,q,h)\mathrm{grad}H = K[u,q,h], \tag{38}$$

where $\theta(u,q,h) := antidiag\left(\dfrac{d}{ds}, \dfrac{d}{ds}, \dfrac{d}{ds}\right)$ is the canonical structure on $M_{(u,q,h)}$. To prove that the mapping of $M_{(u,q,h)}$ onto $M_{(f)}$ is canonical, we use the mapping

$$\tau = \int\limits_{-\infty}^{u(s,t)} dp f(s,p;t)$$

assuming that the free surface is given by

$$h(s,t) = \int\limits_{-\infty}^{u(s,h)} dpf(s,p;t), \ \ (s,t) \in \frac{\mathbf{R}}{2\pi\mathbf{Z}} \times [0,T],$$

and consider the space F of smooth functions on $T^*(R)$ with canonical Poisson bracket

$$\{f,g\}(s,p) := \frac{\partial g}{\partial s}\frac{\partial f}{\partial p} - \frac{\partial f}{\partial s}\frac{\partial g}{\partial p}, (s,p) \in \frac{\mathbf{R}}{2\pi\mathbf{Z}} \times \mathbf{R},$$

which transforms it into a Lie algebra. If we introduce Hilbert space structure on F via the inner product $(\cdot,\cdot)$, then by the Riesz theorem we can identify F^* with F. Note that

$$(f,g) = (g,f) = \int\limits_{\mathbf{R}} ds \int\limits_{\mathbf{R}} dpf(s,p)g(s,p)$$

so that the scalar product is invariant with respect to the Lie – Poisson bracket on F, i.e, $(f,\{g,h\}) = (\{f,g\},h)$ for all $f,g,h \in F$.

We now consider the gradient mapping $\nabla : D(F^*) \to F$ for a functional $\gamma \in D(F^*)$ by $(\nabla\gamma(f),g) := \frac{d}{d\varepsilon}\gamma(f+\varepsilon g) \mid_{\varepsilon=0+}$. Then $\nabla\gamma(f) = \frac{\delta\gamma}{\delta f}$ is an ordinary Euler variational derivative of γ at $f \in F^*$. Clearly $\{\{\gamma,\mu\}\} := (f,[\nabla\gamma,\nabla\mu])$ defines a canonical Hamiltonian structure on F^*.

Consider a functional $H \in D(F^*)$ and its gradient $\nabla H(f) \in F$. Then the vector field $\frac{df}{dt} = ad^*_{\nabla H(f)} f$ on F^* is generated by the coadjoint action of the Lie algebra F on F^*. It follows from the properties of the inner product on F that this vector field is equivalent to the following Lax type equation:

$$df/dt = \{f, \nabla H(f)\}, \tag{39}$$

which coincides with the Hamiltonian equation $df/dt = \{\{H,f\}\}$ on the manifold F^*.

We now make the following identification:

$$M_{(\mathbf{Z}_+)} \ni a_n(s) = \int\limits_{\mathbf{R}} dpp^n f(s,p), \ \ s \in \frac{\mathbf{R}}{2\pi\mathbf{Z}},$$

consistent with the mapping of $M_{(\mathbf{Z}_+)}$ into $M_{(f)}$ introduced above. As a result, the Lie – Poisson bracket $\{\{\cdot,\cdot\}\}$ on F^* transforms into the Lie – Poisson bracket (35) on G_0^*, i.e., the mapping of $M_{(\mathbf{Z}_+,q)}$ into $M_{(f,q)}$ is canonical.

