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HAMILTONIAN FORMULATION FOR THE MOTION OF A TWO FLUID SYSTEM WITH FREE SURFACE

ГАМІЛЬТОНІВ ПІДХІД ДО РУХУ СИСТЕМИ ДВОХ РІДИН ІЗ ВІЛЬНОЮ ПОВЕРХНЕЮ

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In this work a theoretical investigation is performed on modeling interfacial and surface waves in a layered fluid system. The physical system consists of two immiscible liquid layers of different densities $\rho_1 > \rho_2$ with an interfacial and a free surface, inside a prismatic-section tank. By using the potential formulation of the fluid motion, a nonlinear system of partial differential equations is derived applying an Hamiltonian formulation for irrotational flow of the two fluids of different density subject to conservative force. As a consequence of the assumption of potential velocity, the dynamics of the system can be described in terms of variables evaluated only at the boundary of the fluid system, namely the separation surface and the free surface. This Hamiltonian formulation permits to define the evolution equations of the system in a canonical form by using the functional derivatives.

Виконано теоретичне моделювання внутрішніх та поверхневих хвиль у шаровій системі рідин. Фізична система складається з двох рідин, що не змішуються, різних щільностей $\rho_1 > \rho_2$ з внутрішньою та вільною поверхнями в призматичному баці. На основі потенціалу руху рідини за допомогою гамільтонового підходу до безвихрової течії двох рідин різної щільності під дією консервативної сили отримано нелінійну систему диференціальних рівнянь з частинними похідними. Як наслідок припущення про наявність потенціалу швидкості динаміка системи описується за допомогою змінних, заданих лише на межі системи рідин, тобто на поверхні розподілу та вільної поверхні. Такий підхід дав можливість визначити еволюційні рівняння системи в канонічній формі за допомогою функціональних похідних.

1. Introduction. In the present paper a theoretical investigation is performed on a stratified fluid system constituted by two immiscible liquid layers inside a prismatic tank, separated by an interfacial surface and with a free surface in the upper layer.

From the theoretical point of view, several mathematical models on this physical problem have been formulated but many issues need still to be studied in depth. In many problems of applied fluid dynamics, the hypothesis of stratified fluids furnishes a more realistic description of the examined phenomenon: for instance, the determination of dimensioning parameters of the risers of floating offshore platform operating in deep water, where an accurate knowledge of properties of internal and surface waves is necessary.

Interfacial waves dynamics has been largely illustrated in scientific literature.

The experimental work of Kalinichenko [1] on a two layer liquid analyzes the parametric instability of interfacial waves, while Kalinichenko et al. [2] experimentally study the velocity field. The parametric instability of interfacial waves was also analyzed by Benielli and Sommeria [3], both from an experimental and theoretical point of view. Recently, the properties of solitary interfacial and internal waves received attention. In particular, Grue et al. [4, 5] studied both experimentally and theoretically such kind of waves.

From the numerical point of view, there is a huge literary production. It is interesting to mention the work of Michallet and Dias [6], focused on the solitary, interfacial waves, and the work of Wright et al. [7], focused on the numerical study of two-dimensional oscillations of a two layer liquid.

From the analytical-numerical point of view, linear or weakly nonlinear analysis is performed on two layer fluids, expanding the unknown functions, describing kinematic and dynamic quantities, by means of suitable function spaces (Kumar and Tuckermann [8], King and McCready [9]). Perturbative techniques have been and are applied to study both surface and interfacial waves. In particular, concerning the interfacial waves, in the works of Choi et al. [10, 11] perturbative techniques are applied to Euler equation. Among the analytical approaches to the study of the interfacial waves dynamics, variational methods with Hamiltonian formulation for the motion equations have recently obtained great attention, as in the works of Benjamin and Bridges [12], Berning and Rubenchik [13], Craig and Groves [14], Ambrosi [15].

In fact the presence of a separation surface in stratified fluids implies the existence of a constraint which forces the evolution of the system to respect the corresponding physical condition. In particular, the components of the velocity field normal to the separation surface must be equal in each layer. From a mathematical point of view, this constraint is defined by a nonevolutive (in the time sense) condition between the velocities at the interface of the two immiscible fluids. By using a mathematical model which does not implicitly take into account this nonevolutive condition, the evolution of the physical assigned system requires a particular numerical technique, in order to impose the fulfillment of the nonevolutive condition at any instant (LaRocca et al. [16]). When the hypothesis of potential motion is sufficiently realistic, mathematical models which automatically take into account the nonevolutive condition can be constructed. In fact, from the assumption of potential velocities, it follows that the motion of the fluid is completely determined from the motion of the boundary. Moreover, Zakharov [17] showed that the water elevation and the potential at the free surface can be used as canonical variables for free surface wave problems described by an Hamiltonian formalism.

