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GEOMETRIC INTERPRETATION OF QUADRATIC CENTERS

ГЕОМЕТРИЧНА ІНТЕРПРЕТАЦІЯ КВАДРАТИЧНИХ ЦЕНТРІВ

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Polynomial differential equations are considered. We apply Erugin's two-isocline method for the global qualitative analysis of such equations and give a geometric interpretation of the quadratic centers.

Розглянуто поліноміальні диференціальні рівняння. Для глобального якісного аналізу таких рівнянь застосовано двоізоклінний метод Єругіна. Наведено геометричну інтерпретацію квадратичних центрів.

1. Introduction. We consider differential equations

$$\frac{dy}{dx} = \frac{Q_n(x,y)}{P_n(x,y)}, \tag{1}$$

where $P_n(x,y)$ and $Q_n(x,y)$ are polynomials of real variables $x,\ y$ with real coefficients.

It is well-known [1] that existence of a center in the polynomial equation (1) is equivalent to vanishing all Lyapunov's focus quantities of this equation calculated at the singular point having a center in the linear part of (1). Therefore, the center condition for the polynomial equation of degree n is defined by an infinite number of algebraic equations for the finite number of its coefficients. By Hilbert's basis theorem, this infinite system of algebraic equations is equivalent to a finite system, and the problem of distinguishing center and focus for the polynomial equation of degree n is to find this system.

Lyapunov's focus quantities are calculated by means of recurrence relations, and this calculation is programmed. The unsolved part of the problems is formulated as follows.

Problem. *For any degree n of the polynomial equation (1), find $N = N(n)$ such that vanishing the first N focus quantities implies existence of a center.*

For the case of $n = 2$, an unexpectedly simple answer was obtained by H. Dulac [2] and M. Frommer [3], namely: $N(2) = 3$; i. e., if the first three focus quantities vanish, the quadratic equation has a center. For the equation

$$\frac{dy}{dx} = \frac{x + Q_3(x,y)}{-y + P_3(x,y)}, \tag{2}$$

where $P_3(x,y)$ and $Q_3(x,y)$ are homogeneous polynomials of the third degree, $N = 6$ [4]. However in the general case, it is not known even if the number $N(n)$ is less than the dimension of the parameter space of the polynomial equation (1), and we do not know the geometric sense of a center for (1).

In this paper, we will give a geometric interpretation of a center in the quadratic case of the equation (1).

2. Lyapunov's focus quantities for a quadratic equation. Consider the quadratic equation

$$\frac{dy}{dx} = \frac{x + ax^2 + bxy + cy^2}{-y + mxy + ny^2} \tag{3}$$

which was studied in [5–10] and in many other works. Let us write down formulas for the first Lyapunov's quantities L_1, L_2, L_3 for a singular point at the origin [5, 10].

The first focus quantity for the equation (3) has the form

$$L_1 = \frac{1}{4}\,(b(a+c) - n(2c+m)).$$

Under the condition $L_1 = 0$,

$$\begin{aligned} L_2 = \ & \frac{1}{96}\,(4n(c-a-m)(3(a+c)^2 - n^2) + \\ + \ & 6(n(c-a) + bm)(n^2 - (a+c)^2) - \\ - \ & ((n-b)^2 - (c+m-a)^2)(n(2a+m) + b(a+c)) + \\ + \ & 2(n-b)(c+m-a)(n(b+n) + (c-a-m)(a+c) - \\ - \ & 6n(a+c)(n^2 + m^2 - b^2 - (c-a)^2 + \\ + \ & 4(a+c)(b+n)(3n^2 - (a+c)^2)). \end{aligned}$$

Under the conditions $L_1 = 0$, $L_2 = 0$,

$$\begin{aligned} L_3 = \ & -\frac{1}{2048}\,(4(n^2 + (a+c)^2)(16(a+c)((b+n)((n-b)^2 - \\ - \ & (c+m-a)^2) + 2(n-b)(m^2 - (c-a)^2)) + \\ + \ & 16n((c+m-a)((c-a)^2 - m^2 - 2(n^2 - b^2)) - \\ - \ & (n-b)^2(c-a-m)) - \\ - \ & 106(bm + cn - an)(n^2 - (a+c)^2 + \end{aligned}$$

