UDC 517.9

SINGULAR NONLINEAR EIGENVALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL EQUATION WITH ENERGY DISSIPATION*

СИНГУЛЯРНА НЕЛІНІЙНА ЗАДАЧА НА ВЛАСНІ ЗНАЧЕННЯ ДЛЯ ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ДРУГОГО ПОРЯДКУ З ДИСИПАЦІЄЮ ЕНЕРГІЇ

I. O. Parasyuk, S. V. Pozur

Kyiv Taras Shevchenko Nat. Univ.
Volodymyrs'ka St., 64, Kyiv, 01033, Ukraine
e-mail: pio@mechmat.univ.kiev.ua

An eigenvalue problem on the half line for a singular nonlinear ordinary differential operator of the second order is considered. We find sufficient conditions under which this problem has a solution which has a prescribed number of zeroes and vanishes at infinity.

Розглядається задача на власні значення на півосі для нелінійного сингулярного звичайного диференціального рівняння другого порядку. Знайдено достатні умови, при яких ця задача має розв'язок із заданою кількістю нулів, що прямує до нуля на нескінченності.

1. Introduction. The goal of this paper is to establish sufficient conditions for the solvability of the following singular boundary-value problem:

$$y'' + p(x, y, y')y' + q(x, y) = \lambda y, \tag{1}$$

$$\lim_{x\to+0} y^{(k)}(x) < \infty, \quad k = 0, 1, 2, \tag{2}$$

$$\lim_{x\to+\infty} y(x) = 0. \tag{3}$$

Here $p(x, y, z) \in C^1((0, \infty) \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R})$, $q(x, y) \in C^1((0, \infty) \times \mathbb{R} \mapsto \mathbb{R})$ are functions such that

$$|p(x, y, z)| + |q(x, y)| = O(x^{-1}), \quad x \to +0,$$

and $\lambda \geq 0$ is a positive "spectral" parameter. We ask whether there exists $\lambda > 0$ for which the system (1) – (3) has a nontrivial solution. Thus, the above system is treated as a nonlinear eigenvalue problem. Moreover, we will seek for such a λ that there exists a solution to (1) – (3) with a prescribed number of zeroes. The problems of such a kind naturally arise when studying global spherically-symmetric solutions of multidimensional nonlinear evolution equations of the classical field theory. E.g., let us search for a function $u(t, \mathbf{x}; \lambda) = e^{i\lambda t} y(\|\mathbf{x}\|)$, $\lambda \in \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_n)$, satisfying the nonlinear Schrödinger equation (NSE)

$$iu_t + \Delta u + \Phi(|u|)u = 0,$$

* This work was partially supported by the State Fund of Fundamental Research (grant No. 01.07/00047).

where Δ is the n-dimensional Laplace operator and $\Phi(\cdot) \in C(\mathbb{R}_+ \mapsto \mathbb{R})$. We require that $u(t, \mathbf{x}; \lambda)$ represents a global solution which vanishes as $\|\mathbf{x}\| \to \infty$. This requirement will be fulfilled once the function $y(x)$ satisfies the equation

$$y'' + \frac{n-1}{x}y' + \Phi(y)y = \lambda y \tag{4}$$

as well as the boundary conditions (2), (3). There are a lot of papers in which the nonlinear singular boundary-value problems on the half line are studied (see [1–7]). In particular, in [1] the authors considered the equation (4) in the case where $\Phi(y)y = |y|^\alpha \operatorname{sign} y$, $\lambda = 1$, $n - 1 = \gamma$. They have shown that the corresponding problem (1)–(3) has a nontrivial solution iff the parameters α and γ satisfy one of the following conditions: (a) $\gamma > 0$, $1 < \alpha < (\gamma+3)/(\gamma-1)$; (b) $\gamma \in (0,1]$, $\alpha > 1$; (c) $\gamma > 0$, $\alpha < 1$; (d) $\gamma \le 0$, $\alpha > 1$. In the cases (a), (b) any solution either is sign-preserving or has a finite number of zeroes; moreover, there exists a solution with the prescribed number of zeroes. In the case (c) any solution is oscillating, while in the case (d) it is always sign-preserving.

As it was shown in [8], the NSE with $\Phi(|u|) = |u|^k$, $k = 4/n$, by means of the ansätz

$$u = \left(\frac{2}{\sqrt{\lambda}(1+t^2)}\right)^{n/4} \exp\left(-\frac{it\mathbf{x}^2}{2(t^2+1)}\right) y\left(\frac{\sqrt{\lambda}\mathbf{x}^2}{1+t^2}\right),$$

is reduced to the ODE

$$\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + \frac{|y|^k y}{x} = \lambda y, \tag{5}$$

where $a = n/2$. In [7], it has been proved that for any $l \in \mathbb{Z}_+$ the singular problem (5), (2), (3) has a nontrivial solution with exactly l zeroes provided that the following inequality holds:

$$\max(0, 1 - 1/k) < a < 1 + \frac{2}{k}. \tag{6}$$

We will generalize this result for the problem (1)–(3).

The structure of the paper is as follows. In Section 2 we construct a solution $y(x, \lambda)$ to the problem (1), (2), and show that $y(x, \lambda) \in C([0, \infty) \times [0, \infty) \mapsto \mathbb{R})$ if the equation (1) has a property of energy dissipation and the function $q(x, y)$ satisfies certain nonlinearity and convexity conditions.

The main results, — Theorems 1 and 2, — are proved in the Section 3. Here we present a “supremum principle” of finding eigenvalues. This principle is applicable under the assumption that the function $y(x, 0)$ has at least one zero. Although some additional requirements concerning functions $p(x, y, z)$ and $q(x, y)$ are quite numerous, they are satisfied by a wide class of equations (1). Besides, they can be considerably simplified if we restrict ourselves to more special and concrete, but important types of functions $p(x, y, z)$, $q(x, y)$, e.g., $p(x, y, z) = P_1(x)P_2(y)P_3(z)$, $q(x, y) = q_1(x)q_2(y)$. As an example, in Section 4 we consider the eigenvalue problem of the form (1)–(3) for a generalized Emden–Fowler equation.

