UDC 517.9

OSCILLATION RESULTS FOR FOURTH ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

ПРО ОСЦИЛЯЦІЮ ДЛЯ НЕЛІНІЙНОГО ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ЧЕТВЕРТОГО ПОРЯДКУ НЕЙТРАЛЬНОГО ТИПУ З ДОДАТНИМИ ТА ВІД'ЄМНИМИ КОЕФІЦІЄНТАМИ

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Unbounded oscillation and asymptotic behaviour of a class of nonlinear fourth order neutral differential equations with positive and negative coefficients of the form

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = 0$$

and

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = f(t) \qquad (E)$$

are investigated under the assumption

$$\int_0^\infty \frac{t}{r(t)}\,dt < \infty$$

for various ranges of $p(t)$. Sufficient conditions are obtained for the existence of positive bounded solutions of (E).

Досліджено необмежену осциляцію та асимптотичну поведінку класу нелінійних диференціальних рівнянь четвертого порядку нейтрального типу у вигляді

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = 0 \qquad (1)$$

та

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = f(t) \qquad (E)$$

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з додатними та від'ємними коефіцієнтами за умови

$$\int_0^\infty \frac{t}{r(t)}\,dt < \infty$$

для різних областей значень $p(t)$. *Отримано достатні умови існування додатних обмежених розв'язків рівняння* (E).

1. Introduction. In the last few years, there has been an increasing interest in the study of oscillatory behaviour of solutions of neutral delay differential equations with positive and negative coefficients of first and second order, see, for example, [1, 4–8, 10]. However, very little work [10] is available on the study of oscillatory and asymptotic behaviour of solutions of fourth order equations which is due to the technical difficulties arising in its analysis.

In this paper, we consider a class of nonlinear fourth order neutral delay differential equations of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')''+q(t)G(y(t-\alpha))-h(t)H(y(t-\beta)) = 0 \tag{1.1}$$

and

$$(r(t)(y(t)+p(t)y(t-\tau))'')''+q(t)G(y(t-\alpha))-h(t)H(y(t-\beta)) = f(t), \tag{1.2}$$

where r, q and h are continuous and positive on $[0,\infty)$, $p \in C([0,\infty),\mathbb{R})$, $f \in C([0,\infty),\mathbb{R})$, G, $H \in C(\mathbb{R},\mathbb{R})$ with $uG(u) > 0$, $vH(v) > 0$, for $u, v \neq 0$, H is bounded, G is nondecreasing and $\tau, \alpha, \beta > 0$ are constants.

The main objective of this work is to study the oscillatory and asymptotic behaviour of solutions of (1.1) and (1.2), under the assumption

$$\int_0^\infty \frac{t}{r(t)}\,dt < \infty. \tag{A_1}$$

If $h(t) \equiv 0$, then (1.1) and (1.2) reduce to

$$(r(t)(y(t)+p(t)y(t-\tau))'')''+q(t)G(y(t-\alpha)) = 0 \tag{1.3}$$

and

$$(r(t)(y(t)+p(t)y(t-\tau))'')''+q(t)G(y(t-\alpha)) = f(t) \tag{1.4}$$

respectively. In [9], Parhi and Tripathy have studied (1.3) and (1.4), under the assumption (A_1). If $h(t) \not\equiv 0$, then nothing is known about the behaviour of solutions of (1.1)/(1.2). Therefore, an attempt is made here to study (1.1) and (1.2) under the same assumption in addition to

$$\int_0^\infty \frac{s}{r(s)} \int_s^\infty th(t)\,dt\,ds < \infty. \tag{A_2}$$

Because (1.1)/(1.2) is more general than (1.3)/(1.4), it is worth studying. Not only the present work is more illustrative than [9], but also some of the results are generalized and improved.

By a solution of (1.1)/(1.2) we understand a function $y \in C([-\rho, \infty), \mathbb{R})$ such that $(y(t) + +p(t)y(t-\tau))$ is twice continuously differentiable, $(r(t)(y(t) + p(t)y(t-\tau))'')$ is twice continuously differentiable and (1.1)/(1.2) is satisfied for $t \geq 0$, where $\rho = \max\{\tau, \alpha, \beta\}$ and

$$\sup\{|y(t)| : t \geq t_0\} > 0 \quad \text{for every} \quad t \geq t_0.$$

A solution $y(t)$ of (1.1)/(1.2) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. Some preparatory results. To study the nonlinear functional differential equations of the type (1.1)/(1.2), we need the following results for our use in the sequel.

Lemma 2.1 [9]. *Let (A_1) hold. If $u(t)$ is an eventually positive twice continuously differentiable function such that $r(t)u''(t)$ is twice continuously differentiable and $(r(t)u''(t))'' \leq 0, \not\equiv 0$ for large t, where $r \in C([0, \infty), (0, \infty))$, then one of the following cases holds for large t :*

(a) $u'(t) > 0, u''(t) > 0$ and $(r(t)u''(t))' > 0$,
(b) $u'(t) > 0, u''(t) < 0$ and $(r(t)u''(t))' > 0$,
(c) $u'(t) > 0, u''(t) < 0$ and $(r(t)u''(t))' < 0$,
(d) $u'(t) < 0, u''(t) > 0$ and $(r(t)u''(t))' > 0$.

Lemma 2.2 [9]. *Suppose that the conditions of Lemma 2.1 hold. Then (i) the following inequalities hold for large t in the case (c) of Lemma 2.1:*

$$u'(t) \geq -(r(t)u''(t))'R(t), \quad u'(t) \geq -r(t)u''(t)\int\limits_t^\infty \frac{ds}{r(s)},$$

$$u(t) \geq ktu'(t) \quad \text{and} \quad u(t) \geq -k(r(t)u''(t))'tR(t),$$

where $k > 0$ is a constant and $R(t) = \displaystyle\int_t^\infty \frac{s-t}{r(s)}\, ds$ and

(ii) $u(t) \geq r(t)u''(t)R(t)$ for large t in case (d) of Lemma 2.1.

