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SOME UNIQUENESS RESULTS ON CONTROLLABILITY FOR FUNCTIONAL SEMILINEAR DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

ДЕЯКІ РЕЗУЛЬТАТИ ПРО ЄДИНІСТЬ КЕРОВАНОСТІ ДЛЯ ФУНКЦІОНАЛЬНИХ НАПІВЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ У ПРОСТОРАХ ФРЕШЕ

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In this paper, a recent nonlinear alternative for contraction maps in Fréchet spaces, due to Frigon and Granas, is combined with semigroups theory and used to investigate the controllability of some classes of semilinear functional and neutral functional differential equations in Banach spaces on the semiinfinite interval.

Досліджується керованість для деяких класів напівлінійних функціональних і нейтральних функціонально-диференціальних рівнянь у банахових просторах на напівобмеженому інтервалі за допомогою теорії напівгруп, а також нещодавно отриманої Фрігоном і Гранасом нелінійної альтернативи для стискаючих відображень у просторах Фреше.

1. Introduction. This paper is concerned with an application of a recent nonlinear alternative for contraction maps in Fréchet spaces due to Frigon and Granas [1] to the controllability of some classes of initial value problems for first and second order semilinear functional and neutral functional differential equations in Fréchet spaces. In Section 3, we will consider the first order semilinear functional differential equations

$$y'(t) - Ay(t) = f(t, y_t) + (Bu)(t) \quad \text{a. e. } t \in [0, \infty), \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{2}$$

where $r > 0$, $f : J \times C([-r, 0], E) \to E$ is a given function and $\phi \in C([-r, 0], E)$. Also the control function $u(\cdot) \in L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space, B is a bounded linear operator from U to E. Finally A is a densely defined operator generating a semigroup $\{T(t)\}$, $t \geq 0$, of bounded linear operators from E into E and E is real Banach space with norm $|\cdot|$. For any continuous function y defined on $[-r, \infty)$ and any $t \in [0, \infty)$, we denote by y_t the element of $C([-r, 0], E)$ defined by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t. In Section 4, we study the second order semilinear functional differential equations of the form

$$y''(t) - Ay(t) = f(t, y_t) + (Bu)(t) \quad \text{a. e. } t \in [0, \infty), \tag{3}$$

$$y(t) = \phi(t), \;\; t \in [-r, 0], \; y'(0) = \eta, \tag{4}$$

where f and ϕ are as in problem (1), (2), $\eta \in E$, and A is a densely defined operator generating a family of cosinus operators $\{C(t)\}, \; t \geq 0$.

Sections 5 and 6 are concerned with the existence of solutions, of initial value problems for first and second order semilinear neutral functional differential equations. In Section 5 we consider the first order semilinear neutral functional differential equations of the form

$$\frac{d}{dt}[y(t) - g(t, y_t)] = Ay(t) + f(t, y_t) + (Bu)(t) \quad \text{a. e. } t \in [0, \infty), \tag{5}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{6}$$

where $f, \; A$, and ϕ are as in problem (1), (2) and $g : J \times C([-r, 0], E) \to E$. In Section 6 we study the following second order problem

$$\frac{d}{dt}[y'(t) - g(t, y_t)] = Ay(t) + f(t, y_t) + (Bu)(t) \quad \text{a. e. } t \in [0, \infty), \tag{7}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \; y'(0) = \eta, \tag{8}$$

where $f, A, \; \eta$, and ϕ are as in problem (3), (4) and g is as in problem (5), (6). The last section will be devoted to an example illustrating the abstract theory.

Recently the fixed point argument such as the Banach contraction principle and Schaefer's fixed point theorem were applied to the controllability, on compact intervals, of some classes of semilinear differential, integrodifferential, and functional differential equations in Banach spaces in the literature. We mention here the survey paper by Balachandran and Dauer [2] and the references cited therein. In [3] the controllability of a class of first order evolution equations on semiinfinite time horizon was studied by means of an application of Schauder – Tikhonov's fixed point theorem and the semigroup theory.

Our goal here is to give uniqueness results for the above problems. These results can be considered as a contribution to the literature.

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

$C([-r, 0], E)$ is the Banach space of all continuous functions from $[-r, 0]$ into E with the norm

$$\|\phi\| := \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

$B(E)$ is the Banach space of all linear bounded operator from E into E with norm

$$\|N\|_{B(E)} := \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : [0, \infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [4].)

$L^1([0,\infty),E)$ denotes the Banach space of functions $y : [0,\infty) \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^\infty |y(t)|dt.$$

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if:

1) $C(0) = I$ (I is the identity operator in E);

2) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s,t \in \mathbb{R}$;

3) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)yds, \;\; y \in E, \; t \in \mathbb{R}.$$

The infinitesimal generator $A : D(A) \subseteq E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Fattorini [5] and Goldstein [6] and to the papers of Travis and Webb [7, 8]. For properties of semigroup theory, we refer the interested reader to the books of Engel and Nagel [9] and Pazy [10].