In this paper, following the approach of Ambrosi [15], applied to a bounded fluid domain, it is shown that the elevations of the separation surface and the free surface, the jump of momentum potential density evaluated at the fluid interface and the momentum potential density evaluated at the upper free surface can be used as canonical variables. An accurate description of the method of transforming the classical hydrodynamic problem into the Hamiltonian formulation is furnished. The detailed analysis permits to have all the tools necessary for the implementation of an operative mathematical model.

2. Formulation of the hydrodynamic problem. The problem concerns a fluid system constituted by two immiscible perfect fluids of different densities $\rho_1 > \rho_2$ subjected to the gravity force. These are inside of a prismatic tank, separated by an interface $z = \eta_1(x, y, t)$ and with a free surface $z = \eta_2(x, y, t)$ where the pressure is constant. The *x*-*y*-*z* system of coordinates is attached to the tank whose domains is defined by $0 \leq x \leq B$, $0 \leq y \leq L$, $-H_1 \leq z < \infty$ where z defines the vertical direction. At last, the separation surface is at $z = 0$ while the free surface is at $z = H_2$.

The hydrodynamic problem (neglecting surface tension effects) consists in finding two velocity potential functions $\Phi_1(x, y, z, t), \Phi_2(x, y, z, t)$ which satisfy the Laplace equation in the domains $D_1 = \{0 \leq x \leq B, 0 \leq y \leq L, -H_1 \leq z < \eta_1\}$ and $D_2 = \{0 \leq x \leq B, 0 \leq y \leq \leq L, \eta_1 \leq z < \rho_1 > \rho_2 < \eta_2\}$, respectively.

Then the following problem has to be solved:

$$\nabla^2 \Phi_1 = 0, \qquad \left.\frac{\partial \Phi_1}{\partial n}\right|_{\text{on rigid boundaries}} = 0, \tag{1}$$

$$\nabla^2 \Phi_2 = 0, \qquad \left.\frac{\partial \Phi_2}{\partial n}\right|_{\text{on rigid boundaries}} = 0, \tag{2}$$

with the kinematic conditions

$$\left.\frac{\partial \eta_1}{\partial t} + \nabla \eta_1 \cdot \nabla \Phi_1\right|_{z=\eta_1} = 0, \tag{3}$$

$$\left.\frac{\partial \eta_1}{\partial t} + \nabla \eta_1 \cdot \nabla \Phi_2\right|_{z=\eta_1} = 0, \tag{4}$$

$$\left.\frac{\partial \eta_2}{\partial t} + \nabla \eta_2 \cdot \nabla \Phi_2\right|_{z=\eta_2} = 0, \tag{5}$$

and the pressure condition which, in terms of Bernoulli integrals, furnishes

$$\left[\rho_1\left(\frac{\partial \Phi_1}{\partial t} + \frac{1}{2}\nabla \Phi_1 \cdot \nabla \Phi_1 + g\eta_1\right) = \rho_2\left(\frac{\partial \Phi_2}{\partial t} + \frac{1}{2}\nabla \Phi_2 \cdot \nabla \Phi_2 + g\eta_1\right)\right]_{z=\eta_1}$$

$$\left[\frac{\partial \Phi_2}{\partial t} + \frac{1}{2}\nabla \Phi_2 \cdot \nabla \Phi_2 + g\left(\eta_2 - H_2\right) = 0\right]_{z=\eta_2}$$

where g is the gravity acceleration.

From Condition 3 and 4, we obtain the following the nonevolutive condition:

$$\left[\nabla \eta_1 \cdot \left(\nabla \Phi_1 - \nabla \Phi_2\right)\right]_{z=\eta_1} = 0 \tag{6}$$

which imposes the equality between the velocity components of the two fluids normal to the separation surface (La Rocca et al. [16]).