Lemma 2.3 [9]. *If the conditions of Lemma 2.1 hold, then there exist constants $k_1 > 0$ and $k_2 > 0$ such that $k_1R(t) \leq u(t) \leq k_2t$ for large t.*

Lemma 2.4 [9]. *Let (A_1) hold. Suppose that $z(t)$ be a real valued twice continuously differentiable function on $[0, \infty)$, such that $r(t)z''(t)$ is twice continuously differentiable with $(r(t)z''(t))'' \leq \leq 0, \not\equiv 0$ for large t. If $z(t) > 0$ eventually, then one of the following cases holds for large t :*

(a) $z'(t) > 0, z''(t) > 0$ and $(r(t)z''(t))' > 0$,
(b) $z'(t) > 0, z''(t) < 0$ and $(r(t)z''(t))' > 0$,
(c) $z'(t) > 0, z''(t) < 0$ and $(r(t)z''(t))' < 0$,
(d) $z'(t) < 0, z''(t) > 0$ and $(r(t)z''(t))' > 0$.

If $z(t) < 0$ for large t, then either one of the cases (b) – (d) holds or one of the following cases holds for large t :

(e) $z'(t) < 0, z''(t) < 0$ and $(r(t)z''(t))' > 0$,
(f) $z'(t) < 0, z''(t) < 0$ and $(r(t)z''(t))' < 0$.

Lemma 2.5 [3]. *Let* $p,\ y,\ z \in C([0,\infty),\mathbb{R})$ *be such that* $z(t) = y(t) + p(t)y(t-\tau)$, *for* $t \geq \tau \geq 0$, $y(t) > 0$ *for* $t \geq t_1 > \tau$, $\liminf_{t\to\infty} y(t) = 0$ *and* $\lim_{t\to\infty} z(t) = L$ *exists. Let* $p(t)$ *satisfy one of the following conditions:*

$$(i)\ 0 \leq p(t) \leq p_1 < 1, \quad (ii)\ 1 < p_2 \leq p(t) \leq p_3, \quad (iii)\ p_4 \leq p(t) \leq 0,$$

where p_i *is a constant,* $1 \leq i \leq 4$. *Then* $L = 0$.

3. Oscillation results for (1.1). In this section, sufficient conditions are established for unbounded oscillation and asymptotic behaviour of solutions of (1.1) under the assumption (A_1). For our purpose, we need the following assumptions:

(A_3) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u+v)$ for $u, v > 0$, $u, v \in \mathbb{R}$,

(A_4) $G(uv) = G(u)G(v)$, $u, v \in \mathbb{R}$,

(A_5) $G(-u) = -G(u)$, $H(-u) = -H(u)$, $u \in \mathbb{R}$,

(A_6) $\displaystyle\int_{\tau}^{\infty} Q(t)\,d(t) = \infty$, $Q(t) = \min\{q(t), q(t-\tau)\}$, $t \geq \tau$,

(A_7) $\displaystyle\int_{t_0}^{\infty} b(t)Q(t)G(R(t-\alpha))\,dt = \infty$, where $b(t) = \min\{R^\gamma(t), R^\gamma(t-\tau)\}$, $\gamma > 1$, $t_0 \geq \rho > 0$,

(A_8) $\displaystyle\int_{t_0}^{\infty} R^\gamma(t)G(R(t-\alpha))q(t)\,dt = \infty$, $\gamma > 1$, $t_0 \geq \rho > 0$.

Remark 3.1. Since $R(t) < \displaystyle\int_t^\infty \frac{s}{r(s)}\,ds$, we have that $R(t) \to 0$ as $t \to \infty$ in view of (A_1).

Remark 3.2. (A_4) implies that $G(-u) = -G(u)$. Indeed, $G(1)G(1) = G(1)$ and $G(1) > 0$ imply that $G(1) = 1$. Further, $G(-1)G(-1) = G(1) = 1$ implies that $(G(-1))^2 = 1$. Since $G(-1) < 0$ it follows that $G(-1) = -1$. Hence $G(-u) = G(-1)G(-u) = -G(u)$. On the other hand, $G(uv) = G(u)G(v)$ for $u > 0$, $v > 0$, and $G(-u) = -G(u)$ imply that $G(xy) = G(x)G(y)$ for every $x, y \in \mathbb{R}$.

Remark 3.3. The prototype of G satisfying (A_3), (A_4) and (A_5) is

$$G(u) = (a + b|u|^\gamma)|u|^\mu \operatorname{Sgn} u,$$

where $a \geq 0$, $b > 0$, $\gamma \geq 0$ and $\mu \geq 0$ such that $a + b = 1$.

Theorem 3.1. *Let* $0 \leq p(t) \leq a < 1$ *or* $1 < p(t) \leq a < \infty$. *Suppose that* $(A_1)-(A_7)$ *hold. Then every solution of (1.1) either oscillates or tends to zero as* $t \to \infty$.

Proof. Due to Remark 3.1, $b(t) \to 0$ as $t \to \infty$. Hence (A_7) implies that

$$\int_{t_0}^{\infty} Q(t)G(R(t-\alpha))\,dt = \infty. \tag{3.1}$$

Assume that $y(t)$ is a nonoscillatory solution of (1.1). Then $y(t) > 0$ or $y(t) < 0$ for $t \geq t_0 > \rho$. Let $y(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = y(t) + p(t)y(t-\tau), \tag{3.2}$$

$$K(t) = \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (\theta - s)h(\theta)H(y(\theta-\beta))\,d\theta\,ds, \tag{3.3}$$

and

$$w(t) = z(t) - K(t) = y(t) + p(t)y(t-\tau) - K(t), \tag{3.4}$$

we obtain

$$(r(t)w''(t))'' = -q(t)G(y(t-\alpha)) \le 0, \quad \not\equiv 0 \tag{3.5}$$

for $t \ge t_0 + \alpha$. Consequently, $w(t)$, $w'(t)$, $(r(t)w''(t))$, $(r(t)w''(t))'$ are monotonic on $[t_1, \infty)$, $t_1 \ge t_0 + \alpha$. In what follows, we have two cases, viz. $w(t) > 0$ or < 0 for $t \ge t_1$. Suppose the former holds. By the Lemma 2.1, any one of the cases (a), (b), (c) and (d) holds. Suppose that any one of the cases (a), (b) and (d) holds. Upon using (A_3), (A_4) and (A_6), Eq. (1.1) can be viewed as