Section 5 contains auxiliary results which are referred to in the proofs of the main theorems.

2. Existence of solution on the half line. Assume that the following singularity condition at $x = 0$ is fulfilled:

H_1: For any compact set $\mathcal{K} \subset \mathbb{R}^2$ there exist the limits

$$p_0(y, z) := \lim_{x \to +0} xp(x, y, z), \quad q_0(y) := \lim_{x \to +0} xq(x, y),$$

uniform in $(y, z) \in \mathcal{K}$. It is not hard to show that if for a given $\lambda \geq 0$ there exists a solution $y(x)$ of the system (1), (2) then its initial values

$$y_0 := \lim_{x \to +0} y(x), \quad z_0 := \lim_{x \to +0} y'(x)$$

must satisfy the equation

$$p_0(y, z)z + q_0(y) = 0. \tag{7}$$

Basing on this fact, let us adopt the following hypothesis:

H_2: There exists a solution $y = y_0$, $z = z_0$ of the equation (7) such that $y_0 \neq 0$, and in some neighborhood $B(y_0, z_0)$ of the point (y_0, z_0) the functions $p_0(y, z)$, $q_0(y)$ are twice continuously differentiable and satisfy the nondegeneracy condition

$$\frac{\partial p_0(y_0, z_0)}{\partial z} z_0 + p_0(y_0, z_0) > 0; \tag{8}$$

besides, the functions $p(x, y, z)$, $q(x, y)$ admit the representation

$$p(x, y, z) = \frac{p_0(y, z)}{x} + p_1(x, y, z), \quad q(x, y, z) = \frac{q_0(y)}{x} + q_1(x, y),$$

where $p_1(x, y, z)$, $q_1(x, y)$ are continuous on the set $[0, \infty) \times B(y_0, z_0)$, together with their partial derivatives in y, z up to the second order.

Proposition 1. *Let the hypotheses H_1, H_2 be true. Then for any $\lambda_0 > 0$ there exists $\rho > 0$ such that for each $\lambda \in [0, \lambda_0]$ the equation (1) has a solution $y(x, \lambda)$, $x \in [0, \rho]$, satisfying the initial conditions $y(0, \lambda) = y_0$, $y'_x(0, \lambda) = z_0$. As a function of the variables (x, λ), this solution is continuous on $[0, \rho] \times [0, \lambda_0]$.*

Proof. After the change of dependent variable $y = y_0 + z_0 x + v$ in (1), we get the equation

$$v'' + \frac{a}{x} v' + \frac{b}{x} v = f(x, \lambda) + g(x, \lambda)v + h(x)v' + F(x, v, v') \tag{9}$$

where

$$a = p_0(y_0, z_0) + p'_{0z}(y_0, z_0)z_0, \qquad b = p'_{0y}(y_0, z_0)z_0 + q'_0(y_0),$$

$$f(x, \lambda) = \lambda(y_0 + z_0 x) - p(x, y_0 + z_0 x, z_0)z_0 - q(x, y_0 + z_0 x),$$

$$g(x, \lambda) = \lambda + \frac{b}{x} - \frac{\partial p(x, y_0 + z_0 x, z_0)}{\partial y} z_0 - \frac{\partial q(x, y_0 + z_0 x)}{\partial y},$$

$$h(x) = \frac{a}{x} - \frac{\partial p(x, y_0 + z_0 x, z_0)}{\partial z} z_0 - p(x, y_0 + z_0 x, z_0),$$

and

$$F(x,v,v') = p(x,y_0+z_0x,z_0)z_0 + q(x,y_0+z_0x)+$$
$$+\left(\frac{\partial p(x,y_0+z_0x,z_0)}{\partial y}z_0 + \frac{\partial q(x,y_0+z_0x)}{\partial y}\right)v+$$
$$+\left(\frac{\partial p(x,y_0+z_0x,z_0)}{\partial z}z_0 + p(x,y_0+z_0x,z_0)\right)v'-$$
$$-p(x,y_0+z_0x+v,z_0+v')(z_0+v') - q(x,y_0+z_0x+v).$$

In view of H_1, H_2, for any $\lambda \in [0,\lambda_0]$ the functions f,h,g can be extended by continuity on the half line $[0,\infty)$. Now it is easy to see that the assertion to be proved follows from Lemma 1 and Remark 1 below in Section 5. The proposition is proved.

We say that the equation (1) satisfies the *energy dissipation condition* if

H_3: $q'_x(x,y) \le 0,\ p(x,y,z) \ge 0 \quad \forall (x,y,z) \in (0,\infty)\times\mathbb{R}^2.$

Note that the equation (1), with $p \equiv 0$, is a Lagrangian one with the energy function

$$V[y] := \frac{{y'}^2}{2} + Q(x,y) - \frac{\lambda y^2}{2}, \quad \text{where} \quad Q(x,y) := \int\limits_0^y q(x,s)\,ds.$$

When the hypothesis H_3 holds, the derivative $V'[y]$ of this function, in virtue of the equation (1), is nonpositive,

$$V'[y] := -p(x,y,y'){y'}^2 + Q'_x(x,y) \le 0.$$

In the sequel we shall assume that the function $q(x,y)$ satisfies the following *nonlinearity and convexity conditions*:

H_4: For any $x \in (0,\infty)$ there exist the limits

$$\lim_{y\to 0}\frac{q(x,y)}{y} = 0, \quad \lim_{y\to\pm\infty}\frac{q(x,y)}{y} = \infty;$$

besides, there exist $r > 0$ and $R > r$ such that for all $x \in (0,r)\cup(R,\infty)$, $(y,z) \in \mathbb{R}^2$, and $y \ne 0$ there exists the derivative $q''_{yy}(x,y)$ which satisfies the inequality

$$\frac{\partial^2 q(x,y)}{\partial y^2}y > 0 \quad \forall y \ne 0.$$

Proposition 2. *If the hypotheses $H_1 - H_4$ are true, then for each $\lambda \ge 0$ the solution $y(x,\lambda)$ from Proposition 1 is continuable on the half line $[0,\infty)$. Moreover, as a function of the variables (x,λ), this solution is continuous on $[0,\infty)\times[0,\infty)$.*

