UDC 517.9

INTEGRATION OF SHAZY EQUATION WITH CONSTANT COEFFICIENTS
ІНТЕГРУВАННЯ РІВНЯННЯ ШАЗІ ІЗ СТАЛИМИ КОЕФІЦІЄНТАМИ

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The problem of constructing a general solution for the third order Shazy differential equation with six constants parameters is considered. This equation belongs to P-type and is connected with Painleve equations.

Розглядається задача побудови загального розв'язку диференціального рівняння Шазі третього порядку з шістьома сталими параметрами. Це рівняння належить до типу P і пов'язане з рівнянням Пенлеве.

Introduction. Investigating the equations

$$u''' = R(u'', u', u, z), \tag{1}$$

where R is a rational function of u'', u', u with coefficients analytic in z, on whether they are of P-type (solutions of such equations do not have moveable critical singular points), Chazy built the equation [1]

$$u''' = \sum_{k=1}^{6} \frac{(u'-a_k')(u''-a_k'') + A_k(u'-a_k')^3 + B_k(u'-a_k')^2 + C_k(u'-a_k')}{u-a_k} +$$

$$+ Du'' + Eu' + \prod_{k=1}^{6}(u-a_k)\sum_{i=1}^{6}\frac{F_k}{u-a_k}. \tag{2}$$

The 32 coefficients of equation (2) are functions of z, $A_k, B_k, C_k, F_k, D, E, a_k$, $k = \overline{1,6}$. Necessary and sufficient conditions for equation (2) to be of P-type make system (S) [2] that consists of 31 algebraic and differential equations. The unknown functions in system (S) are the functions $A_k, B_k, C_k, F_k, D, E, a_k$, $k = \overline{1,6}$.

Equation (2) is connected quite closely with Painleve equations [3]. Investigation of equation (2) is also connected with the theory of isomonodromy deformation of linear systems, the theory of golonomic quantum fields, and nonlinear evolution equations.

The coefficients A_k, $k = \overline{1,6}$, must have the following form [2]:

$$A_k = -\frac{1}{a_k}, \quad k = \overline{1,6}. \tag{3}$$

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If the coefficients $a_k,\ k = \overline{1,6}$, are constants, then the functions $B_k, C_k, F_k,\ k = \overline{1,6}$, must have the following form [4]:

$$B_k = -\frac{1}{3}D,\ \ C_k = \frac{a_k}{3}\left(E - \frac{1}{3}D' + \frac{2}{9}D^2\right),$$
$$F_k = \frac{a_k}{3\phi(a_k)}\left(\frac{1}{3}D'' - \frac{2}{3}DD' - E' + \frac{2}{3}DE + \frac{4}{27}D^3\right), \tag{4}$$

where $\phi(a_k) \equiv \prod(a_k - a_j),\ k, j = \overline{1,6};\ j \neq k$. Functions (4) satisfy system (S). For $D = 0$ and E being a constant, equation (2) with coefficients (3) and (4) becomes

$$u''' = \sum_{k=1}^{6}\frac{u'u''}{u - a_k} - \sum_{k=1}^{6}\frac{u'^3}{a_k(u - a_k)} + E\left(1 + \frac{1}{4}\sum_{k=1}^{6}\frac{a_k}{u - a_k}\right)u'. \tag{5}$$

By setting

$$\frac{du}{dz} = \eta, \quad \eta^2 = y \tag{6}$$

in (5), we obtain the linear equation

$$\frac{d^2y}{du^2} = \sum_{k=1}^{6}\frac{1}{u - a_k}\,\frac{dy}{du} - 2\sum_{k=1}^{6}\frac{y}{a_k(u - a_k)} + 2E\left(1 + \frac{1}{4}\sum_{k=1}^{6}\frac{a_k}{u - a_k}\right). \tag{7}$$

In [5], a solution of homogeneous linear equation corresponding to equation (7) was built as a generalized power series. It was also shown that for some conditions imposed on coefficients, the power series gives the solution in the form of hypergeometric or even elementary functions.