$$0 = (r(t)w''(t))'' + q(t)G(y(t-\alpha)) + G(a)(r(t-\tau)w''(t-\tau))'' +$$
$$+ G(a)q(t-\tau)G(y(t-\tau-\alpha)) \ge (r(t)w''(t))'' + G(a)(r(t-\tau)w''(t-\tau))'' +$$
$$+ \lambda Q(t)G(y(t-\alpha) + ay(t-\alpha-\tau)) \ge (r(t)w''(t))'' +$$
$$+ G(a)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(z(t-\alpha))$$

for $t \ge t_2 > t_1$, where we have used the fact that $z(t) \le y(t) + ay(t-\tau)$. From (3.3), it follows that $K(t) > 0$, $K'(t) < 0$, and hence $\lim_{t\to\infty} K(t)$ exists due to (A_2). Further, $w(t) > 0$ for $t \ge t_1$ implies that $w(t) < z(t)$ for $t \ge t_2$ and thus the last inequality yields

$$(r(t)w''(t))'' + G(a)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(w(t-\alpha)) \le 0,$$

for $t \ge t_2$, that is,

$$(r(t)w''(t))'' + G(a)(r(t-\tau)w''(t-\tau))'' + \lambda G(k_1)Q(t)G(R(t-\alpha)) \le 0$$

due to (A_4) and Lemma 2.3, for $t \ge t_3 > t_2$. Integrating the above inequality from t_3 to ∞, we get

$$\lambda G(k_1) \int_{t_3}^\infty Q(t)G(R(t-\alpha))\,dt < \infty,$$

a contradiction to (3.1). Next, we suppose that the case (c) holds. Upon using Lemmas 2.2 and 2.3, we have

$$k(-r(t)w''(t))' tR(t) \le w(t) \le k_2 t$$

for $t \ge t_4 > t_3$. Hence

$$-[((-r(t)w''(t))')^{1-\gamma}]' = (\gamma-1)((-r(t)w''(t))')^{-\gamma}(-r(t)w''(t))'' \ge$$
$$\ge (\gamma-1)L^\gamma R^\gamma(t)q(t)G(y(t-\alpha)), \tag{3.6}$$

where $L = \dfrac{k}{k_2} > 0$. Therefore,

$$-[((-r(t)w''(t))')^{1-\gamma}]' - G(a)[((-r(t-\tau)w''(t-\tau))')^{1-\gamma}]' \geq$$
$$\geq (\gamma-1)L^{\gamma}[R^{\gamma}(t)q(t)G(y(t-\alpha)) + G(a)R^{\gamma}(t-\tau)q(t-\tau)G(y(t-\tau-\alpha))] \geq$$
$$\geq \lambda(\gamma-1)L^{\gamma}b(t)Q(t)G(z(t-\alpha)) \geq \lambda(\gamma-1)L^{\gamma}b(t)Q(t)G(w(t-\alpha)) \geq$$
$$\geq \lambda(\gamma-1)L^{\gamma}G(k_1)b(t)Q(t)G(R(t-\alpha))$$

implies that

$$\lambda(\gamma-1)L^{\gamma}G(k_1)\int_{t_4}^{\infty} b(t)Q(t)G(R(t-\alpha))\,dt < \infty,$$

which contradicts (A_7). Hence the latter holds. Consequently, $z(t) < K(t)$ and $K(t)$ is bounded will imply that $y(t)$ is bounded. From Lemma 2.4, it follows that any one of the cases (b) – (f) holds for $t \geq t_2 > t_1$. In the cases (e) and (f) of Lemma 2.4, $\lim_{t\to\infty} w(t) = -\infty$ which contradicts the fact that $y(t)$ is bounded and $\lim_{t\to\infty} w(t)$ exists. Consider the case (b) or (c), where $-\infty < \lim_{t\to\infty} w(t) \leq 0$. Consequently,

$$0 \geq \lim_{t\to\infty} w(t) = \limsup_{t\to\infty}[z(t) - K(t)] \geq \limsup_{t\to\infty}[y(t) - K(t)] \geq$$
$$\geq \limsup_{t\to\infty} y(t) - \lim_{t\to\infty} K(t) = \limsup_{t\to\infty} y(t)$$

implies that $\lim_{t\to\infty} y(t) = 0$. We may note that $\lim_{t\to\infty} K(t) = 0$. Lastly, let the case (d) of Lemma 2.4 hold. Then $\lim_{t\to\infty}(r(t)w''(t))'$ exists. Hence integrating (3.5) from t_2 to ∞, we obtain

$$\int_{t_2}^{\infty} q(t)G(y(t-\alpha))\,dt < \infty,$$

that is,

$$\int_{t_2}^{\infty} Q(t)G(y(t-\alpha))\,dt < \infty. \tag{3.7}$$

If $\liminf_{t\to\infty} y(t) > 0$, then (3.7) yields,

$$\int_{t_2}^{\infty} Q(t)\,dt < \infty,$$

which contradicts (A_6) due to Remark 3.1. Hence $\liminf_{t\to\infty} y(t) = 0$. Because $\lim_{t\to\infty} w(t)$ exists, using Lemma 2.5, $\lim_{t\to\infty} w(t) = 0 = \lim_{t\to\infty} z(t)$. Moreover, $z(t) \geq y(t)$ implies that $\lim_{t\to\infty} y(t) = 0$.

If $y(t) < 0$ for $t \geq t_0$, then we set $x(t) = -y(t)$ for $t \geq t_0$ and

$$(r(t)(x(t) + p(t)x(t-\tau))'')'' + q(t)G(x(t-\alpha)) - h(t)H(x(t-\beta)) = 0.$$

Proceeding as above, we obtain similar conclusion.

Theorem 3.1 is proved.

Remark 3.4. From Theorem 3.1, it revels that $y(t)$ is bounded in the case $w(t) < 0$ for $t \geq t_1$, which further converges to zero as $t \to \infty$. However, this fact is not required in the other case. Hence we have proved the following theorem.