For more details on the following notions we refer to [4]. Let X be a Fréchet space with a family of seminorms $\{\|\cdot\|_n), \; n \in \mathbb{N}\}$. Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that

$$\|y\|_n \le M_n \text{ for all } y \in Y.$$

To X, we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows. For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by $\sim_n$ if and only if $\|x-y\|_n = 0$. We denote $X^n = (X^/ \sim_n, \|\cdot\|)$ the quotient space, the completion of X^n with respect to $\|\cdot\|$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows. For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and define $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $\mathrm{int}_n(Y^n)$, and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of seminorms $\{\|\cdot\|_n\}$ verifies

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \ldots \quad \text{for every } x \in X.$$

Definition 2.1 *A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that*

$$\|f(x) - f(y)\|_n \le k_n\|x-y\|_n \quad \textit{for all } x,y \in X.$$

Theorem 2.1 *(Nonlinear Alternative, [1]). Let X be a Fréchet space and $Y \subset X$ a closed subset in Y let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

C_1) N has a unique fixed point;

C_2) there exists $\lambda \in [0,1),\ n \in \mathbb{N}$, and $x \in \partial_n Y^n$ such that $\|x - \lambda N(x)\|_n = 0$.

In what follows, we will assume that the function $f : [0,\infty) \times C([-r,0],E) \to E$ is an L^1-Carathéodory function, i. e.,

i) $t \longmapsto f(t,u)$ is measurable for each $u \in C([-r,0],E)$;

ii) $u \longmapsto f(t,u)$ is continuous for almost all $t \in [0,\infty)$;

iii) for each $q > 0$, there exists $h_q \in L^1_{\text{loc}}([0,\infty),\mathbb{R}_+)$ such that

$$|f(t,u)| \le h_q(t) \quad \text{for all } \ \|u\| \le q \ \text{ and for almost all } \ t \in [0,\infty).$$

3. Controllability for first order semilinear FDEs. The main result of this section concerns the IVP (1), (2). Before stating and proving it, we first give a definition of a mild solution of the IVP.

Definition 3.1. *A function $y \in C([-r,\infty),E)$ is said a mild solution of (1), (2) if $y(t) = \phi(t),\ t \in [-r,0]$, and for each $t \in [0,\infty)$,*

$$y(t) = T(t)\phi(0) + \int_t^0 T(t-s)f(s,y_s)ds + \int_t^0 T(t-s)(Bu)(s)ds.$$

Definition 3.2. *The system (1), (2) is said to be infinite controllable if for any continuous function ϕ on $[-r,0]$ and any $x_1 \in E$ and for each $n \in \mathbb{N}$ there exists a control $u \in L^2([0,n],U)$ such that the mild solution $y(\cdot)$ of (1) satisfies $y(n) = x_1$.*

Let us introduce the following hypotheses which are assumed hereafter:

H_1) there exists a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1_{\text{loc}}([0,\infty),\mathbb{R}_+)$ such that

$$|f(t,u)| \le p(t)\psi(\|u\|) \ \text{ for a. e. } t \in [0,\infty) \text{ and each } u \in C([-r,0],E)$$

with

$$\int_1^\infty \frac{ds}{\psi(s)} = \infty;$$

H_2) for all $R > 0$ there exists $l_R \in L^1_{\text{loc}}([-r,\infty),\mathbb{R}_+)$ such that

$$|f(t,u) - f(t,\overline{u})| \le l_R(t)\|u - \overline{u}\| \ \text{ for all } u,\ \overline{u} \in C([-r,0],E) \text{ with } \|u\|, \|\overline{u}\| \le R;$$

H_3) there exists $M \ge 1$ such that

$$|T(t)\|_{B(E)} \le M \text{ for each } \ t \ge 0;$$

H$_4$) for every $n > 0$ the linear operator $W : L^2(J_n, U) \to E$ ($J_n = [0, n]$), defined by

$$Wu = \int_0^n T(n-s)Bu(s)ds,$$

has an inverse operator W^{-1} which takes values in $L^2(J_n, U) \backslash \operatorname{Ker} W$ and there exist positive constants $\overline{M}$, $\overline{M}_1$ such that $\|B\| \le \overline{M}$ and $\|W^{-1}\| \le \overline{M}_1$.