Let us set

$$\sum_{k=1}^{6}\frac{1}{x - a_k} = \frac{6x^5 + 4\sigma_2x^3 - 3\sigma_3x^2 + 2\sigma_4x}{x^6 + \sigma_2x^4 - \sigma_3x^3 + \sigma_4x^2 + \sigma_6},$$

$$\sum_{k=1}^{6}\frac{1}{a_k(x - a_k)} = \frac{6x^4 + 4\sigma_2x^2 - 3\sigma_3x + 2\sigma_4}{x^6 + \sigma_2x^4 - \sigma_3x^3 + \sigma_4x^2 + \sigma_6},$$

$$\sum_{k=1}^{6}\frac{a_k}{x - a_k} = \frac{-2\sigma_2x^4 + 3\sigma_3x^3 - 4\sigma_4x^2 - 6\sigma_6}{x^6 + \sigma_2x^4 - \sigma_3x^3 + \sigma_4x^2 + \sigma_6}$$

and consider the next linear equation:

$$y'' - \frac{6x^5 + 4\sigma_2x^3 - 3\sigma_3x^2 + 2\sigma_4x}{x^6 + \sigma_2x^4 - \sigma_3x^3 + \sigma_4x^2 + \sigma_6}\,y' + 2\frac{6x^4 + 4\sigma_2x^2 - 3\sigma_3x + 2\sigma_4}{x^6 + \sigma_2x^4 - \sigma_3x^3 + \sigma_4x^2 + \sigma_6}\,y = 0, \tag{8}$$

where $\sigma_i,\ i = 2, 3, 4, 6$, are elementary symmetric polynomials, consisting of the elements a_k, $k = \overline{1, 6}$. Here we used the results obtained in [2], $\sigma_1 = \sigma_5 = 0$, and the notation $x \equiv u$.

Equation (8) is a homogeneous equation corresponding to equation (7) and belongs to Fuchsian type with six singularities located in the points $x = a_i, a_i \neq a_j,\ i \neq j,\ i, j = \overline{1, 6}$.

Integrating equation (8) we find a solution of equation (7) and then easily integrate equation (5) using substitution (6).

Problem. The aim of the work is to prove existence of a solution of equation (8). Also, we consider a method for constructing such a solution.

Let us consider the following procedure [6] for the linear differential equation of the second order,

$$y'' + p(x)y' + q(x)y = 0, \tag{9}$$

where $p(x)$ and $q(x)$ are analytic functions of z. Suppose we know a partial solution $y_1(x)$ for certain initial conditions $x = x_0,\ \ y(x_0) = y_0, y'(x_0) = y'_0$. We'll consider the partial solution

$$y = \xi(x)y_1 \tag{10}$$

of equation (1), which is linearly dependent on the solution y_1.

If we differentiate equality (10) along solution y_1, then we successively find

$$2\xi' y'_1 + (p\xi' + \xi'')y_1 = 0, \tag{11}$$

$$(3\xi'' - p\xi')y'_1 + (p\xi'' + p'\xi' - 2q\xi' + \xi''')y_1 = 0. \tag{12}$$

Eliminating the values $y_1(x), y'_1(x)$ from equations (11), (12) we construct a Shwarz equation for determining the function $\xi(x)$,

$$2\xi'\xi''' - 3\xi''^2 + (p^2 + 2p' - 4q)\xi'^2 = 0. \tag{13}$$

Let us put, in equation (13),

$$\xi' = \eta, \quad \eta' = w\,\eta. \tag{14}$$

Then we obtain the Riccati equation for $w(x)$,

$$2w' = w^2 - (p^2 + 2p' - 4q). \tag{15}$$

By setting

$$w = v - p$$

in (15), we get the equation

$$2v' = 4q - 2pv + v^2. \tag{16}$$

It follows from (14) – (16) that, in order to find a general solution of equation (9), it is sufficient to find a particular solution of equation (16).