Theorem 3.2. *Let $0 \leq p(t) \leq a < \infty$. Suppose that $(A_1)-(A_7)$ hold, then every unbounded solution of (1.1) oscillates.*

Theorem 3.3. *Let $0 \leq p(t) \leq a < 1$. If (A_1), (A_2), (A_4), (A_5) and (A_8) hold, then every unbounded solution of (1.1) oscillates.*

Proof. Since $R(t) \to 0$ as $t \to \infty$, (A_8) implies that

$$\int_{t_0}^{\infty} G(R(t-\alpha))q(t)\,dt = \infty \tag{3.8}$$

and hence

$$\int_{t_0}^{\infty} q(t)\,dt = \infty. \tag{3.9}$$

Let $y(t)$ be a nonoscillatory solution of (1.1) such that $y(t)$ is unbounded and $y(t) > 0$ for $t \geq t_0 > 0$. The case $y(t) < 0$ for $t \geq t_0 > 0$ is similar. We set $z(t)$, $K(t)$ and $w(t)$ as in (3.2), (3.3) and (3.4) respectively to obtain (3.5) for $t \geq t_0 + \alpha$. Consequently, each of $w(t)$, $w'(t)$, $(r(t)w''(t))$ and $(r(t)w''(t))'$ is of constant sign on $[t_1, \infty)$, $t_1 \geq t_0 + \alpha$. Assume that $w(t) > 0$ for $t \geq t_1$. Then Lemma 2.1 holds. If any one of the cases (a) or (b) holds, then $0 < w'(t) = z'(t) - K'(t)$ implies that $z'(t) > 0$ or < 0 for $t \geq t_1$. We note that $z(t)$ is unbounded due to unbounded $y(t)$. Hence $z'(t) < 0$ doesn't arise. Ultimately, $z'(t) > 0$ and

$$(1-p(t))z(t) < z(t) - p(t)z(t-\tau) = y(t) - p(t)p(t-\tau)y(t-2\tau) < y(t),$$

that is,

$$y(t) > (1-a)z(t) > (1-a)w(t)$$

for $t \geq t_2 > t_1$. Thus (3.5) yields

$$G((1-a)w(t-\alpha))q(t) \leq -(r(t)w''(t))'',$$

that is,

$$G(k_1(1-a))G(R(t-\alpha))q(t) \leq -(r(t)w''(t))'' \tag{3.10}$$

due to Lemma 2.3 and (A_4). Integrating (3.10) from t_2 to ∞, it follows that

$$\int_{t_2}^{\infty} q(t)G(R(t-\alpha))\,dt < \infty,$$

a contradiction to (3.8). For the case (c) of Lemma 2.1, we proceed as in the proof of Theorem 3.1 to obtain (3.6). Using the same type of reasoning as above, (3.6) yields that

$$-[((-r(t)w''(t))')^{1-\gamma}]' \geq (\gamma-1)L^{\gamma}G((1-a)k_1)R^{\gamma}(t)q(t)G(R(t-\alpha))$$

for $t \geq t_2$. Integrating the last inequality from t_2 to ∞, we obtain,

$$\int_{t_2}^{\infty} q(t)R^{\gamma}(t)G(R(t-\alpha))\,dt < \infty,$$

a contradiction to (A_8). In the case (d) of Lemma 2.1, $\lim_{t\to\infty} w(t)$ exists, that is, $\lim_{t\to\infty} z(t)$ exists, a contradiction to our hypothesis. Due to Remark 3.4, the case $w(t) < 0$ doesn't arise.

Theorem 3.3 is proved.

Theorem 3.4. *Let $-1 < a \leq p(t) \leq 0$. If (A_1), (A_2), (A_5) and (A_8) hold, then every solution of (1.1) either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) such that $y(t) > 0$ for $t \geq t_0 > 0$. Setting $z(t)$, $K(t)$ and $w(t)$ as in (3.2), (3.3) and (3.4) we obtain (3.5) for $t \geq t_0 + \alpha$ and hence $w(t)$ is monotone on $[t_1, \infty)$, $t_1 \geq t_0 + \alpha$. Let $w(t) > 0$ for $t \geq t_1$. Suppose that one of the cases (a), (b) and (d) of Lemma 2.1 holds for $t \geq t_1$. From Lemma 2.3, we have $y(t) \geq w(t) \geq k_1 R(t)$ for $t \geq t_2 > t_1$ and hence (3.5) yields

$$\int_{t_3}^{\infty} q(t)G(R(t-\alpha))\,dt < \infty, \quad t_3 > t_2 + \alpha,$$

a contradiction to (3.8). Next we consider the case (c). Proceeding as in the proof of Theorem 3.1, we obtain (3.6). Further, $y(t) \geq w(t) \geq k_1 R(t)$ for $t \geq t_2$ by Lemma 2.3. Consequently, for $t \geq t_3 > t_2 + \alpha$,

$$-[((-r(t)w''(t))')^{1-\gamma}]' \geq (\gamma-1)L^{\gamma}G(k_1)R^{\gamma}(t)q(t)G(R(t-\alpha)).$$

Integrating the above inequality from t_3 to ∞, we get

$$\int_{t_3}^{\infty} q(t)R^{\gamma}(t)G(R(t-\alpha))\,dt < \infty,$$

a contradiction to (A_8).

If $w(t) < 0$ for $t \geq t_1$, then $y(t)$ is bounded ultimately. Hence $z(t)$ is bounded and so also $w(t)$. In what follows, none of the cases (e) and (f) of Lemma 2.4 arises. In the case (b) or (c), $-\infty < \lim_{t\to\infty} w(t) \leq 0$. Using the fact that $\lim_{t\to\infty} K(t) = 0$, we have $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$. Hence

$$0 \geq \lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) = \limsup_{t\to\infty}[y(t) + p(t)y(t-\tau)] \geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(ay(t-\tau)) =$$

$$= \limsup_{t\to\infty} y(t) + a\limsup_{t\to\infty} y(t-\tau) = (1+a)\limsup_{t\to\infty} y(t)$$

implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$. Let the case (d) hold. Since

$$\lim_{t\to\infty}(r(t)w''(t))'$$

exists, (3.5) yields that

$$\int_{t_2}^{\infty} q(t)G(y(t-\alpha))\,dt < \infty. \tag{3.11}$$

If $\liminf_{t\to\infty} y(t) > 0$, then it follows from (3.11) that

$$\int_{t_2}^{\infty} q(t)\,dt < \infty,$$

which contradicts (3.9). Hence $\liminf_{t\to\infty} y(t) = 0$. Using Lemma 2.5, we assert that

$$\lim_{t\to\infty} w(t) = 0 = \lim_{t\to\infty} z(t).$$

Proceeding as above, we may show that $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$.