3. Hamiltonian formulation. In the previous treatment of the problem, a classical fluid dynamic framework has been given. Now, following the approach of Ambrosi [15] and Craig et al. [14] the problem is reformulated introducing an Hamiltonian function defined by the sum of kinetic and potential energy of the fluid system and expressed (as shown in the following) in terms of four variables evaluated only at the moving boundaries.

Two of these variables define the spatial configuration of the fluid system while the other two can be considered as the natural conjugate momenta. For all these variables it is possible to define a set of canonical Hamiltonian equations in a pure evolutive form avoiding the treatment of the nonevolutive Condition 6 which is automatically taken into account by the Hamiltonian formulation of the problem.

The Hamiltonian function is the total energy,

$$H = U + E,$$

where U is the potential energy and E the kinetic energy. It follows that

$$U = \int\limits_{D_1} \rho_1 g z d\tau + \int\limits_{D_2} \rho_2 g z d\tau = \int\limits_{A} dA \left[\int\limits_{-H_1}^{\eta_1} \rho_1 g z dz + \int\limits_{\eta_1}^{\eta_2} \rho_2 g z \, dz \right] =$$

$$= \frac{1}{2} \int\limits_{A} dA \left[\rho_1 g \eta_1^2 + \rho_2 g \left(\eta_2^2 - \eta_1^2 \right) \right],$$

where

$$\int\limits_{A} \cdot dA \equiv \int\limits_{0}^{B} \int\limits_{0}^{L} \cdot dx dy.$$

The kinetic energy E can be expressed as

$$E = \frac{1}{2} \int\limits_{D_1} \rho_1 \nabla\Phi_1 \cdot \nabla\Phi_1 d\tau + \frac{1}{2} \int\limits_{D_2} \rho_2 \nabla\Phi_2 \cdot \nabla\Phi_2 d\tau.$$

Due to the harmonicity of Φ_1 and Φ_2,by using the divergence theorem, the homogeneous Neumann condition on the rigid walls and the nonevolutive Condition 6, it follows after some calculations that

$$E = -\frac{1}{2} \int\limits_{A} \left[(\rho_1 \Phi_1 - \rho_2 \Phi_2) \frac{\partial \Phi_1}{\partial n_1} \right]_{z=\eta_1} \sqrt{1 + (\partial_x \eta_1)^2 + (\partial_y \eta_1)^2} dA -$$

$$- \frac{1}{2} \int\limits_{A} \left[\rho_2 \Phi_2 \frac{\partial \Phi_2}{\partial n_2} \right]_{z=\eta_2} \sqrt{1 + (\partial_x \eta_2)^2 + (\partial_y \eta_2)^2} dA,$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are, respectively, the inward normal vectors to the separation surface and the free surface relatively to the domains D_1 and D_2.

Introducing the new variables $\xi \equiv [(\rho_1\Phi_1 - \rho_2\Phi_2)]_{z=\eta_1}$, $\psi \equiv [\rho_2\Phi_2]_{z=\eta_2}$ it is possible to write the Hamiltonian function in terms only of the four canonical variables $\eta_1, \eta_2, \xi, \psi$ (Ambrosi [15]), namely $H = H(\eta_1, \eta_2, \xi, \psi)$, provided that $\dfrac{\partial \Phi_1}{\partial n_1}$ and $\dfrac{\partial \Phi_2}{\partial n_2}$ are expressible in terms of the same variables. How to do this is shown in the next section.

3.1. Mapping of boundary data into normal derivatives. First of all, two mathematical problems are defined as follows.

In the first one, in the D_1 domain the elliptic boundary-value problem

$$\nabla^2\Phi_1 = 0,$$

$$\Phi_1|_{z=\eta_1} = \chi,$$

$$\left.\frac{\partial \Phi_1}{\partial n}\right|_{\text{on rigid boundaries}} = 0,$$

shows clearly that to the boundary data χ there is associable the harmonic function Φ_1 and, consequently, its derivative $\left.\dfrac{\partial \Phi_1}{\partial n_1}\right|_{z=\eta_1}$.