For equation (16) that corresponds to equation (8), we seek a solution in the form

$$v = \frac{b_0x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5}{x^6 + \sigma_2x^4 - \sigma_3x^3 + \sigma_4x^2 + \sigma_6}, \tag{17}$$

where $b_i,\ i = \overline{0,5},\ \sigma_2, \sigma_3, \sigma_4, \sigma_6$ are unknown coefficients. Substituting (17) into (16) we find an equation of tenth degree in x. Equating the coefficients of this equation to zero we construct a system of eleven equations, which we denote by (A). The first equation in system (A) is obtained from the coefficient at the term of tenth degree and has the form

$$b_0^2 + 14b_0 + 48 = 0. \tag{18}$$

From equation (18), we find

$$b_0 = -8 \tag{19}$$

or

$$b_0 = -6.$$

Remark 1. Below we consider $b_0 = -8$, because if $b_0 = -6$, then it follows from system (A) that $b_i = 0,\ i = \overline{1,5},\ \sigma_2 = \sigma_3 = \sigma_4 = \sigma_6 = 0$. In the case $b_0 = -6$, solution (17) becomes $v \equiv -\dfrac{6}{x}$.

The second equation of system (A) becomes

$$16b_1 + 2b_0b_1 = 0.$$

Using (19) we get from the last equation that b_1 is an arbitrary constant. Let us consider a system formed by all equations of system (A) from the third to the seventh one, inclusively. We find a solution of this system for the unknowns $b_2, b_3, b_4, b_5, \sigma_6$,

$$b_2 = -\frac{1}{2}(b_1^2 + 32\sigma_2),$$

$$b_3 = \frac{1}{4}(b_1^3 + 24b_1\sigma_2 + 56\sigma_3),$$

$$b_4 = -\frac{1}{24}(3b_1^4 + 92b_1^2\sigma_2 + 512\sigma_2^2 + 96b_1\sigma_3 + 320\sigma_4), \tag{20}$$

$$b_5 = \frac{1}{48}(3b_1^5 + 112b_1^3\sigma_2 + 976b_1\sigma_2^2 + 114b_1^2\sigma_3 + 1440\sigma_2\sigma_3 + 160b_1\sigma_4),$$

$$\sigma_6 = -\frac{1}{2048}(5b_1^6 + 220b_1^4\sigma_2 + 2672b_1^2\sigma_2^2 + 6144\sigma_2^3 + 220b_1^3\sigma_3 + $$
$$+ 4032b_1\sigma_2\sigma_3 + 1728\sigma_3^2 + 304b_1^2\sigma_4 + 4096\sigma_2\sigma_4).$$

Substitute (20) into four equations of system (A) (this equation is obtained got from the coefficients of the third-zero power of x). As the result, we have four relations,

$$\frac{27}{256}b_1^7 + \frac{377}{64}b_1^5\sigma_2 + \frac{369}{64}b_1^4\sigma_3 + \frac{677}{4}b_1^2\sigma_2\sigma_3 + 32\sigma_3(27\sigma_2^2 + 7\sigma_4) +$$
$$+ \frac{1}{48}b_1^3(4987\sigma_2^2 + 375\sigma_4) + \frac{1}{12}b_1(6688\sigma_2^3 + 783\sigma_3^2 + 2080\sigma_2\sigma_4) = 0, \quad (21)$$

$$7293b_1^8 + 41292b_1^6\sigma_2 + 52812b_1^5\sigma_3 + 1692864b_1^3\sigma_2\sigma_3 +$$
$$+ 73728b_1\sigma_3(151\sigma_2^2 + 34\sigma_4) + 16b_1^4(46691\sigma_2^2 + 4383\sigma_4) + 64b_1^2(66688\sigma_2^3 +$$
$$+ 14823\sigma_3^2 + 27136\sigma_2\sigma_4) + 8192(160\sigma_2^4 + 837\sigma_2\sigma_3^2 + 416\sigma_2^2\sigma_4 + 256\sigma_4^2) = 0, \quad (22)$$