If $y(t) < 0$ for $t \geq t_0$, then one may proceed as above to obtain $\liminf_{t\to\infty} y(t) = 0$, that is $\lim_{t\to\infty} y(t) = 0$.

Thorem 3.4 is proved.

Theorem 3.5. *Let* $-\infty < p(t) \leq 0$. *If* (A_1), (A_2), (A_5) *and* (A_8) *hold, then every unbounded solution of (1.1) is oscillatory.*

The proof of the theorem follows from the proof of Theorem 3.4. Hence the details are omitted.

4. Oscillation results for (1.2). This section is concerned with the oscillation and asymptotic behaviour of solutions of (1.2) with suitable forcing functions. We restrict our forcing functions which are allowed to change the sign eventually. Let the following hypotheses hold concerning the forcing function $f(t)$ of (1.2):

(A_9) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $F(t)$ changes sign with $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$, $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$.

(A_{10}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $F(t)$ changes sign with $\liminf_{t\to\infty} F(t) = -\infty$, $\limsup_{t\to\infty} F(t) = +\infty$, $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$.

Theorem 4.1. *Let $0 \le p(t) \le a < \infty$. Assume that (A_1), (A_2), (A_3), (A_4), (A_5) and (A_{10}) hold. If*

$$\int_{\alpha}^{\infty} b(t)Q(t)G(F^+(t-\alpha))\,dt = \infty = \int_{\alpha}^{\infty} b(t)Q(t)G(F^-(t-\alpha))dt, \tag{A_{11}}$$

where $F^+(t) = \max\{0, F(t)\}$ and $F^-(t) = \max\{-F(t), 0\}$, then every solution of (1.2) oscillates.

Proof. Suppose on the contrary that $y(t)$ is a nonoscillatory solution of (1.2) such that $y(t) > 0$ for $t \ge t_0 > \rho$. Setting as in (3.2), (3.3) and (3.4), let

$$V(t) = w(t) - F(t) = z(t) - K(t) - F(t). \tag{4.1}$$

Hence for $t \ge t_0 + \alpha$, Eq. (1.2) becomes

$$(r(t)V''(t))'' = -q(t)G(y(t-\alpha)) \le 0, \not\equiv 0. \tag{4.2}$$

Consequently, $V(t)$ is monotone on $[t_1,\infty)$, $t_1 > t_0+\alpha$. Let $V(t) > 0$ for $t \ge t_1$. Then $z(t) - K(t) > F(t)$ implies that $z(t) - K(t) > 0$ due to (A_{10}) and hence $z(t) - K(t) > \max\{0, F(t)\} = F^+(t)$ for $t \ge t_1$, that is

$$z(t) > K(t) + F^+(t) > F^+(t). \tag{4.3}$$

In view of Eq. (1.2), it is easy to verify that

$$\begin{aligned} 0 &= (r(t)V''(t))'' + q(t)G(y(t-\alpha)) + G(a)(r(t-\tau)V''(t-\tau))'' + \\ &+ G(a)q(t-\tau)G(y(t-\alpha-\tau)) \ge (r(t)V''(t))'' + \\ &+ G(a)(r(t-\tau)V''(t-\tau))'' + \lambda Q(t)G(z(t-\alpha)) \end{aligned}$$

due to (A_3) and (A_4). Using (4.3), the last inequality yields

$$0 \ge (r(t)V''(t))'' + G(a)(r(t-\tau)V''(t-\tau))'' + \lambda Q(t)G(F^+(t-\alpha)), \tag{4.4}$$

for $t \ge t_2 > t_1$. Assume that one of the cases (a), (b) and (d) of Lemma 2.1 holds. Then integrating (4.4) from $t_2+\alpha$ to ∞, we obtain

$$\int_{t_2+\alpha}^{\infty} Q(t)G(F^+(t-\alpha))dt < \infty,$$

a contradiction to (A_{11}). We may note that $b(t) \to 0$ as $t \to \infty$ due to Remark 3.1. Consider the case (c) of Lemma 2.1. From Lemma 2.3 it follows that

$$k(-r(t)V''(t))'tR(t) \le V(t) \le k_2 t, \quad t \ge t_3 > t_2.$$

Hence in view of (3.6), we have

$$-[((-r(t)V''(t))')^{1-\gamma}]' \geq (\gamma-1)L^{\gamma}R^{\gamma}(t)q(t)G(y(t-\alpha)),$$

for $t \geq t_3$. Proceeding as in Theorem 3.1, we obtain

$$\lambda(\gamma-1)L^{\gamma}G(k_1)\int_{t_4}^{\infty} b(t)q(t)G(F^+(t-\alpha))\,dt < \infty, \quad t_4 > t_3,$$

which contradicts (A_{11}). Consequently, $V(t) < 0$ for $t \geq t_1$. Thus any one of the cases (b)–(f) of Lemma 2.4 holds. If $V(t) < 0$, $z(t) - K(t) < 0$ ultimately due to (A_{10}). In what follows, $z(t)$ is bounded and so also $y(t)$. Therefore, $\lim_{t\to\infty} V(t)$ exists. Since $z(t) = V(t) + K(t) + F(t)$, we have

$$0 \leq \liminf_{t\to\infty} z(t) = \liminf_{t\to\infty}(V(t)+K(t)+F(t)) \leq$$

$$\leq \limsup_{t\to\infty} V(t) + \liminf_{t\to\infty}(K(t)+F(t)) \leq$$

$$\leq \lim_{t\to\infty} V(t) + \limsup_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) =$$

$$= \lim_{t\to\infty} V(t) + \lim_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \to -\infty,$$

which is absurd.