$$+\ 106n(a+c)(m^2+n^2-b^2-(c-a)^2+$$

$$+\ 69n(c-a-m)(n^2-3(a+c)^2)+$$

$$+\ 69(a+c)(b+n)((a+c)^2-3n^2))+$$

$$+\ 5((b+n)^2+(c-a-m)^2)(3n^2(a+c)(b+n)-$$

$$-\ (a+c)^3(b+n)+n(c-a-m)(3(a+c)^2-n^2)+$$

$$+\ 2(bm+cn-an)(n^2-(a+c)^2+$$

$$+\ 2n(a+c)(b^2+(c-a)^2-m^2-n^2))).$$

By means of these Lyapunov's quantities, the following theorem can be proved (it was proved for the first time by M. Frommer).

Theorem 1 [3]. *The equation (3) has a singular point of the center type at the origin iff one of the following conditions is satisfied:*
1) $b=n=0$;
2) $b=2c+m=0$;
3) $n=a+c=0$;
4) $m=5a+3c$, $b=5n$, $2(a^2+n^2)+a\,c=0$.

3. Classification of the symmetric cases. Let us apply Erugin's two-isocline method [11] for a geometric classification of the center cases. By means of various transformations of the coordinate system (a parallel translation, a rotation and a scaling), the first (linear) approximation of a quadratic equation with a singular point of the center type can always be reduced to the form

$$\frac{dy}{dx}=-\frac{x}{y}, \tag{4}$$

i. e., we can suppose that the center is at the origin and the "inclination" of each isocline is orthogonal to its tangent at this point.

The most obvious cases of a center (their number is overwhelming, as is known) appear when the quadratic equation has a symmetry of the direction field (for example, with respect to the x-axis). We will consider first these cases.

Since in the case of a symmetry there is at least one pair of orthogonal straight lines (or one straight line) among the isoclines, the isoclines

$$P(x,y)=x+ax^2+cy^2, \tag{5}$$

$$Q(x,y)=-y+mxy \tag{6}$$

can be chosen as isoclines of “zero” and “infinity”, respectively, where without loss of generality we will suppose that $m = -1$ or $m = 0$. As a result, for the study of the quadratic equation in the case of a symmetry and with a center at the origin, it is enough to have the equation of the form

$$\frac{dy}{dx} = \frac{x + ax^2 + cy^2}{-y + mxy}, \tag{7}$$

where $m = -1$ or $m = 0$.

Begin with the most typical nondegenerate cases, when the pair of orthogonal straight lines,

$$y = 0, \quad x = -1, \tag{8}$$

is the “infinity” isocline. The second line is an integral line of the equations (7). Let this equation have four singular points. Then we should take an ellipse or a hyperbola as a “zero” isocline. There are two possibilities of intersection of the straight lines (6) with the hyperbola, when: 1) the left branch of the hyperbola passes through the origin; 2) its right branch passes through this point. Consider these cases in more detail.

1) $a < 0,\ c > 0$. It is clear that in this case the points $C_0(-1/a, 0)$, $C_1(-1, -\sqrt{(1-a)/c})$, $C_2(-1, \sqrt{(1-a)/c})$ are saddles, and the point $O(0,0)$ is a center. Depending on properties of the “zero” hyperbola-isocline, we will obtain three possible types of boundary of the domain of the center O: a) if the hyperbola

$$x + ax^2 + cy^2 = 0 \tag{9}$$

is equilateral, i. e., its basis rectangular is a square $(a = -c)$, then the asymptotes of all hyperbola-isoclines will be orthogonal; the lines forming two pairs of straight line isoclines will also be mutually orthogonal. This follows, for example, from the general equation of isoclines:

$$x - c(x^2 - y^2) + ky(x + 1) = 0. \tag{10}$$

The straight lines C_0C_1 and C_0C_2 here are integral lines, and hence the boundary of the center domain is the triangle $C_0C_1C_2$; b) if the hyperbola (5) is expanded along the x-axis (its basis rectangular is expanded: $a + c > 0$), then the isocline portrait of the equation (7) shows that the center domain is bounded by the digon formed by the segment of the straight line and by the arch of the curve C_1C_2; c) if $a + c < 0$, then the boundary of the center domain is a separatrix loop of the saddle C_0. In this case, there are three singular points (nodes) at infinity. The corresponding phase portraits are represented in Fig. 1 (cases 1 – 3).