Proof. Let, for some $\lambda_* \in [0,\infty)$, the function $y_*(x) := y(x,\lambda_*)$ be continuable only on a finite interval $[0,x_*)$. Then $|y_*(x)| \to \infty$, $x \to x_* - 0$. Thus,

$$V[y_*(x)] \geq Q(x, y_*(x)) - \frac{\lambda y_*^2(x)}{2} \geq Q(x_*, y_*(x)) - \frac{\lambda y_*^2(x)}{2} \to \infty, \quad x \to x_* - 0.$$

But on the other hand, from H_3 we get $V[y_*(x)] \leq V[y_*(x_*/2)] < \infty$ if $x \in (x_*/2, x_*)$. The contradiction obtained implies boundedness of $y_*(x)$ on any bounded set of the half line $[0,\infty)$. But then, in view of H_3, $y_*'(x)$ has the same property. Thus the natural domain of definition for $y(x,\lambda)$ is $D := [0,\infty) \times [0,\infty)$. The continuity of $y(x,\lambda)$ at any point $(0,\lambda) \in \partial D$, where $\lambda \in [0,\infty)$, has already been proved in Proposition 1. The continuity of this function at any point (x,λ), where $x > 0$, $\lambda \geq 0$, follows from standard theorems of the general theory of ODEs.

3. The main results. In what follows we shall suppose that

H_5: The function $y(x,0)$ has at least $l+1$ zeroes on $(0,\infty)$ where l is a natural number.

Here we do not discuss additional conditions which the functions $p(x,y,z)$ and $q(x,y)$ must satisfy to guarantee the fulfillment of the hypothesis H_5. There are a lot of papers where such conditions has been established for nonlinear second order differential equations with damping (see, e.g., [9–16]).

Let $\mathcal{L}_m$ be the set of all numbers $\Lambda > 0$ such that the solution $y(x,\lambda)$ has at least $m+1$ zeroes for any $\lambda \in (0,\Lambda)$. Obviously,

$$\mathcal{L}_{m+1} \subseteq \mathcal{L}_m.$$

Let the hypotheses $H_1 - H_5$ be true. Then each set $\mathcal{L}_m$, $m = 0,1,\ldots,l+1$, is nonempty and bounded (Proposition 4 in Section 5). Introduce the numbers

$$\lambda_m = \sup \mathcal{L}_m \quad \text{and} \quad y_m(x) := y(x,\lambda_m), \quad m = 0,\ldots,l.$$

Obviously, $\lambda_l \leq \cdots \leq \lambda_0$.

We are going to show that each λ_m is an eigenvalue and $y_m(x)$ is the corresponding eigenfunction of the problem (1)–(3). We will prove this fact under some additional technical assumptions described below.

The following assumption concerns the behavior of $p(x,y,z)$ for large values of x:

H_6: The function $p(x,0,0)$ has a finite limit when $x \to +\infty$, and

$$\limsup_{x\to+\infty} \sup_{(y,z)\in\mathcal{K}} \left(|p_y'(x,y,z)| + |p_z'(x,y,z)|\right) < \infty$$

for any compact set $\mathcal{K} \subset \mathbb{R}^2$.

We also impose a technical condition on the rate of energy dissipation. Namely,

H_7: For any $\varepsilon > 0$, there exist functions $\alpha_\pm(x,\varepsilon)$ which are continuous with respect to the variable $x \in [R,\infty)$, satisfy the inequalities

$$0 \leq \alpha_\pm(x,\varepsilon) \leq p(x,y,z)z^2 - Q_x'(x,y) \quad \forall (y,z) \in \mathcal{M}_\pm(x,\varepsilon), \quad \forall x \geq R,$$

where

$$\mathcal{M}_{\pm}(x,\varepsilon) := \left\{(y,z) \in \mathbb{R}^2 : \pm\, y \geq \varepsilon x,\ \pm\, z \geq \varepsilon,\ \frac{z^2}{2} + Q(x,y) \geq \varepsilon x^2\right\},$$

and have the following property:

$$\int\limits_R^{\infty} \alpha_{\pm}(x,\varepsilon)\, dx = \infty.$$

Observe that if

$$\frac{\partial p(x,y,z)}{\partial y} y \geq 0, \quad \frac{\partial p(x,y,z)}{\partial z} z \geq 0 \quad \forall x \geq R, \tag{10}$$

then each of the conditions

$$\int\limits_R^{\infty} p(x,0,0)\, dx = \infty \quad \text{or} \quad \int\limits_R^{\infty} p(x,\pm\,\varepsilon x,\pm\,\varepsilon)\, dx = \infty \quad \forall \varepsilon > 0$$

obviously guarantee the fulfillment of H_7.

From H_4 it follows that for any fixed $x \in (0,r) \cup (R,\infty)$ and any $\lambda > 0$ the equation

$$q(x,y) = \lambda y$$

has exactly one positive solution $y := \eta_+(x,\lambda)$ and one negative solution $y := \eta_-(x,\lambda)$. The minimal values of the function $Q(x,y) - \lambda y^2/2$ on the semi-axes $y > 0$ and $y < 0$ are attained at the points $y = \eta_+(x,\lambda)$ and $y = \eta_-(x,\lambda)$ respectively. This minimal values we designate by

$$\mu_{\pm}(x,\lambda) := \left[Q(x,y) - \frac{\lambda y^2}{2}\right]_{y=\eta_{\pm}(x,\lambda)}.$$

Denote by $\xi_+(x,\lambda)$ ($\xi_-(x,\lambda)$) the unique positive (negative) root of the equation $Q(x,y) = \lambda y^2/2$.