For each $n \in \mathbb{N}$ we define in $C([-r, \infty), E)$ the seminorms by

$$\|y\|_n = \sup\{e^{-\tau L_n^*(t)}|y(t)| : t \le n\},$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds$, $\bar{l}_n(t) = \max(Ml_n(t), n\overline{M}M^2\overline{M}_1\|l_n\|_{L^1([0,n])})$, and l_n is the function from H$_2$). Then $C([-r, \infty), E)$ is a Fréchet space with the family of seminorms $\{\|\cdot\|_n\}$. In what follows we will choose τ sufficiently large.

Theorem 3.1. *Suppose that hypotheses H$_1$) – H$_4$) are satisfied. Then problem (1), (2) has a unique solution.*

Proof. Using hypothesis H$_4$) for each $y(\cdot)$ and each $n \in \mathbb{N}$ define the control

$$u_y^n(t) = W^{-1}\left[x_1 - T(n)\phi(0) - \int_0^n T(n-s)f(s, y_s)ds\right](t).$$

Transform the problem (1), (2) into a fixed point problem. Consider the operator $N : C([-r, \infty), E) \to C([-r, \infty), E)$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0]; \\ T(t)\phi(0) + \int_0^t T(t-s)(Bu_y^n)(s)ds + \int_0^t T(t-s)f(s, y_s)ds, & \text{if } t \in [0, \infty). \end{cases}$$

Clearly, the fixed points of the operator N are solutions of the problem (1), (2). Let y be a possible solution of the problem (1), (2). Given $n \in \mathbb{N}$ and $t \leq n$, then

$$|y(t)| \leq |T(t)||\phi(0)| + \int_0^t |T(t-s)||(Bu_y^n)(s)|ds + \int_0^t |T(t-s)||f(s,y_s)|ds \leq$$

$$\leq M|\phi(0)| + M\int_0^t \|B\||u_y^n(s)|ds + M\int_0^t p(s)\psi(\|y_s\|)ds \leq$$

$$\leq M\|\phi\| + n\overline{M}M\overline{M_1}\left[\|x_1\| + M\|\phi\| + nM\int_0^t p(s)\psi(\|y_s\|)ds\right] +$$

$$+ M\int_0^t p(s)\psi(\|y_s\|)ds \leq$$

$$\leq M\|\phi\| + n\overline{M}M\overline{M_1}[\|y_1\| + M\|\phi\|] +$$

$$+ \max\{n^2\overline{M}M^2\overline{M_1}, M\}\int_0^t p(s)\psi(\|y_s\|)ds.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \ \ 0 \leq t \leq n.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,n]$, by the previous inequality we have, for $t \in [0,n]$,

$$\mu(t) \leq M\|\phi\| + n\overline{M}M\overline{M_1}[\|x_1\| + M\|\phi\|] + \max\{n^2\overline{M}M^2\overline{M}_1, M\}\int_0^t p(s)\psi(\mu(s))ds.$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c = v(0) = M\|\phi\| + n\overline{M}M\overline{M_1}[\|x_1\| + M\|\phi\|], \quad \mu(t) \leq v(t), \ \ t \in [0,n],$$

and

$$v'(t) = \max\{n^2\overline{M}M^2\overline{M}_1, M\}p(t)\psi(\mu(t)) \ \text{ a. e. } t \in [0,n].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq \max\{n^2\overline{M}M^2\overline{M}_1, M\}p(t)\psi(v(t)) \ \text{ a. e. } \ t \in [0,n].$$

This implies that for each $t \in [0,n]$

$$\int\limits_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \max\{n^2\overline{M}M^2\overline{M}_1, M\}\int\limits_0^t p(s)ds < \infty.$$

Thus from H_1) there exists a constant K_n such that $v(t) \leq K_n,\ t \in [0,n]$, and hence $\mu(t) \leq \leq K_n,\ t \in [0,n]$. Since for every $t \in [0,n]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_n \leq \max\{\|\phi\|, K_n\} := M_n.$$

Set

$$Y = \{y \in C([-r,\infty),E) : \sup\{|y(t)| : t \leq n\} \leq M_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

Clearly, Y is a closed subset of $C([-r,\infty),E)$. We shall show that $N : Y \to C([-r,\infty),E)$ is a contraction operator. Indeed, consider $y, \overline{y} \in C([-r,\infty),E)$, thus for each $t \in [0,n]$ and $n \in \mathbb{N}$,

$$|N(y)(t) - N(\overline{y})(t)| = \left|\int\limits_0^t T(t-s)[(Bu_y^n)(s) - (Bu_{\overline{y}}^n)(s)]ds + \right.$$

$$\left. + \int\limits_0^t T(t-s)[f(s,y_s) - f(s,\overline{y}_s)]ds\right| \leq$$

$$\leq M\int\limits_0^t \|B\||(u_y^n)(s) - (u_{\overline{y}}^n)(s)|ds +$$

$$+ \int\limits_0^t l_n(s)e^{\tau L(s)}e^{-\tau L(s)}M\|y_t - \overline{y}_t\|dt \leq$$