The operator which realizes this transformation is called Dirichlet – Neumann operator for the domain D_1. Clearly this operator depends nonlinearly on η_1 but its action on the boundary data is linear. Then, denoting by G_1 this operator, we will use the notation

$$G_1\chi = \left.\frac{\partial \Phi_1}{\partial n_1}\right|_{z=\eta_1}.$$

In the second problem, in the domain D_2 the elliptic boundary-value problem

$$\nabla^2\Phi_2 = 0,$$

$$\Phi_2|_{z=\eta_1} = \varphi_1,$$

$$\Phi_2|_{z=\eta_2} = \varphi_2,$$

$$\left.\frac{\partial \Phi_2}{\partial n}\right|_{\text{on rigid boundaries}} = 0$$

shows that the boundary data φ_1, φ_2 is associable with the harmonic function Φ_2 and, consequently, with its derivatives $\left.\dfrac{\partial \Phi_2}{\partial n_1^-}\right|_{z=\eta_1}$, $\left.\dfrac{\partial \Phi_2}{\partial n_2}\right|_{z=\eta_2}$, where $\mathbf{n}_1^- = -\mathbf{n}_1$. As before, introducing the operators

$F_1(\varphi_1, \varphi_2)$, $F_2(\varphi_1, \varphi_2)$ it follows that

$$F_1(\varphi_1, \varphi_2) = \left.\frac{\partial \Phi_2}{\partial n_1^-}\right|_{z=\eta_1}, \qquad F_2(\varphi_1, \varphi_2) = \left.\frac{\partial \Phi_2}{\partial n_2}\right|_{z=\eta_2}.$$

It is useful to split the operator F_1 into the sum of two operators which linearly act on φ_1, φ_2, respectively,

$$F_1(\varphi_1, \varphi_2) = L_1\varphi_1 + L_2\varphi_2,$$

where $L_1\varphi_1 \equiv F_1(\varphi_1, 0)$, $L_2\varphi_2 \equiv F_1(0, \varphi_2)$, as a consequence of linearity of the Laplace equation. For the sake of simplicity, the nonlinear dependence of G_1, F_1, F_2 on the η_1 and η_2 is omitted.

The last step in defining the Hamiltonian in terms of the canonical variables η_1, η_2, ξ, ψ consists in transforming the dependence of the normal derivatives in terms of these canonical variables.

Putting $\widetilde{\Phi}_1 \equiv \Phi_1|_{z=\eta_1}$, $\widetilde{\Phi}_2 \equiv \Phi_2|_{z=\eta_1}$, $\widetilde{\widetilde{\Phi}}_2 \equiv \Phi_2|_{z=\eta_2}$ from the nonevolutive Condition 6 we have

$$\left.\frac{\partial \Phi_1}{\partial n_1}\right|_{z=\eta_1} + \left.\frac{\partial \Phi_2}{\partial n_1^-}\right|_{z=\eta_1} = G_1\widetilde{\Phi}_1 + L_1\widetilde{\Phi}_2 + L_2\widetilde{\widetilde{\Phi}}_2 = 0, \tag{7}$$

and then, remembering that $\xi \equiv \rho_1\widetilde{\Phi}_1 - \rho_2\widetilde{\Phi}_2, \psi \equiv \rho_2\widetilde{\widetilde{\Phi}}_2$ and using eq. 7 we get

$$L_1\xi = \rho_1 L_1\widetilde{\Phi}_1 - \rho_2 L_1\widetilde{\Phi}_2 = \rho_1 L_1\widetilde{\Phi}_1 + \rho_2\left(G_1\widetilde{\Phi}_1 + L_1\widetilde{\Phi}_2\right)$$

this obtaining

$$L_1\xi = (\rho_1 L_1 + \rho_2 G_1)\widetilde{\Phi}_1 + L_2\left(\rho_2\widetilde{\widetilde{\Phi}}_2\right),$$

$$\widetilde{\Phi}_1 = (\rho_1 L_1 + \rho_2 G_1)^{-1}(L_1\xi - L_2\psi).$$

Finally,

$$\left.\frac{\partial \Phi_1}{\partial n_1}\right|_{z=\eta_1} = G_1(\rho_1 L_1 + \rho_2 G_1)^{-1}(L_1\xi - L_2\psi). \tag{8}$$

In the same manner,

$$G_1\xi = \rho_1 G_1\widetilde{\Phi}_1 - \rho_2 G_1\widetilde{\Phi}_2 = \rho_1\left(-L_1\widetilde{\Phi}_2 - L_2\widetilde{\widetilde{\Phi}}_2\right) - \rho_2 G_1\widetilde{\Phi}_2,$$

$$(-\rho_1 L_1 - \rho_2 G_1)\widetilde{\Phi}_2 = G_1\xi + \frac{\rho_1}{\rho_2}L_2\psi.$$