Next we introduce the sets

$$\mathcal{N}_{\pm}(x,\lambda) = \left\{(y,z) \in \mathbb{R}^2 : \pm\, y > \pm\, \eta_{\pm}(x,\lambda),\ \pm\, z > 0,\ \frac{z^2}{2} + Q(x,y) > \frac{\lambda y^2}{2}\right\}$$

and adopt the following assumption:

H_8: For any $\lambda > 0$, there exist functions $\beta_{\pm}(x,\lambda)$ which are continuous with respect to variable $x \in (R,\infty)$, satisfy the inequalities

$$0 \leq \beta_{\pm}(x,\lambda) \leq p(x,y,z) \quad \forall (y,z) \in \mathcal{N}_{\pm}(x,\lambda),\ \forall x > R,$$

and for these functions either

$$\int\limits_R^{\infty} \frac{\exp\left(\int \beta_{\pm}(s,\lambda)\, ds\right)|\mu_{\pm}(s,\lambda)|}{|\xi_{\pm}(s,\lambda) - \eta_{\pm}(s,\lambda)|}\, ds = \infty \quad \forall \lambda > 0, \tag{11}$$

or

$$\int_R^\infty \left|Q'_x(x,y)\right|_{y=\eta_\pm(x,\lambda)} dx = \infty \quad \forall\lambda > 0. \tag{12}$$

In view of the fact that both $|\mu_\pm(x,\lambda)|$ and $|\eta_\pm(x,\lambda)|$ are nondecreasing, this condition is satisfied by a wide class of functions $q(x,y)$ which obey H_3, H_4. It should also be noted that if (10) holds then it is naturally to check the fulfillment of (11) setting $\beta_\pm(x,\lambda) := p(x,\eta_\pm(x,\lambda),0)$.

Theorem 1. *If the hypotheses $H_1 - H_8$ are true, then for any $m = 0,\ldots,l$ the pair $(\lambda_m, y_m(x))$, where $y_m(x) := y(x,\lambda_m)$, is a solution of the problem (1)–(3). Moreover, $y_m(x)y'_m(x) < 0$ for all sufficiently large x, and*

$$|y_m(x)| + |y'_m(x)| = O(e^{-\gamma_m x}), \quad x \to \infty, \tag{13}$$

where γ_m is a positive number.

Proof. Let us show that $V[y_m(x)] > 0$ for all $x \geq 0$. Indeed, if we suppose that there are $x_0 > 0$ and $\varepsilon > 0$ such that $V[y_m(x_0)] < -\varepsilon$, then, according to Proposition 6 (Section 5), for all $\lambda \in \mathcal{L}_m$ sufficiently close to λ_m, the $(m+1)$-st zero, $x_{m+1}(\lambda)$, of $y_m(x)$ satisfies the inequalities $x_{m+1}(\lambda) > x_0$ and $V[y(x_{m+1}(\lambda),\lambda)] \leq V[y(x_0,\lambda)] \leq -\varepsilon/2$. This, however, contradicts Proposition 3 (Section 5).

According to Proposition 5 (Section 5) the function $y_m(x)$ does not vanish for all sufficiently large x. Then from Proposition 7, it follows that $y_m(x)y'_m(x) < 0$ and $|y_m(x)| + |y'_m(x)| \to 0$, $x \to \infty$.

Observe now that $y_m(x)$ can be regarded as a solution of the linear equation

$$y'' + a(x)y' + b(x)y = 0 \tag{14}$$

in which

$$a(x) := \left\{p(x,0,0) + \left[p(x,y_m(x),y'_m(x)) - p(x,0,0)\right]\right\}, \quad b(x) := \lambda_m - \frac{q_m(x,y_m(x))}{y_m(x)}.$$

From H_4, H_6 it follows that this equation satisfy conditions of the Lemma 2 (Section 5).

The theorem is proved.

Theorem 1 asserts that each λ_m is an eigenvalue of the problem (1)–(3), but it does not guarantees that $\lambda_i \neq \lambda_j$ for $i \neq j$. Now we will analyze this question.

Introduce the sets

$$\mathcal{P}_\pm(x,\lambda) := \left\{(y,z) \in \mathbb{R}^2 : \frac{\pm\,\eta_\pm(x,\lambda)}{2} \leq \pm\, y \leq \pm\,\eta(x,\lambda),\ \pm\, z > \sqrt{\lambda y^2 - 2Q(x,y)}\right\}$$

and the following assumptions:

H_9: For any $\lambda > 0$, there exist functions $\pi_\pm(x,\lambda)$ which are continuously differentiable with respect to the variable $x \in (R,\infty)$, satisfy the inequalities

$$0 \leq \pi_\pm(x,\lambda) \leq p(x,y,z) \quad \forall (y,z) \in \mathcal{P}_\pm(x,\lambda),\ \forall x > R,$$

and for which

$$\frac{1}{\pi_\pm^2(x,\lambda)}\frac{\partial \pi_\pm(x,\lambda)}{\partial x} \geq -\omega_\pm(\lambda) \quad \forall x \geq R$$

with some functions $\omega_\pm : (0,\infty) \mapsto (0,\infty)$ bounded on any compact subset of the half line $(0,\infty)$.

Note that if the condition 10 holds then it is natural to set

$$\pi_\pm(x,\lambda) := p\left(x, \frac{\eta_\pm(x,\lambda)}{2}, \pm\sqrt{\frac{|\mu_\pm(x,\lambda)|}{2}}\right).$$

H_{10}: On any compact set of positive values of the parameter λ, the function $\sup\limits_{x\geq R} \dfrac{|\eta_\pm(x,\lambda)|}{\sqrt{|\mu_\pm(x,\lambda)|}}$ is bounded and the limit

$$\lim_{x\to+\infty} \frac{\ln|\eta_\pm(x,\lambda)|}{x} = 0$$

is uniform.

By using a technique which is quite similar to that of the paper [1] we are able to prove the following theorem.