$$\leq M\int\limits_0^t \bar{l}_n(s)e^{\tau L_n^*(s)}dt\|y - \overline{y}\|_n +$$

$$+ M\overline{M}\int\limits_0^t |W^{-1}\left[y_1 - T(n)\phi(0) - \int\limits_0^n T(n-s)f(\omega, y(\omega))d\omega\right]ds -$$

$$- \; W^{-1}\left[y_1 - T(n)\phi(0) - \int_0^n T(n-s)f(\omega,\overline{y}(\omega))d\omega\right]ds \le$$

$$\le \overline{M}M\overline{M}_1\int_0^t M\int_0^n |f(\omega,y(\omega)) - f(\omega,\overline{y}(\omega))|d\omega ds +$$

$$+ \; \frac{M}{\tau}\int_0^t (e^{\tau L_n^*(s)})'dt\|y-\overline{y}\|_n \le$$

$$\le \frac{1}{\tau}e^{\tau L_n^*(t)}\|y-\overline{y}\|_n + \frac{1}{\tau}e^{\tau L^*(t)}\|y-\overline{y}\|_n.$$

Therefore,

$$\|N(y) - N(\overline{y})\|_n \le \frac{2}{\tau}\|y-\overline{y}\|_n,$$

showing that for τ sufficiently large, the operator N is a contraction for all $n \in \mathbb{N}$. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative type [1], we deduce that N_1 has a unique fixed point which is a mild solution to (1), (2).

4. Controllability for second order semilinear FDEs. In this section we give a uniqueness result for the IVP (3), (4).

Definition 4.1. *A function $y \in C([-r,\infty),E)$ is said to be a mild solution of (3), (4) if $y(t) = \phi(t),\ t \in [-r,0],\ y'(0) = \eta$, and*

$$y(t) = C(t)\phi(0) + S(t)\eta + \int_t^0 S(t-s)(Bu)(s)ds + \int_t^0 S(t-s)f(s,y_s)ds.$$

Definition 4.2. *The system (3), (4) is said to be infinite controllable if for any continuous function ϕ on $[-r,0]$ and any $x_1 \in E$ and for each $n \in \mathbb{N}$ there exists a control $u \in L^2([0,n],U)$ such that the mild solution $y(\cdot)$ of (3) satisfies $y(n) = x_1$.*

Theorem 4.1. *Assume H_1), H_2) and the condition:*
H_5) there exists a constant $M_1 \ge 1$ such that

$$\|C(t)\|_{B(E)} \le M_1 \text{ for all } t \in \mathbb{R};$$

H_6) for every $n > 0$ the linear operator $W : L^2(J_n,U) \to E$, defined by

$$Wu = \int_0^n S(n-s)Bu(s)ds,$$

has an inverse W^{-1} which takes values in $L^2(J_n, U)\backslash \operatorname{Ker} W$ and there exist positive constants $\overline{M}^*$, $\overline{M}_1^*$ such that $\|B\| \leq \overline{M}^*$ and $\|W^{-1}\| \leq \overline{M}_1^*$ are satisfied. Then the IVP (3), (4) has a unique mild solution.

Proof. Using hypothesis H$_6$) for each $y(\cdot)$ and each $n \in \mathbb{N}$ define the control

$$u_y^n(t) = W^{-1}\left[x_1 - C(n)\phi(0) - S(n)\eta - \int_0^n S(n-s)f(s,y_s)ds\right](t).$$

Consider the operator $N_1 : C([-r,\infty),E) \to C([-r,\infty),E)$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0];\\ C(t)\phi(0) + S(t)\eta + \displaystyle\int_0^t S(t-s)(Bu_y^n)ds + \\ + \displaystyle\int_0^t S(t-s)f(s,y_s)ds, & \text{if } t \in [0,\infty). \end{cases}$$

Clearly, the fixed points of the operator N_2 are mild solutions of the problem (3), (4).

Let y be a possible solution of (3), (4). Thus for all $t \leq n,\ n \in \mathbb{N}$, we have

$$y(t) = C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)(Bu_y^n)ds + \int_0^t S(t-s)f(s,y_s)ds.$$

