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ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF RAYLEIGH TYPE p-LAPLACIAN EQUATION WITH DEVIATING ARGUMENTS*

ПРО ІСНУВАННЯ ПЕРІОДИЧНИХ РОЗВ'ЯЗКІВ РІВНЯНЬ РЕЛЕЇВСЬКОГО ТИПУ З p-ЛАПЛАСІАНОМ ТА ВІДХИЛЕНИМИ АРГУМЕНТАМИ

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In the present paper, we propose new sufficient conditions for the existence of periodic solutions for a class of Rayleigh type p-Laplacian equation with deviating arguments. Results obtained complement or improve the existing ones.

Запропоновано нові достатні умови існування періодичних розв'язків рівнянь релеївського типу з p-лапласіаном та відхиленими аргументами. Отримані результати доповнюють та покращують існуючі результати.

1. Introduction. In recent years, the problem of the existence of periodic solutions for the Duffing type, Liénard type, and Rayleigh type, p-Laplacian equation with a deviating argument has been received a lot of attention. We refer the reader to [1 – 11] and the references cited therein. However, to the best to our knowledge, fewer papers have considered the Rayleigh type p-Laplacian equation with deviating arguments. We only find that Zong and Liang [12] deal with the Rayleigh type p-Laplacian equation with deviating arguments of the form

$$(\varphi_p(x'(t)))' + f(t, x'(t - \tau_1(t))) + g(t, x(t - \tau_2(t))) = e(t), \tag{1.1}$$

where $p > 1$ and $\varphi_p : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, f and g are continuous and 2π-periodic with respect to the first argument, τ_1, τ_2 and e are continuous and 2π-periodic. Under the assumptions that $f(t, 0) = 0$ and $\int_0^{2\pi} e(t)\,dt = 0$, they proved the following result.

Theorem A. *Suppose there exist positive constants K, d, M such that:*
(H_1) $|f(t, x)| \leq K$ for $(t, x) \in \mathbb{R} \times \mathbb{R}$;
(H_2) $xg(t, x) > 0$ and $|g(t, x)| > K$ for $|x| > d$ and $t \in \mathbb{R}$;
(H_3) $g(t, x) \geq -M$ for $x \leq -d$ and $t \in \mathbb{R}$.
Then (1.1) has at least one solution with period 2π.

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However, upon examining their proof of Theorem A in [12], we have found that the conditions (H_2) and (H_3) can be replaced with a more simple condition (H_2^*) (see to next section). In this paper, we will also discuss the existence of 2π-periodic solutions to Eq. (1.1). By using the Manásevich – Mawhin continuation theorem and some analysis, we establish some new sufficient conditions for the existence of 2π-periodic solution of Eq. (1.1). If applying our results to (1.1), one will find that our results are different from those in [1 – 12]. In particular, an example is also given to illustrate the effectiveness of our results.

2. Main results. The following notations will be used throughout the rest of this paper:

$$|x|_\infty = \max_{t\in[0,2\pi]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,2\pi]} |x'(t)|, \quad \overline{e} = \frac{1}{2\pi}\int_0^{2\pi} e(t)\,dt.$$

Set

$$C_{2\pi} = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+2\pi) = x(t)\}$$

and

$$C_{2\pi}^1 = \{x \in C^1(\mathbb{R},\mathbb{R}) : x(t+2\pi) = x(t)\},$$

which are two Banach spaces with the norms

$$\|x\|_{C_{2\pi}} = |x|_\infty \quad \text{and} \quad \|x\|_{C_{2\pi}^1} = \max\{|x|_\infty, |x'|_\infty\}.$$

For the T-periodic boundary-value problem

$$(\varphi_p(x'(t)))' = \widetilde{f}(t,x,x'), \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{2.1}$$

where $\widetilde{f}(t,x,x')$ is a continuous function and T-periodic in the first variable, we have the following result.

Lemma 2.1 [13]. *Let B be an open ball in C_T^1 of center 0 and radius r. Assume that the following conditions hold:*

(i) For each $\lambda \in (0,1)$, the problem

$$(\varphi_p(x'(t)))' = \lambda\widetilde{f}(t,x,x')$$

has no solution on the boundary of B.

(ii) The continuous function F defined on $\mathbb{R}$ by

$$F(a) := \frac{1}{T}\int_0^T \widetilde{f}(t,a,0)\,dt$$

is such that $F(r)F(-r) < 0$.

Then the periodic boundary-value problem (2.1) has at least one periodic solution in $\overline{B}$.

By using Lemma 2.1, we obtain our main results.

Theorem 2.1. *Let (H_1) hold. Assume that the following condition is satisfied:*
(H_2^*) $x(g(t,x)-\overline{e}) < 0$ and $|g(t,x)-\overline{e}| > K$ for $|x| > d$ and $t \in \mathbb{R}$.
Then Eq. (1.1) has at least one 2π-periodic solution.