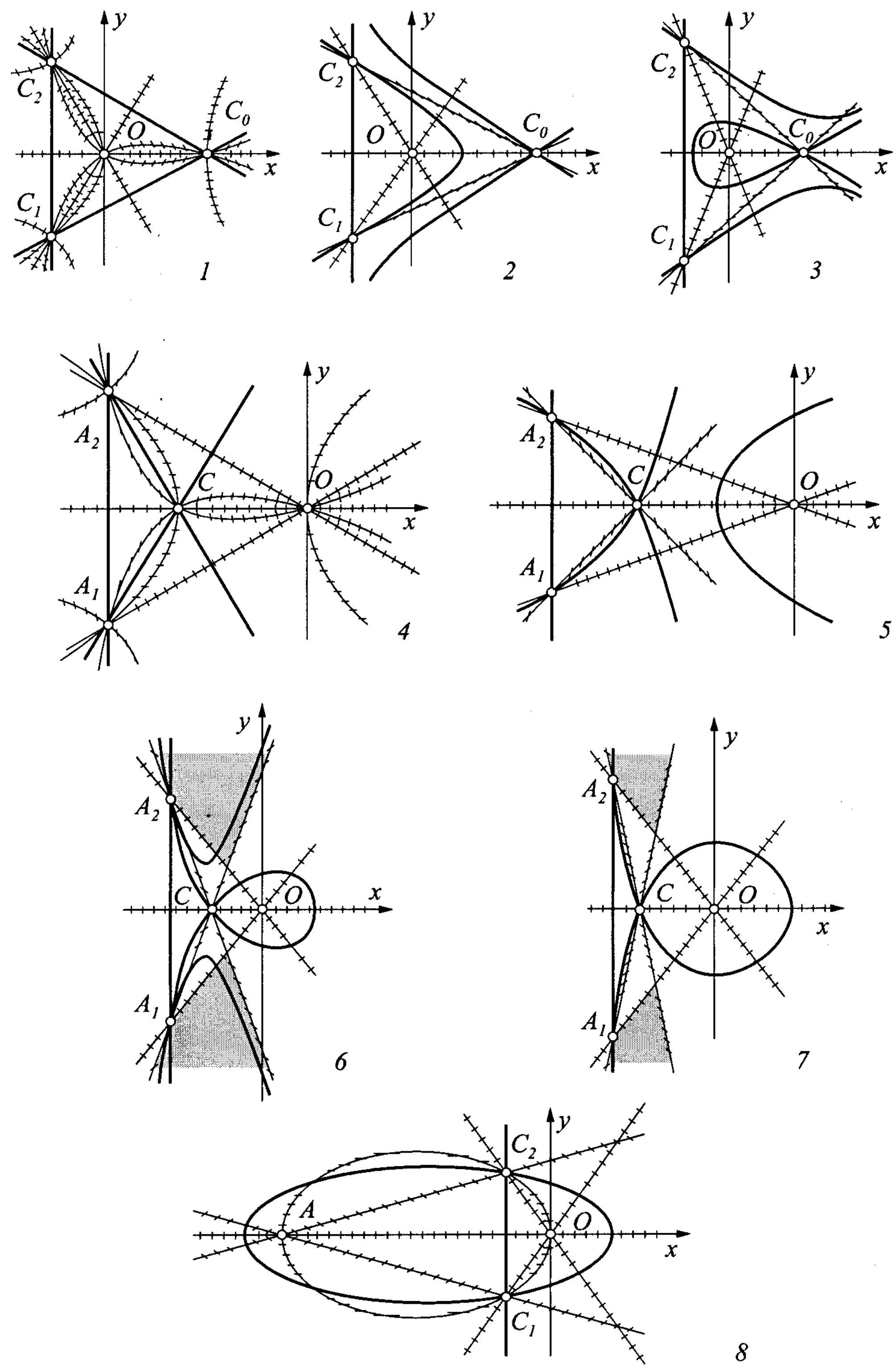

Fig. 1. Phase portraits in the case of a symmetry.