This implies, by H$_1$) and H$_4$, that for each $t \in [0,n]$ we have

$$|y(t)| \leq M_1\|\phi\| + nM_1|\eta| + nM_1\int_0^t \|B\||u_y^n(s)|ds + nM_1\int_0^t p(s)\psi(\|y_s\|)ds \leq$$

$$\leq M_1\|\phi\| + nM_1|\eta| + n\overline{M^*}M_1\overline{M_1^*}\Bigg[\|x_1\| + M_1|\phi(0)| + nM_1\eta +$$

$$+\ nM_1\int_0^t p(s)\psi(\|y_s\|)ds\Bigg] + nM_1\int_0^t p(s)\psi(\|y_s\|)ds \leq$$

$$\leq M_1\|\phi\| + nM_1|\eta| + n\overline{M^*}M_1\overline{M_1^*}[\|x_1\| + M_1|\phi(0)| + nM_1\eta] +$$

$$+\ \max\{n^2M_1^2\overline{M}^*\overline{M}_1^*, nM_1\}\int_0^t p(s)\psi(\|y_s\|)ds.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \ \ 0 \leq t \leq n.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, n]$, by the previous inequality we have, for $t \in [0, n]$,

$$\mu(t) \leq M_1\|\phi\| + nM_1|\eta| + n\overline{M^*}M_1\overline{M_1^*}[\|x_1\| + M_1|\phi(0)| + nM_1\eta] +$$

$$+ \ \max\{n^2M_1^2\overline{M}^*\overline{M}_1^*, nM_1\} \int\limits_0^t p(s)\psi(\mu(s))ds.$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c = v(0) = M_1\|\phi\| + nM_1|\eta| + n\overline{M^*}M_1\overline{M_1^*}[\|x_1\| + M_1|\phi(0)| + nM_1\eta], \quad \mu(t) \leq v(t), \ \ t \in [0, n],$$

and

$$v'(t) = \max\{n^2M_1^2\overline{M}^*\overline{M}_1^*, nM_1\}p(t)\psi(\mu(t)) \ \text{ a. e. } t \in [0, n].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq \max\{n^2M_1^2\overline{M}^*\overline{M}_1^*, nM_1\}p(t)\psi(v(t)) \ \text{ a. e. } \ t \in [0, n].$$

This implies that for each $t \in [0, n]$

$$\int\limits_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \max\{n^2M_1^2\overline{M}^*\overline{M}_1^*, nM_1\} \int\limits_0^t p(s)ds < \infty.$$

Thus from H$_1$) there exists a constant K_n such that $v(t) \leq K_n, \ t \in [0, n]$, and hence $\mu(t) \leq$ $\leq K_n, \ t \in [0, n]$. Since for every $t \in [0, n]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_n \leq \max\{\|\phi\|, K_n\} := M_n.$$

Set

$$Y = \{y \in C([-r, \infty), E) : \sup\{|y(t)| : t \leq n\} \leq M_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

Thus

$$\|y\|_n \leq \max\{\|\phi\|, K_n^*\} := M_n^*.$$

Let

$$Y = \{y \in C([-r, \infty), E) : \sup\{|y(t)| : t \leq n\} \leq M_n^* + 1 \text{ for all } n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, we define in $C([0, \infty), E)$ the seminorms by

$$\|y\|_n = \sup\{e^{-\tau\tilde{L}_n(t)}|y(t)| : t \leq n\},$$

where $\tilde{L}_n(t) = \int_0^t \tilde{l}_n(s)ds$ and $\tilde{l}_n(t) = \max(nM_1 l_n(t), n\overline{M^*}M_1^2\overline{M_1^*}\|l_n\|_{L^1([0,n])})$. As in Theorem 3.1 we can show that $N_1 : Y \to C([-r,\infty), E)$ defined by

$$N_1(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0]; \\ C(t)\phi(0) + S(t)\eta + \int\limits_0^t S(t-s)(Bu)(s)ds + \\ + \int\limits_0^t S(t-s)f(s,y_s)ds, & \text{if } t \in [0,\infty), \end{cases}$$

is a contraction operator. From the choice of Y there is no $y \in \partial Y$ such that $y = \lambda N_1(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative [1] we deduce that N_2 has a unique fixed point which is a mild solution to (3), (4).

5. Controllability for first order semilinear neutral FDEs. Let us start by defining what we mean by a solution of IVP (5), (6).

Definition 5.1. *A function $y \in C([-r,\infty), E)$ is said to be a mild solution of (5), (6) if $y(t) = \phi(t),\ t \in [-r,0]$, the restriction of $y(\cdot)$ to the interval $[0,\infty)$ is continuous, and for each $t \in [0,\infty)$ the function $AT(t-s)g(s,y_s)$, $s \in [0,t)$, and*

$$y(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,y_t) + \int\limits_0^t AT(t-s)g(s,y_s)ds +$$

$$+ \int\limits_0^t T(t-s)(Bu)(s)ds + \int_0^t T(t-s)f(s,y_s)ds.$$

Let us introduce the following hypotheses which are assumed hereafter:

A$_1$) for each $R > 0$, there exists a function $\widetilde{l}_R \in L^1_{\text{loc}}(J,E) \cap C(J,E)$ such that $|g(t,u) - g(t,\overline{u})| \le \widetilde{l}_R(t)\|u-\overline{u}\|,\ t \in [0,\infty),\ u,\overline{u} \in C([-r,0],E),$ with $\|u\|,\ \|\overline{u}\| \le R$;