$$243b_1^9 + 16524b_1^7\sigma_2 + 14580b_1^6\sigma_3 + 575712b_1^4\sigma_2\sigma_3 +$$
$$+ 25344b_1^2\sigma_3(249\sigma_2^2 + 43\sigma_4) + 16b_1^5(25063\sigma_2^2 + 2133\sigma_4) +$$
$$+ 46080\sigma_3(416\sigma_2^3 - 27\sigma_3^2 + 160\sigma_2\sigma_4) + 128b_1^3(31843\sigma_2^3 + 1377\sigma_3^2 + 9151\sigma_2\sigma_4) +$$
$$+ 512b_1(28640\sigma_2^4 + 2241\sigma_2\sigma_3^2 + 17056\sigma_2^2\sigma_4 + 2240\sigma_4^2) = 0, \quad (23)$$

$$243b_1^{10} + 18144b_1^8\sigma_2 + 18468b_1^7\sigma_3 + 933264b_1^5\sigma_2\sigma_3 +$$
$$+ 2304b_1^3\sigma_3(6607\sigma_2^2 + 322\sigma_4) + 16b_1^6(31022\sigma_2^2 + 1269\sigma_4) +$$
$$+ 18432b_1\sigma_3(4448\sigma_2^3 - 27\sigma_3^2 + 400\sigma_2\sigma_4) + 61b_1^4(91645\sigma_2^3 + 5265\sigma_3^2 + 10753\sigma_2\sigma_4) +$$
$$+ 256b_1^2(96416\sigma_2^4 + 34641\sigma_2\sigma_3^2 + 16768\sigma_2^2\sigma_4 + 1376\sigma_4^2) -$$
$$- 98304(96\sigma_2^5 - 648\sigma_2^2\sigma_3^2 + 160\sigma_2^3\sigma_4 + 27\sigma_3^2\sigma_4 + 64\sigma_2\sigma_4^2) = 0. \quad (24)$$

Below we find a solution of system (21) – (24). We use the resultant to eliminate σ_4 from equations (21), (22) and from equations (23), (24). In the result we have two equations. From this two equations, we eliminate σ_3. After transformations we obtain the equation

$$b_1^6(3b_1^2 + 64\sigma_2)^5(9b_1^2 + 256\,\sigma_2)^4 P_1(b_1,\sigma_2)P_2(b_1,\sigma_2) = 0, \tag{25}$$

where

$$\begin{aligned} P_1(b_1,\sigma_2) \equiv\ & 30649071407558418\, b_1^{30} + 12114936173440097280\, b_1^{28}\sigma_2 + \\ & + 2195536741376273255457\, b_1^{26}\sigma_2^2 + 242545992862227840359064\, b_1^{24}\,\sigma_2^3 + \\ & + 18297161691160879588644864\, b_1^{22}\,\sigma_2^4 + 999655093084027958425843712\, b_1^{20}\,\sigma_2^5 + \\ & + 40899809155897515619716431872\, b_1^{18}\,\sigma_2^6 + 1276939511343798846743623237632\, b_1^{16}\,\sigma_2^7 + \\ & + 30689212561899625725713285382144\, b_1^{14}\,\sigma_2^8 + 567987097532735199786827071356928\, b_1^{12}\,\sigma_2^9 + \\ & + 2^{45}\cdot 3\cdot 2535107\cdot 30014232359263\, b_1^{10}\,\sigma_2^{10} + 2^{53}\cdot 19\cdot 12329\cdot 40392774440657\, b_1^{8}\,\sigma_2^{11} + \\ & + 2^{60}\cdot 3\cdot 11\cdot 883\cdot 19558829049743\, b_1^{6}\,\sigma_2^{12} + 2^{69}\cdot 3^2\cdot 654105344721161\, b_1^{4}\,\sigma_2^{13} + \\ & + 2^{77}\cdot 3^3\cdot 37\cdot 311\cdot 601\cdot 399523\, b_1^{2}\,\sigma_2^{14} + 2^{88}\cdot 3^4\cdot 5^3\cdot 7^2\cdot 11\cdot 17\cdot 19\cdot 31\,\sigma_2^{15}. \end{aligned} \tag{26}$$