Using now the Hamiltonian

$$H = \int\limits_{\mathbf{R}} ds(a_2 + 2qa_0),$$

we find the kinetic Boltzmann – Vlasov equation for the distribution function $f \in M_{(f)}$ and the field function $q \in M_{(q)}$:

$$\begin{aligned} \frac{df}{dt} &= -pf_s + q_s f_p, \\ \frac{dq}{dt} &= -\int\limits_{\mathbf{R}} dpf_s(s,p,t). \end{aligned} \tag{40}$$

The consistent system of evolution equations (40) is important and has interesting applications to the theory of kinetic processes. Consider a system of interacting particles on an axis $\mathbf{R}$, and assume that its density ρ is a constant ($=1$) and the potential of particle interaction has the form $\Phi(s-s')$, $s, s' \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}}$. Then the distribution function satisfies the kinetic Boltzmann – Vlasov equation

$$\frac{df}{dt} = \int_0^{2\pi} ds' \int_{\mathbf{R}} dp' \Phi(s-s') f(s', p'; t), \tag{41}$$

provided that there is no multiparticle correlation. Comparing (40) with (41), we see that the following identification obtains:

$$q(s,t) = \int_0^{2\pi} ds' \int_{\mathbf{R}} dp' \Phi(s-s') f(s', p'; t), \tag{42}$$

which can be used to reduce the second equation in (40) to the linear integral equation

$$\int_0^{2\pi} ds' \int_{\mathbf{R}} dp' \Phi(s-s') p f(s', p') - \int dp f(s,p) = \mathrm{const} \tag{43}$$

being valid for all $f \in M_{(f)}$ satisfying (40). But owing to the definition of the distribution function f, (29) and the fact that the density

$$\rho(s,t) = \int_{\mathbf{R}} dp f(s,p;t) \equiv 1$$

for all $t \in [0,T]$, we obtain that

$$\int_0^{2\pi} ds' \int_{\mathbf{R}} dp \Phi(s-s') p f(s', p) - 1 = \mathrm{const} \tag{44}$$

for all $s \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}}$. Since, in general, $\Phi(s-s') \to 0$ as $|s-s'| \to \infty$, it follows from (43) and (44) that $\mathrm{const} = 1$; hence, for all $s \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}}$

$$\int_{\mathbf{R}} dp p f(s,p) = 0.$$

Thus, $f(s,p) = f(s,-p)$ for all $(s,p) \in \dfrac{\mathbf{R}}{2\pi\mathbf{Z}} \times \mathbf{R}$, and the solution (42) is consistent with the initial dynamical system (30). A complete description of all possible solutions to (30) and (40) will be presented elsewhere.

5. Here we consider some differential-geometric aspects of a swept volume manifold $S_{t_0}(V)$ generated by a Euclidean motion in $\mathbf{R}^3$. We shall use the basic notation from Theorem 2 of Subsection 2.

We define the following system of one-forms (20) generated by a group action $G \times Y \to Y$ on a manifold Y:

$$\beta^j := dy^j + \sum_{i=1}^{n} \xi_i^j(y)\bar{\omega}^i(a, da). \tag{45}$$

For this system to be completely Frobenius integrable, the canonical one-forms $\{\bar{\omega}^i : 1 \le i \le n\}$ must satisfy the Maurer – Cartan equations

$$d\bar{\omega}^j + \frac{1}{2}\sum_{i,k=1}^{n} C_{ik}^j \bar{\omega}^i \wedge \bar{\omega}^k := \bar{\Omega}^j = 0 \tag{46}$$

for all $1 \le j \le n$, where the C_{ik}^j are the structure constants. If the canonical one-forms $\{\bar{\omega}^i\}$ are defined via the scheme

$$\begin{array}{ccc} T^*(M) & \overset{\eta^*}{\longleftarrow} & T^*(G) \\ \downarrow & & \downarrow \\ M & \overset{\eta}{\longrightarrow} & G \end{array}, \tag{47}$$

where M is a smooth finite-dimensional manifold and η is a smooth mapping, then (46) takes the simple form

$$\eta^*\bar{\Omega}^j \mid_{\bar{M}} = 0, \ 1 \le j \le n, \tag{48}$$

on some integral submanifold $\bar{M} \subset M$ that is diffeomorphic to $S_{t_0}(V)$.