A$_2$) there exists a constant $L > 0$ such that

$$|g(t,u)| \le L \ \text{ for each } t \in [0,\infty),\ u \in C([-r,0],E);$$

A$_3$) A is the infinitesimal generator of the semigroup $\{T(t) : t \ge 0\}$ such that

$$\|AT(t)\|_{B(E)} \le M_2 \ \text{ for some } \ M_2 > 0;$$

A$_4$) for every $n > 0$ the linear operator $W : L^2(J_n,U) \to E$, defined by

$$Wu = \int\limits_0^n T(n-s)Bu(s)ds,$$

has an inverse W^{-1} which takes values in $L^2(J_n, U)\backslash \mathrm{Ker}\, W$ and there exist positive constants $\overline{M},\ \overline{M}_1$ such that $\|B\| \le \overline{M}$ and $\|W^{-1}\| \le \overline{M}_1$.

Let $\overline{L}_n(t) = \int_0^t \widehat{l}_n(s)ds$, where

$$\widehat{l}_n(t) = \max(\tilde{l}_n(t), M_2\tilde{l}_n(t), n\overline{M}_1 M\overline{M}M_2\|\tilde{l}_n\|, n\overline{M}_1 M^2\overline{M}M_2\|l_n\|).$$

For each $n \in \mathbb{N}$ we define in $C([-r,\infty), E)$ the seminorms by

$$\|y\|_n = \sup\{e^{-\tau\overline{L}_n(t)}|y(t)| : t \le n\}.$$

Then $C([-r,\infty), E)$ is a Fréchet space with a family of seminorms $\{\| \cdot \|_n\}$.

Theorem 5.1. *Assume that hypotheses H_1), H_2) and A_1)–A_4) hold. If for each $n \in \mathbb{N}$ we have*

$$\left(\frac{4}{\tau} + \sup_{t\in[0,n]} \tilde{l}_n(t)\right) < 1,$$

then the problem (5), (6) has a unique mild solution.

Proof. Using hypothesis A_4) for each $y(\cdot)$ and each $n \in \mathbb{N}$ define the control

$$u_y^n(t) = W^{-1}\Bigg[x_1 - T(n)[\phi(0) - g(0,\phi)] + g(n, y_n) +$$

$$+ \int_0^n AT(n-s)g(s,y_s)ds + \int_0^n AT(n-s)f(s,y_s)ds\Bigg](t).$$

Transform the problem (5), (6) into a fixed point problem. Consider the operator $N_2 : C([-r,\infty), E) \to C([-r,\infty), E)$ defined by

$$N_3(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r,0]; \\ T(t)[\phi - g(0,\phi)] + g(t,y_t) + \int_0^t A(t-s)g(s,y_s)\,ds + \\ + \int_0^t T(t-s)(Bu_y^n)(s)\,ds + \\ + \int_0^t T(t-s)f(s,y_s)\,ds, & \text{if } t \in [0,\infty). \end{cases}$$

Remark 5.1. It is clear that the fixed points of N_2 are mild solutions to (5), (6).

Let y be a possible solution of the problem (5), (6). Given $n \in \mathbb{N}$ and $t \leq n$, we have

$$y(t) = T(t)[\phi(0) - g(0, \phi(0))] + g(t, y_t) + \int_0^t AT(t-s)g(s, y_s)ds +$$

$$+ \int_0^t T(t-s)(Bu_y^n)(s)\,ds + \int_0^t T(t-s)f(s, y_s)\,ds.$$

This implies by H$_3$), A$_2$), A$_3$), and A$_4$) that for each $t \in [0, n]$ we have

$$|y(t)| \leq M|\phi(0)| + M|g(0, \phi(0))| + |g(t, y_t)| +$$

$$+ M\int_0^t \|B\||u_y^n(s)|ds + nM_2L + M\int_t^0 p(s)\psi(\|y_s\|)ds \leq$$

$$\leq M\|\phi\|_\infty + (M+1)L + nM_2L + n\overline{M}M\overline{M_1}[\|x_1\| +$$

$$+ M\|\phi\| + (M+1)L + nM_2L] +$$

$$+ \max\{n\overline{M}M^2\overline{M_1}, M\}\int_0^t p(s)\psi(\|y_s\|)ds.$$

Consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \ \ 0 \leq t \leq n.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, n]$, by the previous inequality we have, for $t \in [0, n]$,

$$\mu(t) \leq M\|\phi\|_\infty + (M+1)L + nM_2L + n\overline{M}M\overline{M_1}\big[\|x_1\| + M\|\phi\| + (M+1)L +$$