If $y(t) < 0$ for $t \geq t_0$, we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$ and

$$(r(t)(x(t)+p(t)x(t-\tau))'')'' + q(t)G(x(t-\alpha)) - h(t)H(x(t-\beta)) = \tilde{f}(t)$$

due to (A_5) where $\tilde{f}(t) = -f(t)$. If we set $\tilde{F}(t) = -F(t)$, then $\tilde{F}(t)$ changes sign. Further, $\tilde{F}^+(t) = F^-(t)$ and $(r(t)\tilde{F}''(t))'' = \tilde{f}(t)$. Proceeding as above we obtain a contradiction.

Theorem 4.1 is proved.

Remark 4.1. In Theorem 4.1, $V(t) < 0$ implies that $z(t)$ and $y(t)$ are bounded simultaneously. This fact is unlikely true due to our assumption (A_{10}). If (A_9) is replaced by (A_{10}), then bounded $y(t)$ doesn't provide any conclusion about the oscillatory behaviour of the solutions of (1.2). Hence with unbounded $y(t)$, we have proved the following theorem:

Theorem 4.2. *Let $0 \leq p(t) \leq a < \infty$. If (A_1) –(A_5), (A_9) and (A_{11}) hold, then every unbounded solution of (1.2) is oscillatory.*

Theorem 4.3. *Let $-1 < p(t) \leq 0$. Suppose that (A_1), (A_2), (A_5) and (A_{10}) hold. If*

$$\int_{\alpha}^{\infty} R^{\gamma}(t)q(t)G(F^+(t-\alpha))\,dt = \infty = \int_{\alpha}^{\infty} R^{\gamma}(t)q(t)G(F^-(t-\alpha))\,dt \quad \gamma > 1, \qquad (A_{12})$$

then (1.2) is oscillatory.

Proof. For the sake of contradiction, let $y(t)$ be a nonoscillatory solution of (1.2) such that $y(t) > 0$ for $t \geq t_0 > \rho$. The case $y(t) < 0$ can be similarly dealt with. For $t \geq t_1 > t_0$, $y(\alpha(t)) > 0$ and $y(\beta(t)) > 0$. Let's set $V(t)$ as in (4.1), so that we get (4.2). Consequently, $V(t)$ is monotone on $[t_1, \infty)$. Let $V(t) > 0$ for $t \geq t_1$. Then one of the cases (a) – (d) of Lemma 2.1 holds. Indeed, $V(t) > 0$, that is $z(t) - K(t) > F(t)$ implies that $z(t) - K(t) > 0$ due to (A_{10}). Hence (4.3) holds. Further, $z(t) - K(t) > 0$ yields that $z(t) > K(t) > 0$. Thus

$$y(t) > z(t) > K(t) + F^+(t) > F^+(t) \tag{4.5}$$

for $t \geq t_2 > t_1$. If any one of the cases (a), (b) and (d) holds, then using (4.5) in (4.2), we obtain

$$\int_{t_3}^{\infty} q(t)G(F^+(t-\alpha))\,dt < \infty, \quad t_3 > t_2 + \alpha,$$

a contradiction to (A_{12}). Assume that case (c) holds. Proceeding as in Theorem 3.4 and upon using (4.5) in (3.6), we get

$$-[((-r(t)V''(t))')^{1-\gamma}]' \geq (\gamma - 1)L^\gamma G(k_1)R^\gamma(t)q(t)G(F^+(t-\alpha)), \tag{4.6}$$

for $t \geq t_2 > t_1$. Integrating (4.6) from t_2 to ∞, we obtain a contradiction to (A_{12}).

Next, we suppose that $V(t) < 0$ for $t \geq t_1$. Then one of the cases (b) – (f) of Lemma 2.4 holds. Indeed, $z(t) - K(t) < F(t)$ implies that $z(t) - K(t) < 0$ ultimately, due to (A_{10}). Thus $z(t)$ is bounded. Since $V(t)$ is monotone, $\lim_{t\to\infty} V(t)$ exists. Therefore, $z(t) < K(t) + F(t)$ implies that

$$\liminf_{t\to\infty} z(t) < \liminf_{t\to\infty}(K(t) + F(t)) \leq \limsup_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) =$$

$$= \lim_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \to -\infty,$$

which is absurd.

Theorem 4.3 is proved.

Theorem 4.4. *Let $-1 < b \leq p(t) \leq 0$. If (A_1), (A_2), (A_5), (A_9), and (A_{12}) hold, then every unbounded solution of (1.2) oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (1.2) such that $y(t) > 0$ for $t \geq t_0 > \rho$. The case when $y(t) < 0$ for $t \geq t_0 > \rho$ is similar. Proceeding as in the proof of Theorem 4.3, we have the required contradiction when $V(t) > 0$ for $t \geq t_1$.

Next, we suppose that $V(t) < 0$ for $t \geq t_1$. As a result, $z(t) - K(t) < 0$ due to (A_9). Ultimately, we have two cases on $z(t)$, viz. $z(t) > 0$ or $z(t) < 0$. If the former holds, and since $y(t)$ is unbounded, then there exists $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$, $y(\eta_n) \to \infty$ as $n \to \infty$ and

$$y(\eta_n) = \max\{y(t) : t_1 \leq t \leq \eta_n\}.$$

We may choose n large enough such that $\eta_n - \tau > t_1$. Hence it happens that

$$z(\eta_n) \geq (1 + b)y(\eta_n).$$

By Lemma 2.4, any one of the cases (b) – (f) holds. Assume that either case (b) or (c) holds true. Then $\lim_{t\to\infty}|V(t)| < \infty$ and $z(t) = V(t) + K(t) + F(t)$ implies that

$$\infty = (1+b)\lim_{n\to\infty} y(\eta_n) \le \lim_{n\to\infty}[|V(\eta_n)| + K(\eta_n) + |F(\eta_n)|] < \infty, \tag{4.7}$$

which is absurd. Suppose that any of the cases (d), (e) and (f) holds. For each of the cases $V(t)$ is nonincreasing. Let $\lim_{t\to\infty} V(t) = \mu$, $\mu \in [-\infty, 0)$. If $-\infty < \mu < 0$, then the conclusion follows from (4.7). It happens from (4.7) that $\infty \le -\infty$ if $\mu = -\infty$. Hence the latter holds. As a result, $y(t) < y(t-\tau)$ for $t \ge t_1$, that is, $y(t)$ is bounded for $t \ge t_1$ which contradicts to our hypothesis.