Then

$$\widetilde{\Phi}_2 = -\left(\rho_1 L_1 + \rho_2 G_1\right)^{-1}\left(G_1\xi + \frac{\rho_1}{\rho_2} L_2 \psi\right)$$

and, finally,

$$\left.\frac{\partial \Phi_2}{\partial n_2}\right|_{z=\eta_2} = F_2(\widetilde{\Phi}_2, \widetilde{\widetilde{\Phi}}_2) = F_2\left(\left(-\rho_1 L_1 - \rho_2 G_1\right)^{-1}\left(G_1\xi + \frac{\rho_1}{\rho_2} L_2\psi\right), \frac{\psi}{\rho_2}\right). \tag{9}$$

So the Hamiltonian function is defined in terms of the four canonical variables $\eta_1, \eta_2, \xi, \psi$ and can be expressed by the following relation ($\mathcal{H}(\eta_1, \eta_2, \xi, \psi)$ being the Hamiltonian density):

$$\begin{aligned} H(\eta_1, \eta_2, \xi, \psi) = \int\limits_A dA\, \mathcal{H}(\eta_1, \eta_2, \xi, \psi) = \frac{1}{2}\int\limits_A dA \Big[\rho_1 g \eta_1^2 + \rho_2 g\left(\eta_2^2 - \eta_1^2\right) - \\ - \xi G_1 \left(\rho_1 L_1 + \rho_2 G_1\right)^{-1}\left(L_1\xi - L_2\psi\right)\sqrt{1 + (\partial_x \eta_1)^2 + (\partial_y \eta_1)^2} - \\ - \psi F_2\left(-\left(\rho_1 L_1 + \rho_2 G_1\right)^{-1}\left(G_1\xi + \frac{\rho_1}{\rho_2} L_2\psi\right), \frac{\psi}{\rho_2}\right)\sqrt{1 + (\partial_x \eta_2)^2 + (\partial_y \eta_2)^2}\Big]. \end{aligned} \tag{10}$$

By using the functional derivatives (Goldstein [18]) of the Hamiltonian functional in terms of the canonical variables, the following Hamilton equations hold (Ambrosi [15], Craig et al. [14]):

$$\begin{aligned} \frac{\partial \eta_1}{\partial t} &= \frac{\delta H}{\delta \xi}, \qquad & \frac{\partial \xi}{\partial t} &= -\frac{\delta H}{\delta \eta_1}, \\ \frac{\partial \eta_2}{\partial t} &= \frac{\delta H}{\delta \psi}, \qquad & \frac{\partial \psi}{\partial t} &= -\frac{\delta H}{\delta \eta_2}. \end{aligned} \tag{11}$$

As can be seen, this system is a purely evolutive differential system which takes automatically into account the nonevolutive Condition 6, used in deriving the system itself.

From an operator point of view, taking into account the necessity to approximate both the operators (G_1, F_1, F_2) and the unknowns $\eta_1, \eta_2, \xi, \psi$, it is useful to write variational principle a equivalent to the (11) (Goldstein [18]),

$$\delta \int\limits_{t_1}^{t_2} dt \int\limits_A dA(\dot{\eta}_1 \xi + \dot{\eta}_2 \psi - \mathcal{H}(\eta_1, \eta_2, \xi, \psi)) = 0, \tag{12}$$

where $\dot{(\#)} \equiv \partial(\#)/\partial t$. This form is particularly useful when an expansion of the canonical variables $\eta_1, \eta_2, \xi, \psi$ in terms of unknown coefficients time-depending with a finite number of modes is adopted.

3.2. Conclusion. In this work a theoretical investigation is performed on modeling interfacial and surface waves in a layered fluid system. The physical system consists of two immiscible liquid layers of different densities $\rho_1 > \rho_2$ with an interfacial and a free surface, inside a prismatic-section tank. By using the potential formulation of the fluid motion, a nonlinear systems of partial differential equations is derived applying an Hamiltonian formulation for irrotational flow of the two fluids of different density subject to conservative force.

This Hamiltonian formulation permits to define the evolution equations of the system in a canonical form by using the functional derivatives.

An accurate and detailed description of the nature of the Neumann – Dirichlet operators necessary to implementation of an operative mathematical model is furnished.

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