Proof. Consider the homotopic equation of (1.1)

$$(\varphi_p(x'(t)))' + \lambda f(t, x'(t-\tau_1(t))) + \lambda g(t, x(t-\tau_2(t))) = \lambda e(t), \quad \lambda \in (0,1). \tag{2.1$_\lambda$}$$

Let $S \subset C^1_{2\pi}$ be the set of all possible 2π-periodic solutions of (2.1_λ). If $S = \phi$, the proof is ended. Suppose $S \neq \phi$, and let $x \in S$; then integrating both sides of (2.1_λ) from 0 to 2π, we get

$$\int_0^{2\pi} \big\{ f(t, x'(t-\tau_1(t))) + [g(t, x(t-\tau_2(t))) - \overline{e}] \big\}\, dt = 0. \tag{2.2}$$

By the integral mean value theorem, there is a $\xi \in [0, 2\pi]$ such that

$$f(\xi, x'(\xi-\tau_1(\xi))) + [g(\xi, x(\xi-\tau_2(\xi))) - \overline{e}] = 0. \tag{2.3}$$

By applying condition (H_1) we have

$$|g(\xi, x(\xi-\tau_2(\xi))) - \overline{e}| = |-f(\xi, x'(\xi-\tau_1(\xi)))| \le K, \tag{2.4}$$

and from condition (H_2^*) we can obtain $|x(\xi-\tau_2(\xi))| \le d$. Let $\xi - \tau_2(\xi) = 2n\pi + t_0$, where $t_0 \in [0, 2\pi]$, and n is an integer. Then, we obtain $|x(t_0)| \le d$.

Hence, for any $t \in [t_0, t_0+2\pi]$, we have

$$|x(t)| = \left| x(t_0) + \int_{t_0}^{t} x'(s)\, ds \right| \le |x(t_0)| + \int_{t_0}^{t} |x'(s)|\, ds$$

and

$$|x(t)| = \left| x(t_0+2\pi) + \int_{t_0+2\pi}^{t} x'(s)\, ds \right| \le |x(t_0)| + \left| -\int_{t}^{t_0+2\pi} x'(s)\, ds \right| \le |x(t_0)| + \int_{t}^{t_0+2\pi} |x'(s)|\, ds.$$

Now combining the above two inequalities, we obtain

$$|x|_\infty = \max_{t\in[0,2\pi]} |x(t)| = \max_{t\in[t_0,t_0+2\pi]} |x(t)| \le$$

$$\le \max_{t\in[t_0,t_0+2\pi]} \left\{ |x(t_0)| + \frac{1}{2} \int_{t_0}^{t_0+2\pi} |x'(s)|\, ds \right\} \le d + \frac{1}{2} \int_0^{2\pi} |x'(s)|\, ds. \tag{2.5}$$

Denote

$$E_1 = \{t : t \in [0, 2\pi],\ |x(t-\tau_2(t))| > d\}$$

and

$$E_2 = \{t : t \in [0, 2\pi], |x(t - \tau_2(t))| \le d\}.$$

Then $E_1 \cup E_2 = [0, 2\pi]$.

Since $x(t)$ is 2π-periodic, multiplying both sides of (2.1_λ) by $x(t)$, integrating over $[0, T]$, and applying (H_2^*), we have

$$\int_0^{2\pi} |x'(t)|^p \, dt = -\int_0^{2\pi} (\varphi_p(x'(t)))' x(t) \, dt = \lambda \int_0^{2\pi} f(t, x'(t - \tau_1(t))) x(t) \, dt +$$
$$+ \lambda \int_0^{2\pi} [g(t, x(t - \tau_2(t))) - e(t)] x(t) \, dt = \lambda \int_0^{2\pi} f(t, x'(t - \tau_1(t))) x(t) \, dt +$$
$$+ \lambda \int_{E_1} [g(t, x(t - \tau_2(t))) - e(t)] x(t) \, dt + \lambda \int_{E_2} [g(t, x(t - \tau_2(t))) - e(t)] x(t) \, dt \le$$
$$\le \int_0^{2\pi} |f(t, x'(t - \tau_1(t)))| |x(t)| \, dt +$$
$$+ \int_0^{2\pi} \max\{|g(t, x(t - \tau_2(t))) - e(t)| : t \in \mathbb{R}, |x(t - \tau_2(t))| \le d\} |x(t)| \, dt \le$$
$$\le 2\pi D |x|_\infty, \tag{2.6}$$

where $D = \max\{|g(t, x(t - \tau_2(t))) - e(t)| : t \in \mathbb{R}, |x(t - \tau_2(t))| \le d\} + K$.