Theorem 4.4 is proved.

Theorem 4.5. *Let* $-\infty < p(t) \le -1$. *If all the conditions of Theorem 4.4 hold, then every bounded solution of (1.2) oscillates.*

The proof of theorem follows from the proof of Theorem 4.4. Hence the details are omitted.

Theorem 4.6. *Let* $1 < b_1 \le p(t) \le b_2 < \frac{1}{2}b_1^2$ *and* (A_2) *hold. Suppose that* (A_9) *holds with*

$$-\frac{b_1-1}{16b_2} \le F(t) \le \frac{b_1-1}{8b_2}.$$

If

$$\int_0^\infty \frac{s}{r(s)}\int_s^\infty tq(t)\,dt\,ds < \infty,$$

then (1.2) admits a positive bounded solution.

Proof. It is possible to choose T_0 large enough such that

$$\int_{T_0}^\infty \frac{s}{r(s)}\int_s^\infty tq(t)\,dt\,ds < \frac{b_1-1}{16b_2G(1)}$$

and

$$\int_{T_0}^\infty \frac{s}{r(s)}\int_s^\infty th(t)\,dt\,ds < \frac{b_1-1}{4b_1H(1)}.$$

Let $X = BC([T_0,\infty),\mathbb{R})$. Then X is a Banach space with respect to supremum norm defined by

$$\|x\| = \sup_{t\ge T_0}\{|x(t)|\}.$$

Let

$$S = \left\{x \in X : \frac{b_1-1}{8b_1b_2} \le x(t) \le 1, t \ge T_0\right\}.$$

Hence S is a closed bounded convex subset of X. Define two maps Ω_1 and Ω_2 on S as follows;

$$(\Omega_1 y)(t) = \begin{cases} (\Omega_1 y)(T_1), & T_0 \le t \le T_1, \\ -\dfrac{y(t+\tau)}{p(t+\tau)} + \dfrac{2b_1^2 + b_1 - 1}{4b_1 p(t+\tau)}, & t \ge T_1, \end{cases}$$

and

$$(\Omega_2 y)(t) = \begin{cases} (\Omega_2 y)(T_1), & T_0 \le t \le T_1, \\ \dfrac{F(t+\tau)}{p(t+\tau)} + \dfrac{K(t+\tau)}{p(t+\tau)} - \\ -\dfrac{1}{p(t+\tau)} \displaystyle\int_{t+\tau}^{\infty} \left(\dfrac{s-(t+\tau)}{r(s)} \int_s^{\infty} (u-s) q(u) G(y(u-\alpha))\, du \right) ds, & t \ge T_1, \end{cases}$$

where $K(t)$ is defined in (3.3). Indeed,

$$K(t) = \int_t^{\infty} \frac{s-t}{r(s)} \int_s^{\infty} (u-s) h(u) H(y(u-\beta))\, du\, ds \le H(1) \int_t^{\infty} \frac{s}{r(s)} \int_s^{\infty} u h(u)\, du\, ds < \frac{b_1 - 1}{4b_1}$$

implies that

$$(\Omega_1 y)(t) + (\Omega_2 y)(t) \le \frac{2b_1^2 + b_1 - 1}{4b_1^2} + \frac{b_1 - 1}{8b_1 b_2} + \frac{b_1 - 1}{4{b_1}^2} = \frac{{b_1}^2 + b_1 - 1}{2{b_1}^2} + \frac{b_1 - 1}{8b_1 b_2} \le$$

$$\le \frac{{b_1}^2 + b_1 - 1}{2{b_1}^2} + \frac{b_1 - 1}{8{b_1}^2} = \frac{4{b_1}^2 + 5b_1 - 5}{8{b_1}^2} < 1$$

and

$$(\Omega_1 y)(t) + (\Omega_2 y)(t) \ge -\frac{1}{b_1} + \frac{2{b_1}^2 + b_1 - 1}{4b_1 b_2} - \frac{b_1 - 1}{16 b_1 b_2} - \frac{b_1 - 1}{16 b_1 b_2} =$$

$$= -\frac{1}{b_1} + \frac{2{b_1}^2 + b_1 - 1}{4b_1 b_2} - \frac{b_1 - 1}{8 b_1 b_2} =$$

$$= -\frac{1}{b_1} + \frac{4{b_1}^2 + b_1 - 1}{8b_1 b_2} \ge \frac{b_1 - 1}{8 b_1 b_2},$$

that is, $\Omega_1 y + \Omega_2 y \in S$. It is easy to verify that Ω_1 is a contraction mapping.

Next, we show that Ω_2 is continuous. Let $\{y_j(t)\}$ be the sequence of continuous functions defined on S such that $\|y_j - y\| = 0$ for all $j \to \infty$. Because S is closed and bounded, $(y_j - y) \in$

$\in S$ and

$$|(\Omega_2 y_j)(t) - (\Omega_2 y)(t)| \leq$$

$$\leq \frac{1}{p(t+\tau)} \int\limits_{t+\tau}^{\infty} \frac{s-t-\tau}{r(s)} \int\limits_{s}^{\infty} (u-s)h(u)|H(y_j(u-\beta)) - H(y(u-\beta))|\,du\,ds+$$

$$+\frac{1}{p(t+\tau)} \int\limits_{t+\tau}^{\infty} \frac{s-t-\tau}{r(s)} \int\limits_{s}^{\infty} (u-s)q(u)|G(y_j(u-\alpha)) - G(y(u-\alpha))|\,du\,ds.$$