Let

$$\lambda = \frac{\sigma_2}{b_1^2}.$$

Then the polynomial $P_1(b_1,\sigma_2)$ is a polynomial of 15-th degree in λ. The polynomial $P_2(b_1,\sigma_2)$ is a polynomial of 24-th degree in λ. We don't show this polynomial for its large size.

Solutions of equation (25) are

$$b_1 = 0$$

or

$$\sigma_2 = -\frac{3}{64}\,b_1^2, \tag{27}$$

or

$$\sigma_2 = -\frac{9}{256}\,b_1^2,$$

or roots of the equation

$$P_1(\lambda) = 0 \tag{28}$$

or

$$P_2(\lambda) = 0. \tag{29}$$

Cases (27) and also the case of remark 1, $b_0 = -6$, corresponde to the situation where some of six poles a_j, $j = \overline{1,6}$, coincide. Taking into account the character of the solved problem we don't consider this cases.

Let us consider equation (28). This equation does not have rational roots. Below we write the roots of this equation with the precision of twenty digits,

$$\begin{gathered}
\lambda = -0,07320121649118275031, \\
\lambda = -0,07082863896615321427 \mp 0,00019613284001135784\, I, \\
\lambda = -0,04717955463774, \\
\lambda = -0,04709244625007, \\
\lambda = -0,04686596251523 \mp 0,00016284672972\, I, \\
\lambda = -0,04035287961726453 \mp 0,00156250716732476\, I, \\
\lambda = -0,03815572721541967, \\
\lambda = -0,036772571395259496, \\
\lambda = -0,03375957925135584779 \mp 0,00599574993519825017\, I, \\
\lambda = -0,02627569670739842404\,12, \\
\lambda = -0,014306917591318076788943\,85.
\end{gathered} \tag{30}$$

The interval where the real parts of the roots (30) are located lies in the interval $[-0,074; -0,014]$.

Let us consider the second case corresponding to equation (29). This equation is an algebraic equation of 24-th degree in λ. Equation (29) does not have rational roots. Below we write the roots of this equation with the precision of twenty digits,

$$\lambda = -0,05973182469049811563,$$

$$\lambda = -0,05924753382551228670 \mp 0,00182156033988359224\, I,$$

$$\lambda = -0,05696256387908262978 \mp 0,00261736440986289830\, I,$$

$$\lambda = -0,05239705906429039678 \mp 0,00537643598666310768\, I,$$

$$\lambda = -0,04747230324687809597 \mp 0,00072094501657168285\, I,$$

$$\lambda = -0,04613932974796169946 2,$$

$$\lambda = -0,04512936129632223196 4,$$

$$\lambda = -0,04337787890350930667 \mp 0,00035738165795966455\, I,$$

$$\lambda = -0,03764480589616372658, \qquad (31)$$

$$\lambda = -0,03579279750781801977 \mp 0,00947339513257097077\, I,$$

$$\lambda = -0,02296144667645378004 \mp 0,00835932269884471527\, I,$$

$$\lambda = -0,02226493882167980806,$$

$$\lambda = 0,00498921439984309221 \mp 0,00598656611957620107\, I,$$

$$\lambda = 0,08137504883311225792,$$

$$\lambda = 0,19203841474092165564,$$

$$\lambda = 0,24383263714533381017.$$

The interval where the real parts of the roots (31) are located is in the interval $[-0,06;\ 0,25]$.