Theorem 2. *Let the hypotheses $H_1 - H_{10}$ be true. Then*

$$0 < \lambda_l < \cdots < \lambda_1 < \lambda_0,$$

and for any $m = 0,1,\ldots,l$ the eigenfunction $y_m(x) := y(x,\lambda_m)$ has exactly m zeroes (the function $|y_0(x)|$ is positive).

Proof. Let m be an arbitrary nonnegative integer not exceeding l. According to Proposition 6 (Section 5) the $(m+1)$-st zero $x_{m+1}(\lambda)$ of the function $y(x,\lambda)$ tends to infinity when $\lambda \to \lambda_m - 0$. For this reason there exists an integer $k \in [1, m+1]$ such that $\limsup_{\lambda\to\lambda_m-0} x_k(\lambda) = \infty$ but, if $k > 1$, $\limsup_{\lambda\to\lambda_m-0} x_{k-1}(\lambda) < \infty$. Our goal is to show that one can choose such a positive $\lambda < \lambda_m$ near λ_m that the function $y(x,\lambda)$ has only k zeroes. Obviously, this is possible only if $k = m+1$, and hence, $\lambda_{m+1} < \lambda_m$ for any $m < l$.

Let $\{\lambda^{(i)}\}_{i=1,2,\ldots}$ be a sequence such that $\lambda^{(i)} \to \lambda_m - 0$ and $x_k(\lambda^{(i)}) \to \infty$ as $i \to \infty$. Suppose, on the contrary, that for each natural i the function $y(x,\lambda^{(i)})$ has a zero $x_{k+1}(\lambda^{(i)}) > x_k(\lambda^{(i)})$. Without loss of generality we may assume that

$$|y(x,\lambda^{(i)})| > 0 \quad \forall x \in \big(x_k(\lambda^{(i)}), x_{k+1}(\lambda^{(i)})\big).$$

Under this assumption we are going to show that for sufficiently large i the function $V[y(x,\lambda^{(i)})]$ vanishes on $(x_k(\lambda^{(i)}), x_{k+1}(\lambda^{(i)}))$. In view of Proposition 3 this will give us the desired contradiction.

Let us choose a sufficiently small $\delta > 0$ in such a way that

$$\delta < |\eta_\pm(R,\lambda_m)|, \quad Q(R,y) < \frac{\lambda_m y^2}{4} \quad \forall y \in (-\delta,\delta).$$

Then for all sufficiently large i we have the inequalities

$$\delta < |\eta_{\pm}(x,\lambda^{(i)})| \quad \forall x \geq R, \quad \frac{\lambda^{(i)} y^2}{2} - Q(x,y) > \frac{\lambda_m y^2}{4} \quad \forall y \in (-\delta,\delta).$$

Next we choose $x_\delta > R$ so large that $x_\delta > \sup_{i\geq 1} x_{k-1}(\lambda^{(i)})$ if $k > 1$ and

$$|y_m(x)| < \delta, \quad y_m(x)y_m'(x) < 0, \quad V[y_m(x)] < \delta \quad \forall x \geq x_\delta.$$

Then for all sufficiently large i the following inequalities holds:

$$|y(x_\delta,\lambda^{(i)})| < \delta, \quad y(x_\delta,\lambda^{(i)})y'(x_\delta,\lambda^{(i)}) < 0, \quad V[y(x_\delta,\lambda^{(i)})] < \delta, \quad x_k(\lambda^{(i)}) > x_\delta.$$

Now it is not hard to show that $|y(x,\lambda^{(i)})|$ is monotone decreasing to zero on $[x_\delta, x_k(\lambda^{(i)})]$ and simultaneously

$$\frac{y'^2(x,\lambda^{(i)})}{2} > \frac{\lambda_m y^2(x,\lambda^{(i)})}{4} \quad \forall x \in [x_\delta, x_k(\lambda^{(i)})].$$

From this it follows that

$$|y(x,\lambda^{(i)})| < e^{-\lambda_m (x-x_\delta)/\sqrt{2}}\delta \quad \forall x \in [x_\delta, x_k(\lambda^{(i)})].$$

Besides, since $|y(x,\lambda^{(i)})| < |\eta_{\pm}(x,\lambda^{(i)})|$ for $x \in [x_\delta, x_k(\lambda^{(i)})]$, we have $y''(x,\lambda^{(i)})y'(x,\lambda^{(i)}) <$ < 0 and

$$|y'(x,\lambda^{(i)})| < |y'(x_\delta,\lambda^{(i)})| < \sqrt{2(\lambda_m\delta^2 + \delta)} \quad \forall x \in [x_\delta, x_k(\lambda^{(i)})].$$

Now for sufficiently small δ and all sufficiently large i one can extend the functions

$$a(x) = a_i(x) := p(x, y(x,\lambda^{(i)}), y'(x,\lambda^{(i)})), \quad b(x) = b_i(x) := -\lambda^{(i)} + \frac{q(x,y(x,\lambda^{(i)}))}{y(x,\lambda^{(i)})}$$

on the semi-axis $[x_\delta,\infty)$ in such a way that they satisfy the conditions of Lemma 2 (Section 5) with $x_0 = x_\delta$ and with the numbers α, β, ε independent of i. But then taking into account that $y(x,\lambda^{(i)})$ is a solution of the equation

$$y'' + a_i(x)y' + b_i(x)y = 0,$$

we get the estimate

$$|y'(x_k(\lambda^{(i)}),\lambda^{(i)})| \leq K_\delta e^{-\gamma_m x_k(\lambda^{(i)})}$$

with some positive constants K_δ and γ_m. Thus,

$$V[y(x_k(\lambda^{(i)}))] \leq \frac{K_\delta^2}{2} e^{-2\gamma_m x_k(\lambda^{(i)})}. \tag{15}$$