2) $a > 1, c < 0$. The point $O(0,0)$ is a center, $A_1(-1, -\sqrt{(1-a)/c})$, $A_2(-1, \sqrt{(1-a)/c})$ are nodes, and $C(-1/a, 0)$ is a saddle (see cases 4–7 in Fig. 1). The boundaries of the center domain here are the following: a) if $a + c = 0$, they are straight lines, more precisely, rays of straight lines passing through one of the nodes A_1 or A_2 and intersecting in the saddle C; b) if $a + c < 0$, the boundary of the center domain is a curve leaving to infinity. The isocline portrait shows that this curve leaves to saddles at infinity; c) if $a + c > 0$, the boundary is a separatrix loop of the saddle C. Two cases are possible here depending on how many singular points there are at infinity. If, in the shaded domain in Fig. 1 (cases 6, 7), the "inclination" of the hyperbola-isocline coincides with the inclination of its asymptote, then there are three singular points (a node and two saddles) at infinity; if there is no such coincidence, then there is only point (a saddle) at infinity. Both possibilities are realized. The equation of infinite singular points

$$u^2 = -\frac{a}{c+1}, \tag{11}$$

other than $u = \infty$, shows that (there are three singular points at infinity for $c < -1$ and only one for $-1 < c < 0$).

Consider the center case when four singular points are obtained by intersection of the straight lines (6) with an ellipse.

3) $0 < a < 1, c > 0$. There are two centers $O(0,0)$, $A(-1/a, 0)$ and two saddles $C_1(-1, -\sqrt{(1-a)/c})$, $C_2(-1, \sqrt{(1-a)/c})$ in this case. The isocline portrait of the equation (7) shows that both center domains are bounded here by the segment of the straight line C_1C_2 and the closed curve intersecting this straight line at the points C_1, C_2 (case 8 in Fig. 1).

Suppose now that an ellipse (a pair of parallel straight lines) intersects the lines (6) (more precisely, the line $y = 0$) in two singular points.

4) $a > 1, c \geq 0$ or $a < 0, c \leq 0$. The point O here is still a center, the point C is a saddle. In both cases, the center domain is bounded by a separatrix loop of the saddle C. In the first case, there is one singular point (a node) at infinity; in the second case, there are either three points (two nodes and a saddle: $c < -1$) or only one (a node: $-1 \leq c < 0$). If a hyperbola (a pair of parallel straight lines) intersects the line $y = 0$ in two points, we will also obtain two cases.

5) $-1 \leq c < 0$ and $c < -1$, where $0 < a < 1$. In the first case, the domains of the centers $O(0,0)$ and $A(-1/a, 0)$ are bounded by the straight line $x = -1$; in the second case, they are bounded by curves leaving to saddles at infinity.

The listed cases will be regarded as basic (nondegenerate). Consider intermediate (degenerate) cases. If a parabola intersects the straight lines (6) (in three points), i. e., in other words, if in case 3 the point A leaves to infinity, we will obtain a new case of a center.

6) $a = 0, c > 0$, where there is a complex singular point with an open saddle domain, a closed node domain and two adjoining node domains at infinity.

Distinguish further two cases, when the line $x = -1$ is tangent to a hyperbola or an ellipse in the point of its intersection with the line $y = 0$, i. e., when there are a simple singular point (a center) and a complex (triple) point in a finite part of the plane.

7) $a = 1, c < 0$.

8) $a = 1, c > 0$.

In case 7, three subcases can be distinguished: a) $-1/2 < c < 0$, when the center domain is bounded by a closed curve tangent the line $x = -1$ in a complex singular point having, together with a saddle domain, a closed node domain and two adjoining node domains; b) $-1 \leq c \leq -1/2$, when the closed curve turns into the line $x = -1$; c) $c < -1$, when the boundary is a curve leaving to saddles at infinity (in this case, unlike the previous ones, there are three singular points at infinity, two saddles and a node). In case 8, the the center domain is bounded by a closed curve tangent the line $x = -1$ in a complex singular point (a triple saddle).

Suppose there are two simple singular points in the finite plane and they are obtained by intersection of a hyperbola or an ellipse (a pair of parallel straight lines) with a line which is an “infinity” isocline $(m = 0)$.

9) $a > 0, c < 0$ $(a < 0, c > 0)$.

10) $a > 0, c \geq 0$ $(a < 0, c \leq 0)$.