$$+ nM_2L\big] + \max\{n\overline{M}M^2\overline{M_1}, M\}\int_0^t p(s)\psi(\|y_s\|)ds.$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c = v(0) = M\|\phi\|_\infty + (M+1)L + nM_2L + n\overline{M}M\overline{M_1}[\|x_1\| +$$

$$+ M\|\phi\| + (M+1)L + nM_2L], \quad \mu(t) \leq v(t), \ \ t \in [0, n],$$

and

$$v'(t) = \max\{n\overline{M}M^2\overline{M_1}, M\}p(t)\psi(\mu(t)) \text{ a. e. } t \in [0, n].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq \max\{n\overline{M}M^2\overline{M_1}, M\}p(t)\psi(v(t)) \text{ a. e. } t \in [0, n].$$

This implies that for each $t \in [0, n]$

$$\int\limits_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \max\{n\overline{M}M^2\overline{M_1}, M\} \int\limits_0^t p(s)ds < \infty.$$

Thus from H_1) there exists a constant K_n such that $v(t) \leq K_n,\ t \in [0, n]$, and hence $\mu(t) \leq \leq K_n,\ t \in [0, n]$. Since for every $t \in [0, n]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_n \leq \max\{\|\phi\|, K_n\} := M'_n.$$

Let

$$Y_1 = \{y \in C([-r, \infty), E) : \sup\{|y(t)| : t \leq n\} \leq M'_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

We shall show that $N_3 : Y \to C([-r, \infty), E)$ is a contraction operator. Indeed, consider $y, \overline{y} \in \in C([-r, \infty), E)$, thus for each $t \in [0, \infty)$ such that $t \leq n$, $n \in \mathbb{N}$,

$$|N_3(y)(t) - N_3(\overline{y})(t)| = \Bigg| \int\limits_t^0 AT(t-s)[g(s, y_s) - g(s, \overline{y}_s)]ds +$$

$$+ \int\limits_0^t T(t-s)[(Bu_y^n)(s) - (Bu_{\overline{y}}^n)](s)ds +$$

$$+\ g(t, y_t) - g(t, \overline{y}_t) + \int\limits_0^t T(t-s)[f(s, y_s) - f(s, \overline{y}_s)]ds \Bigg| \leq$$

$$\leq M_2 \int\limits_t^0 \tilde{l}_n(s)\|y_s - \overline{y}_s\|ds + \tilde{l}_n(t)\|y_t - \overline{y}_t\| +$$

$$+\ M\|B\|\int\limits_0^t |(u_y^n)(s)-(u_{\overline{y}}^n)(s)|ds + M\int\limits_0^t l_n(s)\|y_s-\overline{y}_s\|ds \leq$$

$$\leq \frac{1}{\tau}e^{\tau\overline{L}_n(t)}\|y-\overline{y}\|_n + e^{\tau\overline{L}_n(t)}\widehat{l}_n\|y-\overline{y}\|_n +$$

$$+\ M\overline{M}\|W^{-1}\|\int\limits_0^t [nM_2\|\tilde{l}_n(s)\|\|y-\overline{y}\|_\infty ds +$$

$$+\ M_2\|l_n\|_{L([0,n])}\|y-\overline{y}\|_\infty]ds + e^{\tau\overline{L}_n(t)}\frac{1}{\tau}\|y-\overline{y}\|_n \leq$$

$$\leq \frac{1}{\tau}e^{\tau\overline{L}_n(t)}\|y-\overline{y}\|_n + \frac{1}{\tau}e^{\tau\overline{L}_n(t)}\|y-\overline{y}\|_n + \frac{1}{\tau}e^{\tau\overline{L}_n(t)}\|y-\overline{y}\|_n +$$

$$+ \frac{1}{\tau}e^{\tau\overline{L}_n(t)}\|y-\overline{y}\|_n + \sup_{t\in[0,n]}|\tilde{l}(t)|\|y-\overline{y}\|_n.$$

Therefore,

$$\|N_3(y)-N_3(\overline{y})\|_n \leq \left(\frac{4}{\tau} + \sup_{t\in[0,n]}|\tilde{l}(t)\right)\|y-\overline{y}\|_n,$$

showing that N_3 is a contraction for all $n\in\mathbb{N}$. From the choice of Y_1 there is no $y\in\partial Y_1$ such that $y=\lambda N_3(y)$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative [1] we deduce that N_3 has a unique fixed point which is a mild solution to (5), (6).

6. Controllability for second order semilinear neutral FDEs. In this section we study the initial value problem (7), (8).