Let $\left\{\alpha^j \in \Lambda^2(M) : 1 \le j \le m\right\}$ be a basis of two-forms generating the ideal $I(\alpha)$ over the two-forms (46). Using this basis we can formulate the Cartan criterion for the system of one-forms (45) to define a group action of G on $S_{t_0}(V)$. The ideal $I(\alpha, \beta)$ generated by both (45) and $\left\{\alpha^j\right\}$ over the prolonged locally defined manifold $M \times Y$ must be completely integrable; therefore

$$d\beta^j = \sum_{k=1}^{m} f_k^j \alpha^k + \sum_{i=1}^{n} g_i^j \wedge \beta^i, \tag{49}$$

where $f_k^j \in \Lambda^0(M \times Y)$, $g_i^j \in \Lambda^1(M \times Y)$, $1 \le i, j \le n$, $1 \le k \le m$. We note that the ideal $I(\alpha, \beta)$ over the two-forms (46) should also be completely integrable via the Cartan criterion, because it follows from the equations $d\bar{\Omega}^j = 0$, $1 \le j \le n$, that $dI(\alpha) \subset I(\alpha)$ and $I(\alpha) \mid_{\bar{M}} = 0$.

The above result enables us to interpret the locally defined manifold $M \times S_{t_0}(V)$ as the adjoint of a principal fiber bundle $P(M; G)$ over the base manifold M with structure group G acting on the fibered manifold P. By representing a point locally as $(z, a) \in P(M; G)$, we obtain the local representation of the connection one-form as

$$\omega(z, a) := \bar{\omega}(a, da) + Ad_{a^{-1}} \left\langle \Gamma(z), dz \right\rangle, \tag{50}$$

where $\Gamma(z) \in G$ is the Christoffel elements, $z \in M$,

$$\bar{\omega}(a, da) := \sum_{i=1}^{n} \bar{\omega}^i(a, da) A^i$$

and $\{A^i\}$ is a basis for G.

Hence, the curvature two-form (defined just on M) is

$$\Omega := d\omega + \omega \wedge \omega = \frac{1}{2} Ad_{a^{-1}} \sum_{i,j=1}^{m} F_{ij}(z) dz^i \wedge dz^j, \tag{51}$$

where for all $z \in M$, $i, j = 1, ..., m = \dim M$,

$$F_{ij}(z) := \frac{\partial \Gamma_j}{\partial z^i} - \frac{\partial \Gamma_i}{\partial z^j} + [\Gamma_i, \Gamma_j]. \tag{52}$$

The results obtained above for the one-forms (45) imply that

$$\Omega \mid_{\bar{M}} = 0 \Leftrightarrow F_{ij} \mid_{\bar{M}} = 0 \tag{53}$$

for all $1 \leq i, j \leq m$, and this is equivalent to

$$\frac{1}{2} \sum_{i,j=1}^{m} F_{ij}(z) dz^i \wedge dz^j \in \mathcal{I}(\alpha). \tag{54}$$

Thus we can find the structure constants of the Lie group G by a simple yet tedious computation of the holonomy algebra of the connection (50), which is generated [12, 13] by all linear combinations of elements $F_{ij}, \nabla_k F_{ij}, \nabla_k \nabla_l F_{ij}, ..., i, j, k, l = 1, ..., m$, where $\nabla_k := \frac{\partial}{\partial z^k} - \Gamma_k$, are the appropriate covariant derivatives in $T^*(M)$.

Whence, we obtain the following result:

Theorem 4. *Suppose that the holonomy Lie algebra G associated with a Euclidean motion of the solid object V in E^3 is generated by parallel transport along a two-dimensional integral submanifold $\bar{M} \subset M$ with local coordinates $(s, t) \in \frac{\mathbf{R}}{2\pi\mathbf{Z}} \times [0, T)$. Then owing to (48), the following system is compatible on $\bar{M}$:*

$$\frac{\partial f}{\partial s} = -\Gamma_{(s)} f, \quad \frac{\partial f}{\partial t} = -\Gamma_{(t)} f, \tag{55}$$

for all f spanning a linear representation space for the Lie algebra G, where $\Gamma_{(s)}, \Gamma_{(t)}$ are the nontrivial Christoffel matrices that define the curvature form Ω over M.

The above theorem gives rise to a new way of constructing exact forms of a group actions $G \times Y \to Y$ which produce Euclidean motions in $\mathbf{R}^3$. Some interesting applications of this new approach are presented in [14].

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