Distinguish the following possibilities in case 9: a) $a + c = 0$, when the center domain is bounded by rays of two intersecting straight lines leaving to saddle-nodes at infinity; b) $a + c > 0$, when the boundary is a curve leaving to saddle-nodes at infinity; c) $a + c < 0$, when the center $O(0, 0)$ is bounded by a separatrix loop of the saddle $C(-1/a, 0)$. In case 10, the center domain can be bounded only by a separatrix loop of the saddle C, and there is only singular point at infinity in this case: a node (of multiplicity-5 for $c = 0$). Consider the case of one (isolated) singular point in a finite part of the plane. If $m = -1$, we obtain the following cases.

11) $a = 0, c < 0$.

12) $a = 1, c = 0$.

If $m = 0$, we have a trivial (linear) case.

13) $a = c = 0$.

Distinguish a few subcases for case 11: a) $-1/2 < c < 0$, when there is a complex singular point with open saddle and closed node domains at the “ends” of the x-axes and there is a simple saddle at the “ends” of the straight line $x = -1$; b) $-1 < c < -1/2$, when the saddle domain becomes closed and two node domains adjoin it; c) $c = -1$, when infinity is filled with singular points in all directions; d) $c < -1$, when there is a complex saddle with a closed domain at the “ends” of the x-axes and there is a simple node at the “ends” of the line $x = -1$. In case 12, the whole straight line $x = -1$ consists of singular points, and if to reduce the common cofactor $(x + 1)$ both in the numerator and in the denominator of the right-hand side of the corresponding equation (7), then the last 13th case of a center with an axial symmetry of the direction field will be obtained. The phase portraits of all these cases are given in Fig. 2.

It is clear, that the considered cases 1 – 13 do not settle all center cases for the quadratic equation (1). The most general form of the equation with a center, taking into account the reasoning mentioned above, will be the form (3), where $m = -1$ or $m = 0$. However, for many problems, it is sufficient to know only these center cases (for example, for the classification of separatrix cycles), because other cases can be considered only as possible bifurcations under the change of stability of a focus (the separatrix cycles will be determined uniquely in these cases).

4. A geometric interpretation. Let us list all possible center cases of the equation (3) [6 – 10] and give their geometric interpretation (see Fig. 2 and Fig. 3).

Theorem 2. *The equation (3) has a singular point of the center type at the origin iff one of the following geometric conditions is valid:*

1) axial symmetry of the inclination field;