Definition 6.1. *A function $y\in C([-r,\infty),E)$ is said to be a mild solution of (7), (8) if $y(t)=\phi(t),\ t\in[-r,0], y'(0)=\eta$, and*

$$y(t) = C(t)\phi(0) + S(t)[\eta - g(0,\phi)] + \int\limits_t^0 C(t-s)g(s,y_s)ds +$$

$$+\ \int\limits_0^t S(t-s)(Bu)(s)ds + \int\limits_0^t S(t-s)f(s,y_s)ds.$$

Theorem 6.1. *Assume that hypotheses H_1), H_2), H_5), H_6) and A_1), A_2) hold. Then the IVP (7), (8) has a unique mild solution.*

Proof. Define the control

$$u_y^n(t) = W^{-1}\Bigg[x_1 - C(n)\phi(0) + [\eta - g(0,\phi)]S(n) +$$

$$+ \int\limits_0^n C(n-s)g(s,y_s)ds + \int\limits_0^n S(n-s)f(s,y_s)ds\Bigg].$$

Transform the problem (7), (8) into a fixed point problem. Consider the operator N_3 : $C([-r,\infty),E) \to C([-r,\infty),E)$ defined by:

$$N_3(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0]; \\ C(t)\phi(0) + [\eta - g(0,\phi)]S(t) + & \\ + \int\limits_t^0 C(t-s)g(s,y_s)ds + & \\ + \int\limits_0^t S(t-s)(Bu_y^n)(s)ds + & \\ + \int\limits_0^t S(t-s)f(s,y_s)ds, & \text{if } t \in [0,\infty). \end{cases}$$

We can easily show (as in the previous theorems with minor appropriate modifications) that the operator n_4 is a contraction. The details are left to the reader.

7. An example. As an application of our results we consider the following partial neutral functional differential equation of the form

$$\frac{\partial}{\partial t}[z(t,x) - p(t, z(t-r,x))] = \frac{\partial^2}{\partial x^2} z(t,x) +$$

$$+ Q(t, z(t-r,x), z_x(t-r,x)) + Bu(t), \quad 0 \le x \le \pi, t \in [0,\infty), \tag{9}$$

$$\begin{aligned} z(t,0) &= z(t,\pi), \ \ t \ge 0, \\ z(t,x) &= \phi(t,x), \ \ -r \le t \le 0, \end{aligned} \tag{10}$$

where ϕ is continuous. Let

$$g(t,w_t)(x) = p(t, w(t-x)), \ 0 \le x \le \pi,$$

and

$$f(t,w_t)(x) = Q\left(t, w(t-x), \frac{\partial}{\partial x} w(t-x)\right), \ 0 \le x \le \pi.$$

Take $E = L^2[0,\pi]$ and define $A : D(A) \subset E \to E$ by $Aw = w''$ with domain

$$D(A) = \{w \in E, w, w' \quad \text{are absolutely continuous}, \ w'' \in E, w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \ w \in D(A),$$

where (,) is the inner product in L^2 and $w_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, \ n = 1, 2, \ldots$, is the orthogonal set of eigenvectors of A. It is well known (see [10]) that A is the infinitesimal generator of an analytic semigroup $T(t), \ t \geq 0$, in E and is given by

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \ w \in E.$$

Since the analytic semigroup $T(t)$ is compact there exist constants $m_1 \geq 1$ and $m_2 > 0$ such that

$$\|T(t)\| \leq m_1 \quad \text{and} \quad \|AT(t)\| \leq m_2.$$

Assume that the operator $B : U \to Y, U \subset [0, \infty)$, is a bounded linear operator and for each $b > 0$ the operator

$$Wu = \int_0^b T(b - s) Bu(s) ds$$

has a bounded inverse W^{-1} which takes values in $L^2([0, \infty), U) \backslash \ker W$. Assume that there exists a constant $L > 0$ such that

$$|p(t, w(t - x)| \leq L.$$

Also assume that there exists an integrable function $\sigma : J \to [0, \infty)$ such that

$$|q(t, w(t - x))| \leq \sigma(t) \Omega(\|w\|)$$

where $\Omega : [0, \infty) \to (0, \infty)$ is continuous and nondecreasing with

$$\int_1^{\infty} \frac{ds}{s + \Omega(s)} = +\infty.$$

Assume that for each $R > 0$, there exists a function $\tilde{l}_R \in L^1_{\text{loc}}(J, E)$ such that

$$|q(t, w(t - x)) - q(t, \overline{w}(t - x))| \leq \tilde{l}_R(t) \|w - \overline{w}\|,$$

$$t \in [0, \infty), \ w, \overline{w} \in E, \ \text{with } \|w\|, \|\overline{w}\| \leq R.$$

We can show that problem (5), (6) is an abstract formulation of problem (9), (10). Since all the conditions of Theorem 5.1 are satisfied, the problem (9), (10) has a unique solution z on $[-r, \infty) \times [0, \pi]$.

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