Without loss of generality we assume that $y'(x_k(\lambda^{(i)})) > 0$. Taking into account that $V[y(x,\lambda^{(i)})] > 0$ for $x \in \left[x_k(\lambda^{(i)}), x_{k+1}(\lambda^{(i)})\right]$, it is easy to establish the existence of the nearest to $x_k(\lambda^{(i)})$ value $x = x_*(\lambda^{(i)}) \in (x_k(\lambda^{(i)}), x_{k+1}(\lambda^{(i)}))$ for which

$$\left[y(x,\lambda^{(i)}) = \eta_+(x,\lambda^{(i)})\right]_{x=x^*(\lambda^{(i)})}.$$

Then there exists $x_*(\lambda^{(i)})$ such that

$$\Big[y(x,\lambda^{(i)}) = \frac{\eta_+(x,\lambda^{(i)})}{2}\Big]_{x=x^*(\lambda^{(i)})},$$

and

$$\frac{\eta_+(x,\lambda^{(i)})}{2} < y(x,\lambda^{(i)}) < \eta_+(x,\lambda^{(i)}), \quad y'(x,\lambda^{(i)}) > \sqrt{\lambda^{(i)}y^2 - 2Q(x,y)}\Big|_{y=y(x,\lambda^{(i)})}$$

for all $x \in (x_*(\lambda^{(i)}), x^*(\lambda^{(i)}))$.

Now for such values of x, using the convexity property of the function $f(y) := := \sqrt{\lambda y^2 - 2Q(x,y)}$ (Lemma 3, Section 5) and inequality (15), we have

$$V'[y(x,\lambda)] \le -\pi_+(x,\lambda){y'}^2(x,\lambda) \le$$

$$\le -\pi_+(x,\lambda)\frac{|\mu_+(x_*(\lambda),\lambda)|V[y(x,\lambda)]}{4V[y(x_*(\lambda),\lambda)]} -$$

$$-\pi_+(x,\lambda)\sqrt{\frac{|\mu_+(x_*(\lambda),\lambda)|}{8}}y'(x,\lambda) \le$$

$$\le -\omega_+(\lambda)\pi_+(x,\lambda)V[y(x,\lambda)] - \pi_+(x,\lambda)\frac{\sqrt{|\mu_+(R,\lambda)|}}{3}y'(x,\lambda),$$

provided that $\lambda = \lambda^{(i)}$ and i is sufficiently large. Integrating the above differential inequality we get

$$V[y(x^*(\lambda),\lambda)] \le C(\lambda)\Bigg[V[y(x_*(\lambda),\lambda)] -$$

$$- \frac{\sqrt{|\mu_+(R,\lambda_m)|}}{3}\int\limits_{x_*(\lambda)}^{x^*(\lambda)} e^{\omega_+(\lambda)\int_{x_*(\lambda)}^{x}\pi_+(s,\lambda)\,ds}\pi_+(x,\lambda)y'(x,\lambda)\,dx\Bigg],$$

where

$$C(\lambda) = e^{-\omega_+(\lambda)\int_{x_*(\lambda)}^{x^*(\lambda)}\pi_+(s,\lambda)\,ds}.$$

Since H_9 is true, the function $e^{\omega_+(\lambda)\int_{x_*(\lambda)}^{x}\pi_+(s,\lambda)\,ds}\pi_+(x,\lambda)$ is nondecreasing. Thus,

$$V[y(x^*(\lambda),\lambda)] \le C(\lambda)\Bigg[\frac{K_\delta^2}{2}e^{-2\gamma_m x_k(\lambda)} -$$

$$- \frac{\sqrt{|\mu_+(R,\lambda_m)|}\pi_+(x_*(\lambda),\lambda)}{3}\big(y(x^*(\lambda),\lambda) - y(x_*(\lambda),\lambda)\big)\Bigg] \le$$

$$\le C(\lambda)\left[\frac{K_\delta^2}{2}e^{-2\gamma_m x_k(\lambda)} - \frac{\sqrt{|\mu_+(R,\lambda)|}\eta_+(R,\lambda)}{6}\pi_+(x_*(\lambda),\lambda)\right]$$

where $\lambda = \lambda^{(i)}$.

From H_9 it follows that

$$\pi_+(x,\lambda) \geq \frac{\pi_+(R,\lambda)}{\pi_+(R,\lambda)\omega_+(\lambda)(x-R)+1}$$

and Proposition 8 (Section 5) allows to obtain the estimate $x^*(\lambda^{(i)}) = O\big(x_k(\lambda^{(i)})\big)$ as $i \to \infty$ provided that H_{10} is true. Now it is easy to show that $V\big[y(x^*(\lambda^{(i)}),\lambda^{(i)})\big] < 0$ for sufficiently large i. This is just the required contradiction. The theorem is proved.

4. Singular eigenvalue problem for generalized Emden – Fowler equation. In this section we consider the following boundary-value problem:

$$y'' + \frac{a}{x}y' + f(x)|y|^k y = \lambda y, \tag{16}$$

$$\lim_{x\to+0} y(x) = y_0, \quad \lim_{x\to+0} y^{(k)}(x) < \infty, \quad k = 1,2, \tag{17}$$

$$\lim_{x\to+\infty} y(x) = 0 \tag{18}$$

where a and k are positive constants and $f(x) \in C^1((0,\infty) \mapsto (0,\infty))$.

By applying Theorem 2, we arrive at the following generalization of results obtained in [7].

Theorem 3. *Let the function $f(x)$ satisfy the following conditions:*

1) there exists the limit $\lim\limits_{x\to+0} xf(x) = f_0 \geq 0$;

2) $f'(x) \leq 0$;

3) $\int\limits_1^\infty xf(x)\,dx = \infty$ *if* $a \in (0,1)$; $\int\limits_1^\infty xf(x)\ln x\,dx = \infty$ *if* $a = 1$; $\int\limits_1^\infty x^{1-(a-1)k}f(x)\,dx =$ $= \infty$ *if* $a > 1$;

4) $\lim\limits_{x\to\infty} \dfrac{\ln f(x)}{x} = 0$.

Then for any $y_0 \neq 0$ and arbitrary natural l there exists a solution of the problem (16) – (18) with exactly l zeroes.