We can calculate 39 roots of equations (30), (31) an with arbitrary precision. So, we can find partial solutions (17) with an arbitrary precision.

Suppose that the precision of the calculation for solutions of equations (30), (31) is known. Suppose also that we have found the partial solution (17) of equation (16). To find a general solution of equation (5) we construct the following procedure.

Let us represent the function $v(x) - p(x)$ as a sum of six elementary fractions,

$$\frac{4x^5 + b_1x^4 + (b_2 - 4\sigma_2)x^3 + (b_3 - 3\sigma_3)x^2 + (b_4 - 2\sigma_4)x + b_5}{(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)(x-a_6)} = \sum_{i=1}^{6} \frac{k_i}{x-a_i}. \tag{32}$$

To evaluate the unknowns $k_i,\ i = \overline{1,6}$, from (32), we have the system of equations,

$$\begin{gathered}
k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = -2, \\
a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4 + a_5k_5 + a_6k_6 = b_1, \\
a_1^2k_1 + a_2^2k_2 + a_3^2k_3 + a_4^2k_4 + a_5^2k_5 + a_6^2k_6 = b_2 - 2\sigma_2, \\
a_1^3k_1 + a_2^3k_2 + a_3^3k_3 + a_4^3k_4 + a_5^3k_5 + a_6^3k_6 = b_3 - b_1\sigma_2 - 5\sigma_3, \\
a_1^4k_1 + a_2^4k_2 + a_3^4k_3 + a_4^4k_4 + a_5^4k_5 + a_6^4k_6 = b_4 + b_1\sigma_3 - (b_2 + 2\sigma_2)\sigma_2, \\
a_1^{-1}k_1 + a_2^{-1}k_2 + a_3^{-1}k_3 + a_4^{-1}k_4 + a_5^{-1}k_5 + a_6^{-1}k_6 = -b_5/\sigma_6.
\end{gathered} \tag{33}$$

Let us consider system (33) with the unknowns k_1, k_2, k_3, k_4, k_5. Using properties of the Vandermonde determinant we find a solution of this system in the form

$$\begin{aligned}
k_j = {} & \frac{1}{a_6^3 m}(a_6^4 a_i b_4 + a_6^4 a_i (a_6 + a_i) b_3 + \\
& + a_i(a_6^5 a_i + a_6^4(a_i^2 - \sigma_2) + a_6^3\sigma_3 - a_6^4\sigma_4 - \sigma_6)b_2 + \\
& + 2(a_6^5 a_i(a_i\sigma_2 - \sigma_3 - a_6^2 a_i \sigma_2\sigma_4 + a_6^4 a_i(a_i^2\sigma_2 - \sigma_2^2 - a_i\sigma_3 + \sigma_4) + \\
& + a_6^3(a_i\sigma_2\sigma_3 - \sigma_6) - a_i\sigma_2\sigma_6) - a_6k_6(2a_6^7a_i + 2a_6^6a_i^2 + 2a_6^5a_i^3 + a_6^4a_i(a_i^3 + \sigma_3) + \\
& + a_6^3a_i(a_i\sigma_3 - 2\sigma_4) - 2a_6a_i\sigma_6 - a_i^2\sigma_6 + a_6^2(\sigma_6 - a_i^2\sigma_4))), \quad j = \overline{1,5},
\end{aligned}$$

$$\begin{aligned}
k_6 = {} & \frac{1}{m}(a_6^3 b_5 + a_6^4 b_4 + a_6^5 b_3 - (a_6^4\sigma_2 - a_6^3\sigma_3 + a_6^2\sigma_4 + \sigma_6)b_2 - \\
& - a_6(a_6^4\sigma_2 - a_6^3\sigma_3 + a_6^2\sigma_4 + \sigma_4)b_1 - \\
& - 2(a_6^5\sigma_3 + a_6^4(\sigma_2^2 - \sigma_4) - a_6^3\sigma_2\sigma_3 + a_6^2(\sigma_2\sigma_4 - \sigma_6) + \sigma_2\sigma_6)),
\end{aligned} \tag{34}$$