Because G and H are continuous functions, then it follows that $\|\Omega_2 y_j - \Omega_2 y\| = 0$ as $j \to \infty$. We know that Ω_2 is uniformly bounded, there exist $t_1, t_2 > 0$ such that for $t_1 > t_2 \geq T_1$ and for all $y(t) \in S$,

$$|\Omega_2 y(t_1) - \Omega_2 y(t_2)| \leq \left|\frac{F(t_1+\tau)}{p(t_1+\tau)}\right| + \left|\frac{F(t_2+\tau)}{p(t_2+\tau)}\right| + \left|\frac{K(t_1+\tau)}{p(t_1+\tau)}\right| + \left|\frac{K(t_2+\tau)}{p(t_2+\tau)}\right| +$$

$$+\left|\frac{1}{p(t_1+\tau)} \int\limits_{t_1+\tau}^{\infty} \frac{s-t_1-\tau}{r(s)} \int\limits_{s}^{\infty} (u-s)q(u)G(y(u-\alpha))\,du\,ds\right| +$$

$$+\left|\frac{1}{p(t_2+\tau)} \int\limits_{t_2+\tau}^{\infty} \frac{s-t_2-\tau}{r(s)} \int\limits_{s}^{\infty} (u-s)q(u)G(y(u-\alpha))\,du\,ds\right| \leq$$

$$\leq 2\left(\frac{b_1-1}{8b_1b_2}\right) + 2\left(\frac{b_1-1}{4{b_1}^2}\right) + 2\left(\frac{b_1-1}{16b_1b_2}\right) \leq \frac{7(b_1-1)}{8{b_1}^2}$$

implies that Ω_2 is precompact. Hence verifying all the required conditions of Krasnosel'skii's fixed point theorem it yields that $\Omega_1 + \Omega_2$ has a fixed point in S, that is,

$$y(t) = -\frac{y(t+\tau)}{p(t+\tau)} + \frac{2{b_1}^2 + b_1 - 1}{4b_1 p(t+\tau)} + \frac{K(t+\tau)}{p(t+\tau)} -$$

$$-\frac{1}{p(t+\tau)} \int\limits_{t+\tau}^{\infty} \left(\frac{s-(t+\tau)}{r(s)} \int\limits_{s}^{\infty} (u-s)q(u)G(y(u-\alpha))du\right) ds + \frac{F(t+\tau)}{p(t+\tau)}.$$

Clearly, $y(t)$ is a solution of (1.2) on $\left[\frac{b_1-1}{8b_1b_2}, 1\right]$.

Theorem 4.6 is proved.

Remark 4.2. Theorems similar to Theorem 4.6 can be proved in the other ranges of $p(t)$.

5. Examples and discussions.

Example 5.1. Consider

$$(e^t(y(t)+e^{-4t}y(t-\pi))'')''+8e^{t+2\pi}y(t-2\pi)-$$
$$-50e^{-3t+\frac{\pi}{2}}\left(1+e^{2t-3\pi}\cos^2 t\right)\frac{y\left(t-\frac{3\pi}{2}\right)}{1+y^2\left(t-\frac{3\pi}{2}\right)}=6e^{2t}\cos t. \quad (5.1)$$

Indeed, if we choose $F(t)=\left(\dfrac{e^t}{25}\right)(9\sin t-12\cos t)$, then it is easy to verify that $(r(t)F''(t))''=$ $=f(t)=6e^{2t}\cos t$. Since

$$F^+(t-2\pi)=\begin{cases}0, & t\in[(2n+3)\pi+\theta_1,(2n+4)\pi+\theta_1],\\ \dfrac{3}{5}e^{t-2\pi}\sin(t-2\pi-\theta_1), & t\in[2(n+1)\pi+\theta_1,(2n+3)\pi+\theta_1],\end{cases}$$

and

$$F^-(t-2\pi)=\begin{cases}-\dfrac{3}{5}e^{t-2\pi}\sin(t-2\pi-\theta_1), & t\in[(2n+3)\pi+\theta_1,(2n+4)\pi+\theta_1],\\ 0, & t\in[2(n+1)\pi+\theta_1,(2n+3)\pi+\theta_1],\end{cases}$$

for $n=0,1,2,\ldots$, then

$$\int_{2\pi}^{\infty}b(t)Q(t)F^+(t-2\pi)\,dt=\frac{24}{5}e^{-\pi}e^{\theta_1/2}\sum_{n=0}^{\infty}\int_{2(n+1)\pi}^{(2n+3)\pi}e^{\frac{z}{2}}\sin z\,dz=$$
$$=\frac{48}{25}e^{-\pi}e^{\theta_1/2}\sum_{n=0}^{\infty}\left(2e^{\frac{(2n+3)\pi}{2}}+2e^{(n+1)\pi}\right)=\infty,$$

where $F(t)=\dfrac{3}{5}e^t\sin(t-\theta_1)$, $\theta_1=\tan^{-1}\left(\dfrac{4}{3}\right)$ and $z=t-\theta_1$. Clearly, $(A_1)-(A_5)$ and (A_{10}) is satisfied. Hence by Theorem 4.1, every solution of (5.1) is oscillatory. In particular, $y(t)=e^t\sin t$ is such an oscillatory solution of (5.1).

Example 5.2. Consider

$$(e^t(y(t)+(1+e^{-t})y(t-2\pi))'')''+e^{3t}y(t-4\pi)-e^{-t}\frac{y(t-6\pi)}{1+y^2(t-6\pi)}=0. \quad (5.2)$$

Clearly, $(A_1)-(A_7)$ are satisfied. Hence by Theorem 3.1 every solution of (5.2) oscillates or tends to zero.

It is learnt that the solution space of (1.1)/(1.2) is divided for bounded and unbounded solutions. Due to the method incorporated here, we could not stop the bounded solutions of (1.1) as converging to zero. However, in case of unbounded solution, it oscillates.

It is interesting to notice the solution space of forced equation (1.2) pertaining to (A_9) or (A_{10}). Emphasis will be given to forcing function as compared to the results concerning (1.1). It reveals that every unbounded solutions of (1.2) oscillates if (A_9) holds except $p(t) \leq 1$.

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