Proof. From the assumptions 1 and 2 it follows that the hypotheses $H_1 - H_4$, H_6 and H_9 are true. In fact, the equation (7) takes the form $az + f_0|y|^k y = 0$, and thus $z_0 = -f_0|y_0|^k y_0/a$ for any y_0. To verify H_7 and H_9 it is sufficient to put $\alpha_\pm(x,\varepsilon) = \pi_\pm(x,\lambda) = a/x$.

As it was shown in [17], any nontrivial solution of the equation $(r(x)y')' + s(x)|y|^\gamma \operatorname{sgn} x = 0$, where $r(x), s(x) \in C([1,\infty) \mapsto (0,\infty))$, $\gamma > 1$, is oscillatory on $(0,\infty)$ provided that

$$\int\limits_1^\infty \frac{1}{r(x)}\,dx = \infty \quad \text{and} \quad \int\limits_1^\infty s(x)\int\limits_1^x \frac{1}{r(\xi)}\,d\xi\,dx = \infty$$

or

$$\int\limits_1^\infty \frac{1}{r(x)}\,dx < \infty \quad \text{and} \quad \int\limits_1^\infty s(x)\left(\int\limits_x^\infty \frac{1}{r(\xi)}\,d\xi\right)^\gamma dx = \infty.$$

Now letting $r(x) = x^a$, $s(x) = x^a f(x)$ one can easily verify that the assumption 3 guarantees the oscillatory property for any solution of the equation (16) with $\lambda = 0$. Thus the hypothesis H_5 is true.

Next, one can easily calculate that

$$\eta_\pm(x,\lambda) = \pm\left(\frac{\lambda}{f(x)}\right)^{1/k}, \quad \xi_\pm(x,\lambda) = \pm\left(\frac{(k+2)\lambda}{kf(x)}\right)^{1/k},$$

$$|\mu_\pm(x,\lambda)| = \frac{\lambda k}{2(k+2)}\left(\frac{\lambda}{f(x)}\right)^{2/k}.$$

Thus, in view of condition 2, we get $\inf_{x\geq 1}|\xi_\pm(x,\lambda) - \eta_\pm(x,\lambda)| > 0$, and H_8 is true.

Lastly, H_{10} follows from the assumption 4. The theorem is proved.

It should be noted that to satisfy the condition 3 it is necessary to require $a \leq 1 + 2/k$. In the special case $f(x) = 1/x$ the assumption 3 takes the form $0 < a \leq 1 + 1/k$. This shows that the condition $a > 1 - 1/k$ in (6) is superfluous. On the other hand, Theorem 3 does not allow us to obtain the best upper bound for a. The reason is that our requirement that all solutions of (5) are oscillatory is stronger then the hypothesis H_5.

5. Auxiliary results.

Lemma 1. *Consider the equation*

$$v'' + \frac{a}{x}v' + \frac{b}{x}v = f(x) + g(x)v + h(x)v' + F(x,v,v') \tag{19}$$

where $a > 0$, $b \in \mathbb{R}$, $f, g, h \in C([0,r] \mapsto \mathbb{R})$, $F \in C((0,r] \times B(0,0) \mapsto \mathbb{R})$, $F(x,0,0) \equiv 0$, *and* $B(0,0)$ *is a neighborhood of the origin in* $\mathbb{R}^2$. *Suppose that there exists a constant* $L > 0$ *such that*

$$|xF(x,v_1,w_1) - xF(x,v_2,w_2)| \leq L\max\{|v_1|,|v_2|,|w_1|,|w_2|\}\big(|v_1 - v_2| + |w_1 - w_2|\big)$$

for all (x,v,w), (x,v_1,w_1), $(x,v_2,w_2) \in (0,r] \times B(0,0)$. *Then there exist numbers* $\rho \leq r$, $H > 0$ *and a solution* $v(x) \in C^2([0,\rho] \mapsto \mathbb{R})$ *of the equation (19) satisfying the inequalities*

$$|v(x)| \leq Hx^2, \quad |v'(x)| \leq Hx \quad \forall x \in [0,\rho]. \tag{20}$$

Proof. The equation

$$v'' + \frac{a}{x}v' + \frac{b}{x}v = 0 \tag{21}$$

has a regular singular point at $x = 0$. The roots of the corresponding indicial equation are 0 and $1 - a$.

As is well known, if $a \notin \mathbb{N}$, the equation (21) has a pair of linearly independent solutions $v_1(x)$, $v_2(x)$ with asymptotic representation

$$v_1(x) = 1 + O(x), \qquad v_2(x) = x^{1-a}\left(1 + O(x)\right), \quad x \to 0.$$

In the resonant case where $a \in \mathbb{N}$ there exists a pair of linearly independent solutions with asymptotic representation

$$v_1(x) = 1 + O(x), \qquad v_2(x) = \alpha v_1(x) \ln x + 1 + O(x), \quad x \to 0,$$

where α is a real number. In both cases the Wronskian $W(x)$ of these solutions equals x^{-a}/ω where ω is a real number.

It is not hard to show that if for some $\rho \in (0, r]$, we have that $H > 0$, a function $v(x) \in \in C^1[0,\rho]$ satisfies the inequalities (20) and the integral equation

$$v(x) = A[v](x) := \int_0^x G(x,s)\big[f(s) + g(s)v(s) + h(s)v'(s) + F(s, v(s), v'(s))\big]\, ds, \tag{22}$$

where

$$G(x,s) := \frac{v_2(x)v_1(s) - v_1(x)v_2(s)}{W(s)} \equiv \omega s^a\big(v_2(x)v_1(s) - v_1(x)v_2(s)\big),$$

then this function is a solution on $(0,\rho]$ of differential equation (19).