$$m \equiv a_6^2(2a_6^2 - a_6 + a_6^3\sigma_3 - 2a_6^2\sigma_4 - 3\sigma_6).$$

Using notations (34) for known values $b_j,\ j = \overline{0,5},\ \sigma_2, \sigma_3, \sigma_4, \sigma_6$ (these values can be calculated for each of the 39 roots of equations (30), (31) showen above we find $k_i,\ i = \overline{1,6}$.

Let us set, in equation (15),

$$W = \sum_{i=1}^{6} \frac{k_i}{x - a_i} + V.$$

To find the function V, we have the following equation:

$$2V' = V^2 + 2\sum_{i=1}^{6} \frac{k_i}{x - a_i}\, V$$

from which we find that

$$V = \frac{2\prod_{i=1}^{6}(x - a_i)^{k_i}}{C_1 - \int \prod_{i=1}^{6}(x - a_i)^{k_i} dx},$$

and, consequently,

$$W = \sum_{i=1}^{6} \frac{k_i}{x - a_i} + \frac{2\prod_{i=1}^{6}(x - a_i)^{k_i}}{C_1 - \int \prod_{i=1}^{6}(x - a_i)^{k_i} dx}. \tag{35}$$

By substituting (35) into formulas (14), we find

$$\eta(x) = C_2 \frac{\prod_{i=1}^{6}(x - a_i)^{k_i}}{[C_1 - \int \prod_{i=1}^{6}(x - a_i)^{k_i} dx]^2} \tag{36}$$

and

$$\xi(x) = C_3 + C_2 \frac{1}{-C_1 + \int \prod_{i=1}^{6}(x - a_i)^{k_i} dx}, \tag{37}$$

where C_1, C_2, C_3 are arbitrary constants.

From (11) and (14) we get

$$y_1 = \exp\left(-\frac{1}{2}\int_0^w \left(p(\tau) + \frac{\eta'(\tau)}{\eta(\tau)}\right) d\tau\right)$$

or, accounting for (36), (37), we find

$$y_1(x) = \frac{1}{\sqrt{\eta}}\, e^{-\frac{1}{2}F}, \tag{38}$$

where

$$F \equiv \int p(w)\, dw, \ \ p(w) = -\frac{6x^5 + 4\sigma_2 x^3 - 3\sigma_3 x^2 + 2\sigma_4 x}{x^6 + \sigma_2 x^4 - \sigma_3 x^3 + \sigma_4 x^2 + \sigma_6}.$$

Substituting (37) and (38) into (10) we find $y(x)$. The preceding gives the following theorem.

Theorem 1. *A general solution of equation (8) has the form*

$$y(x) = \xi(x)\, y_1(x),$$

where $\xi(x)$ is determined by (37) and $y_1(x)$ – by (38).

To construct a solution of equation (7), we use the formula

$$y = \xi(w)y_1 + y_2 \int h(w)e^F\, y_1 dw - y_1 \int h(w)e^F\, y_2\, dw, \tag{39}$$

where

$$y_2 = y_1 \int \frac{e^{-F}}{{y_1}^2}\, dw,\ h(w) \ \equiv\ 2E\left(1 + \frac{1}{4}\sum_{k=1}^{6} \frac{a_k}{w - a_k}\right). \tag{40}$$

Using substitution (6) we finally obtain the next theorem.

Theorem 2. *General integral of equation (5) is*

$$\int \frac{dw}{\sqrt{y(w)}} = z + C,$$

where $y(w)$ is determined by (39), (40) and C is an arbitrary constant.

Remark 2. During the calculations we used $CAS\ \ Mathematica$.

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