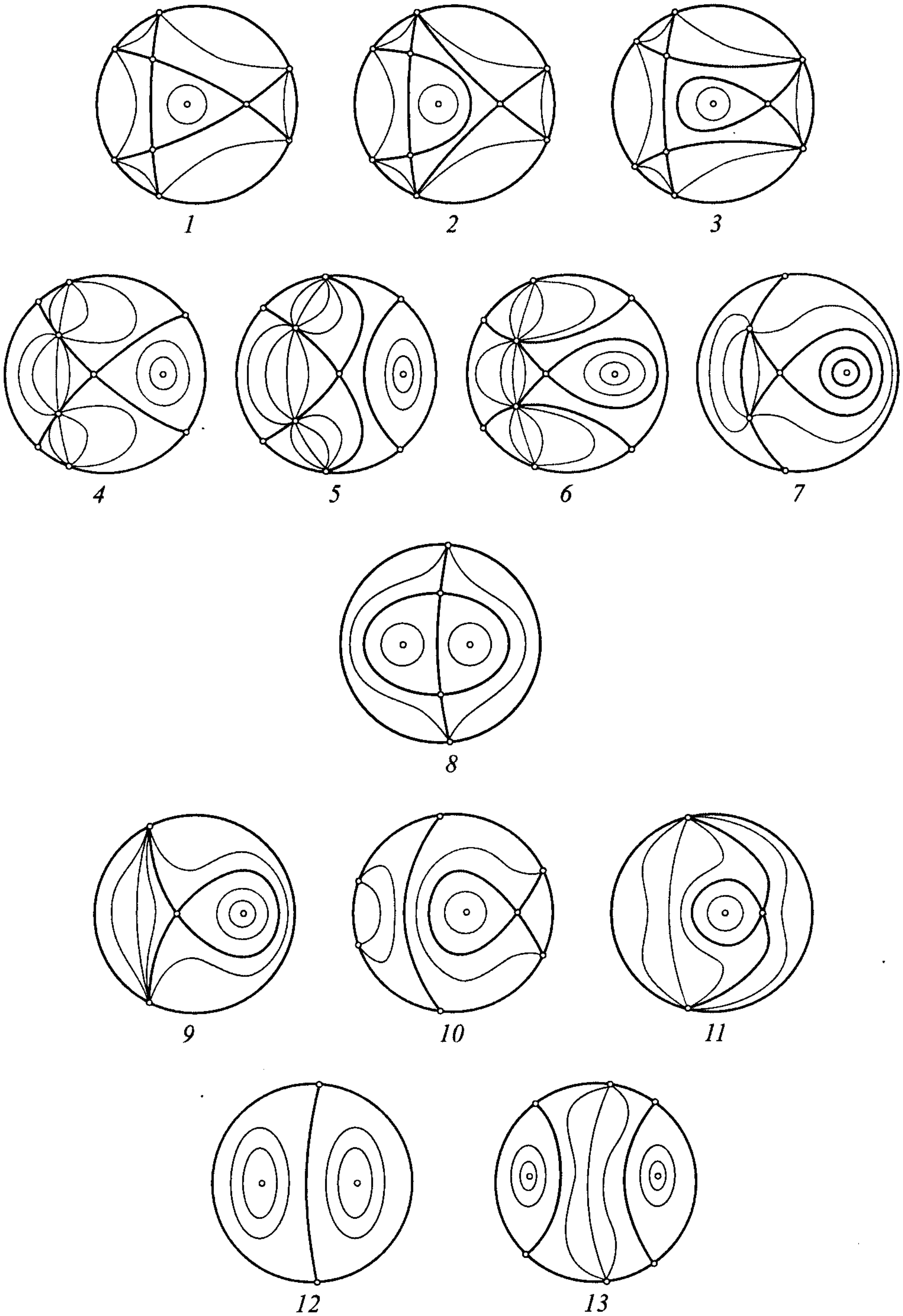

Fig. 2. A center in the case of a symmetry.