For $x(t) \in C(\mathbb{R}, \mathbb{R})$ with $x(t + 2\pi) = x(t)$, and $0 < r \le s$, by using Hölder inequality, we obtain

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |x(t)|^r \, dt \right)^{\frac{1}{r}} \le \left(\frac{1}{2\pi} \left(\int_0^{2\pi} (|x(t)|^r dt)^{\frac{s}{r}} \right)^{\frac{r}{s}} \left(\int_0^{2\pi} 1 \, dt \right)^{\frac{s-r}{s}} \right)^{\frac{1}{r}} = \left(\frac{1}{2\pi} \int_0^{2\pi} |x(t)|^s \, dt \right)^{\frac{1}{s}},$$

which implies that

$$|x|_r \le (2\pi)^{\frac{s-r}{rs}} |x|_s \quad \text{for} \quad 0 < r \le s. \tag{2.7}$$

Then, in view of (2.5)–(2.7), we have

$$\left(\int_0^{2\pi} |x'(t)|dt\right)^p \leq (2\pi)^{p-1}|x'(t)|_p^p = (2\pi)^{p-1}\int_0^{2\pi} |x'(t)|^p\, dt \leq$$

$$\leq (2\pi)^{p-1}2\pi D|x|_\infty \leq (2\pi)^p D\left(d + \frac{1}{2}\int_0^{2\pi} |x'(s)|\, ds\right). \qquad (2.8)$$

Since $p > 1$, (2.8) yields that we can choose some positive constant M_1 such that

$$\int_0^{2\pi} |x'(t)|dt \leq M_1, \quad |x|_\infty \leq d + \frac{1}{2}\int_0^{2\pi} |x'(s)|ds \leq M_1.$$

As $x(0) = x(2\pi)$, there exists $t_1 \in [0, 2\pi]$, such that $x'(t_1) = 0$, and since $\varphi_p(0) = 0$, we have

$$|x'|_\infty^{p-1} = \max_{t\in[0,2\pi]}\{|\varphi_p(x'(t))|\} = \max_{t\in[0,2\pi]}\left\{\left|\int_{t_1}^{t} (\varphi_p(x'(s)))'\, ds\right|\right\} =$$

$$= \max_{t\in[0,2\pi]}\left\{\left|\int_{t_1}^{t} [\lambda f(s, x'(s-\tau_1(s))) + \lambda g(s, x(s-\tau_2(s))) - \lambda e(s)]\, ds\right|\right\} \leq$$

$$\leq \int_0^{2\pi} |f(s, x'(s-\tau_1(s)))|\, ds + \int_0^{2\pi} |g(s, x(s-\tau_2(s))) - e(s)|\, ds \leq$$

$$\leq 2\pi M + 2\pi \max\{|g(t, x(t-\tau_2(t))) - e(t)| : t \in \mathbb{R}, |x(t-\tau_2(t))| \leq M_1\} := M_1^*. \qquad (2.9)$$

Thus, we can get some positive constant $M_2 > M_1 + M_1^* + 1$ such that, for all $t \in R$,

$$|x'(t)| \leq M_2.$$

Hence, taking $r = M_1 + M_2 + d + 1$, we have that $S \subset B$. On the other hand, it is clear that, in our case, $F(a) = -\frac{1}{2\pi}\int_0^{2\pi}[g(t,a) - e(t)]\, dt$. From (H_2^*), it follows that $F(-r)F(r) < 0$. As a consequence, we can apply the Manásevich–Mawhin continuation theorem to deduce that Eq. (1.1) has at least one solution in $\overline{B}$.

Theorem 2.1 is proved.

3. Example and remark. In this section, we give an example to demonstrate the results obtained in the previous section.

Example. *Let* $f(t,x) = (1+\cos^2 t)e^{-x^2(t)}$, $g(t,x) = (3+\sin^2 t)\arctan x(t)$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}$. *Then, the Rayleigh type* p*-Laplacian equation with deviating arguments*

$$(\varphi_p(x'(t)))' + f(t, x'(t-\sin^2 t)) + g(t, x(t-\cos t)) = 2\sin^2 t \tag{3.1}$$

has at least one periodic solution with period 2π.

Proof. By (3.1), we have $\overline{e} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} 2\sin^2 t\,dt = 1$ and choose $K = 2$, and $d = \dfrac{\pi}{4}$. It is obvious that the assumptions (H_1) and (H_2^*) hold. Hence, by Theorem 2.1, Eq. (3.1) has one periodic solution with period 2π.

Remark. It is easy to see that all the results in [1–12], and the references cited therein, cannot be applied to Eq. (3.1) for securing the existence of 2π-periodic solutions. This implies that the results of this paper are essentially new.

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