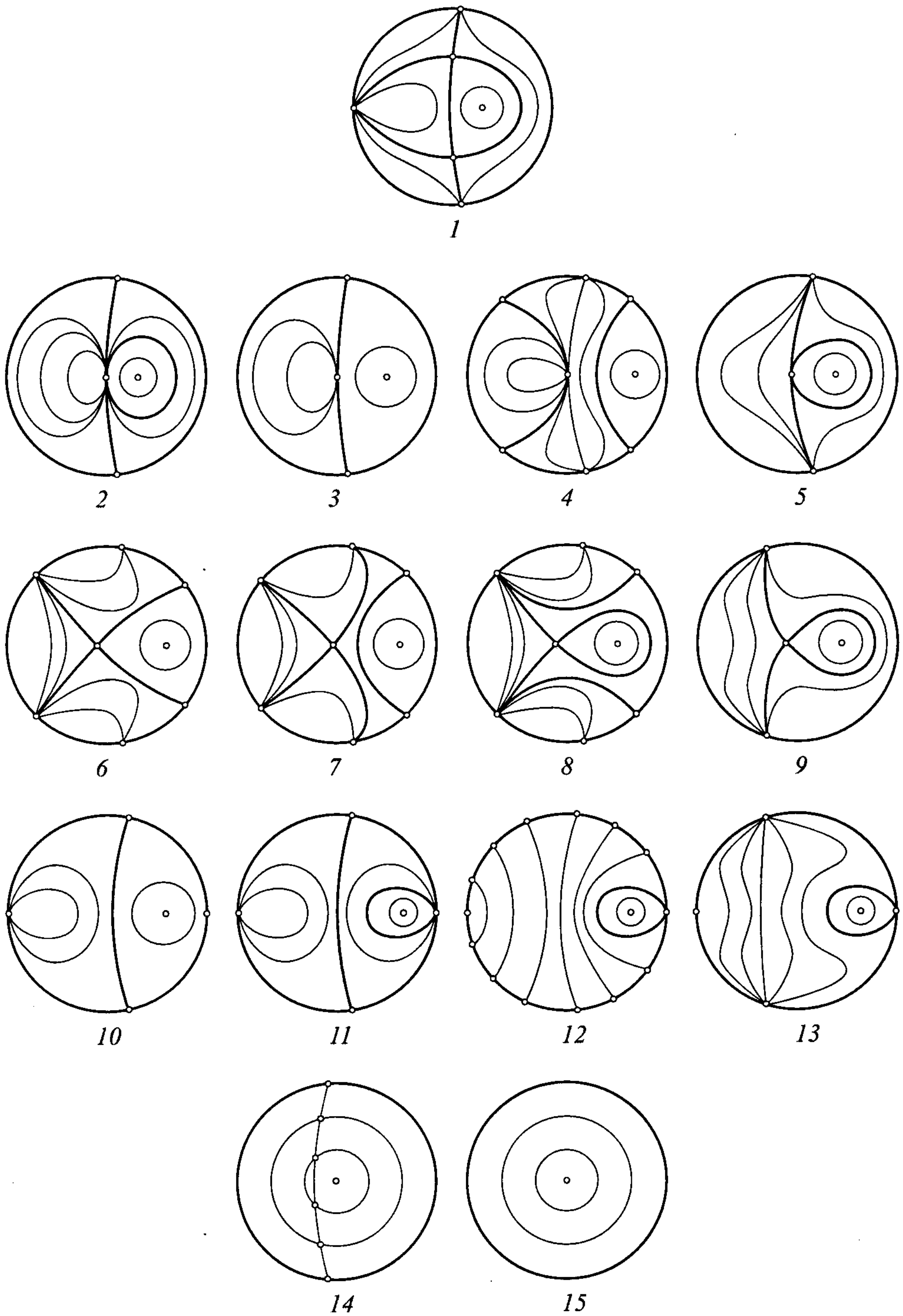

Fig. 2. A center in the case of a symmetry (continuation).

2) local symmetry (zero divergence) of the corresponding vector field;

3) orthogonality of the asymptotes of the hyperbolas (or straight lines) forming the family of isoclines;

4) orthogonality of the asymptotes of the saddles at infinity.

Proof. 1. Axial symmetry (see the equation (7)):

$$\frac{dy}{dx} = \frac{x + ax^2 + cy^2}{-y + mxy}.$$

In this case we have the x-axial symmetry of the inclination field of (7) for the function

$$F(x, y) \equiv \frac{Q_2(x, y)}{P_2(x, y)} :$$

$$F(x, -y) = -F(x, y).$$

2. Local symmetry (zero divergence) on the whole phase plane (the Hamiltonian case):

$$\frac{dy}{dx} = \frac{x + ax^2 + cy^2}{-y - 2cxy + ny^2}. \tag{12}$$

Here the divergence of the corresponding vector field

$$f(x, y) \equiv P_2(x, y)\frac{\partial}{\partial x} + Q_2(x, y)\frac{\partial}{\partial y}$$

vanishes on the whole phase plane:

$$\operatorname{div} f(x, y) = -2\,c\,y + 2\,c\,y \equiv 0.$$

3. Orthogonality of the asymptotes of the hyperbolas (or straight lines) forming the family of isoclines (the Lotka – Volterra case):

$$\frac{dy}{dx} = \frac{x + ax^2 + bxy - ay^2}{-y + mxy}. \tag{13}$$

It is easy to show that there are only hyperbolas with orthogonal asymptotes or orthogonal straight lines among the isoclines of this equation:

$$x + ky + ax^2 + (b - km)xy - ay^2 = 0, \quad k \in \mathbb{R}.$$

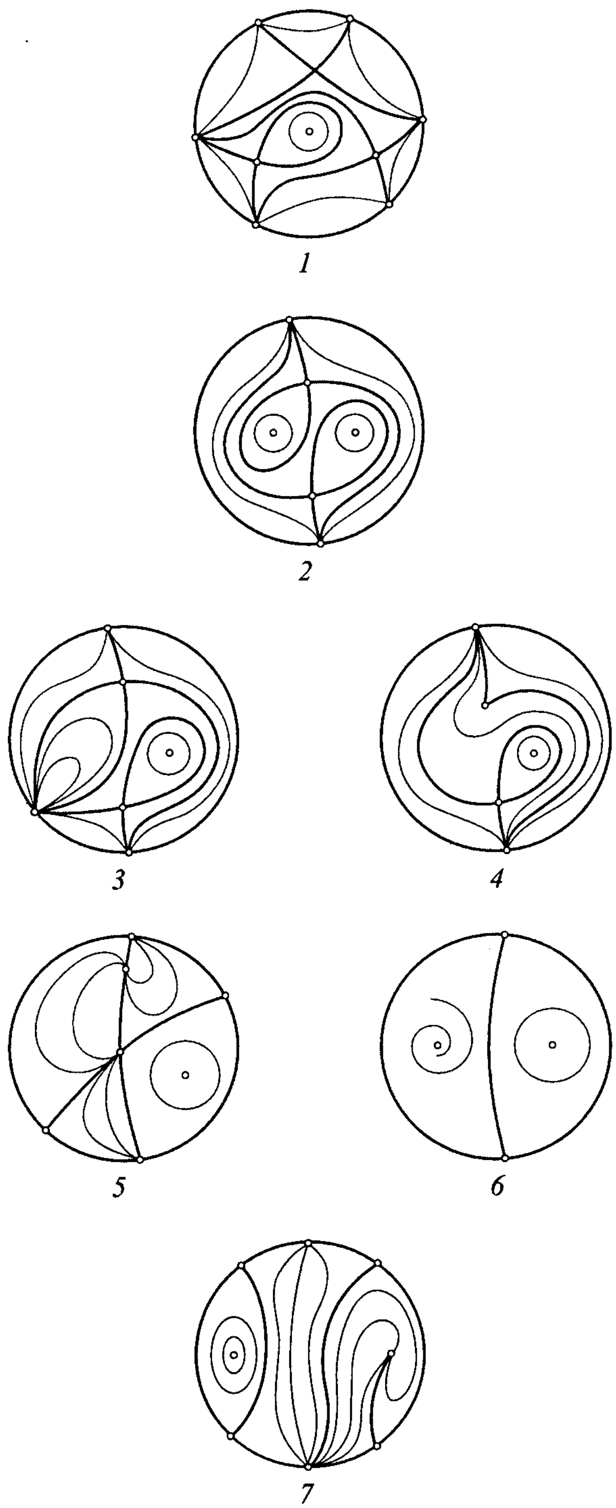

Fig. 3. Nonsymmetric cases of a center.

4. Orthogonality of the asymptotes of the saddles at infinity:

$$\frac{dy}{dx} = \frac{x - x^2 - 5nxy + 2(n^2+1)y^2}{-y + (6n^2+1)xy - ny^2}. \tag{14}$$

Here, without loss of generality, we had put the second finite singularity (which is a node for any value n) into the point $(1, 0)$. Then an infinite node lies in the direction $u = -1/n$ and two infinite saddles are determined by the equation

$$u^2 - 4nu - 1 = 0$$

which gives orthogonal directions. Theorem 2 is proved.

Applying the isocline portraits like in Fig. 1, it is easy to see that for all considered cases, the direction field of the equation (3) is rotated in such a way that the separatrix cycles of the equation will be maintained under the variation of the parameters b and n (with the corresponding compensation by the parameters $a,\ c,\ m$), namely: 1) $b = n = 0$, i. e., neither the parameter b nor n is present; 2) only the parameter n is present $(b = 0)$; 3) only the parameter is present b $(n = 0)$; 4) both parameters b and n are present. This geometric properties can be used for a geometric center criterion for an arbitrary polynomial equation (1).

5. Conclusion. Studying contact and rotation properties of the isoclines we can also construct the simplest (canonical) systems containing limit cycles.

Theorem 3 [12]. *Any quadratic system with limit* (*separatrix*) *cycles can be reduced to one of the systems either*

$$\dot{x} = -y\,(1+x) + \alpha\, Q(x,y), \quad \dot{y} = Q(x,y) \tag{15}$$

or

$$\dot{x} = -y + \nu y^2, \quad \dot{y} = Q(x,y), \quad \nu = 0, 1, \tag{16}$$

where

$$Q(x,y) = x + \lambda y + ax^2 + \beta y(1+x) + cy^2.$$

The advantage of the systems (15) and (16) is that they contain the minimal number of essential parameters and some of these parameters (α, β, λ) rotate the vector fields of (15), (16).

By means of these systems and using the center cases, we can carry out the classification of separatix cycles and can study the limit cycle bifurcations solving the most complicated problems of the qualitative investigation of the polynomial equation (1) [12, 13].

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