Next we show that there exists such $H_1 > 0$ that

$$\int_0^x |G(x,s)|\, ds \le H_1 x^2, \quad \int_0^x |G'_x(x,s)|\, ds \le H_1 x \quad \forall x \in [0, r].$$

Now it turns out that we can choose sufficiently large $H > 0$ and sufficiently small $\rho > 0$ in such a way that the mapping $A : \mathcal{M}_{H,\rho} \mapsto \mathcal{M}_{H,\rho}$ be a contraction in the space

$$\mathcal{M}_{H,\rho} = \{v(x) \in C^1[0,\rho] : |v(x)| \le Hx^2,\ |v'(x)| \le Hx,\ x \in [0,\rho]\}$$

supplied with the standard metric

$$\varrho(v(x), w(x)) = \max_{x\in[0,\rho]} \big[|v(x) - w(x)| + |v'(x) - w'(x)|\big].$$

(Note that for any $v(x) \in \mathcal{M}_{H,\rho}$ we set, by definition, $A[v](0) = 0$.)

To prove that the fixed point $v(x)$ of the mapping A belongs to $C^2[0,\rho]$, it remains only to observe the following property of the kernel $G(x,s)$: for any function $\varphi(x) \in C[0,\rho]$ the function

$$\int_0^x G''_{xx}(x,s)\varphi(s)\, ds = \left(-\frac{av_2'(x)}{x} - \frac{bv_2(x)}{x}\right) \int_0^x \omega s^a v_1(s)\varphi(s)\, ds +$$

$$+ \left(-\frac{av_1'(x)}{x} - \frac{bv_1(x)}{x}\right) \int_0^x \omega s^a v_2(s)\varphi(s)\, ds$$

can be extended by continuity onto the whole segment $[0,\rho]$. The lemma is proved.

Remark 1. If in equation (19) the functions $f = f(x,\lambda)$, $g = g(x,\lambda) \in C([0,r] \times [0,\lambda_0])$ additionally depend on a parameter λ then the solution $v(x,\lambda)$ satisfying (20) is a continuous function of the variables (x,λ) on the set $[0,\rho] \times [0,\lambda_0]$.

Proposition 3. *Let the hypotheses H_3, H_4 be true. If for a nontrivial solution $y(x)$, $x \geq 0$, of the equation (1) there exists $x_0 \geq 0$ such that $V[y(x_0)] \leq 0$ then $y(x)$ does not vanish on $[x_0,\infty)$.*

Proof. In fact, if there exists $x_1 \geq x_0$ such that $y(x_1) = 0$, then $V[y(x_1)] = \big(y'(x_1)\big)^2/2 > 0$. But $V[y(x)]$ is nonincreasing, and thus we arrive at the contradiction, $V[y(x_0)] > 0$.

Proposition 4. *If the hypotheses $H_1 – H_5$ are true, then for $m = 0,1,\ldots,l+1$ each set $\mathcal{L}_m$ is nonempty and bounded.*

Proof. From $q(x,0) \equiv 0$ it follows that $y \equiv 0$ is a solution of (1). As a nontrivial solution, the function $y(x,0)$ has only simple zeroes: if $y(x_i,0) = 0$ then $y'(x_i,0) \neq 0$, $i = 1,\ldots,l+1$. Then by implicit function theorem there exists $\lambda_l > 0$ such that for any $\lambda \in (0,\lambda_l)$ the function $y(x,\lambda)$ has at least $l+1$ simple zeroes. Thus $\mathcal{L}_l \neq \varnothing$. Now we are going to show that one can choose $\lambda_* \geq \lambda_l$ in such a way that the function $y_*(x) := y(x,\lambda_*)$ does not vanish on $[0,\infty)$. If $z_0 = 0$, then $q_0(y_0) = 0$ and $q_0(y) = 0$ for $y \in [0,y_0]$. So $\lim_{x\to+0} Q(x,y_*(x)) = \int_0^{y_0} q_1(0,y)\,dy$, and

$$\lim_{x\to+0} V[y_*(x)] = \int\limits_0^{y_0} q_1(0,y)\,dy - \frac{\lambda_* y_0}{2} < 0$$

if λ_* is sufficiently large. From this it follows that $V[y_*(x)] < 0$ when $x > 0$ and, in virtue of Proposition 3, $y_*(x)$ does not vanish on $[0,\infty)$. Now let $z_0 \neq 0$. Without loss of generality we assume that $z_0 < 0$. Then $y_0 > 0$. Note that in view of (8) there is a neighborhood of (y_0,z_0) in which those points of the (y,z)-plane whose coordinates satisfy the equation (7) form the graph of a function $Z(y) \in C^1([y_0-\delta,y_0+\delta] \mapsto \mathbb{R})$, where δ is a positive number. We can choose $\delta \leq y_0/2$ in such a way that $Z(y) < 0$ for all $y \in [y_0-\delta,y_0]$. For a fixed λ_*, the function $y_*(x) := y(x,\lambda_*)$ is decreasing on some interval $[0,x_1)$. Choose x_1 as such a maximal number that ${y_*}'(x) < 0$ and $y_0-\delta < y_*(x) \leq y_0$ for all $x \in [0,x_1)$. Analyzing the nontrivial case where $x_1 < \infty$, let us show that if λ_* is sufficiently large, then

$$ {y_*}'(x) > Z(y_*(x)) \tag{23}$$

for all $x \in (0,x_1)$, i.e., the curve given by the equations

$$y = y_*(x), \quad z = {y_*}'(x), \quad x \in (0,x_1),$$

is contained inside the figure $\mathcal{F}$ bounded by the lines $y = y_0-\delta$, $y = y_0$, $z = 0$, $z = Z(y)$.

It is not hard to calculate that

$$y''_{xx}(+0,\lambda) \sim \frac{\lambda y_0}{\frac{\partial p_0(y_0,z_0)}{\partial z} z_0 + p_0(y_0,z_0)}, \quad \lambda \to \infty.$$

This implies

$$\frac{d}{dx}\Big|_{x=+0}\Big(y'_*(x)\Big) > \frac{d}{dx}\Big|_{x=+0}\Big(Z(y_*(x))\Big) = Z'(y_0)z_0,$$

provided that λ_* is sufficiently large. Taking into account that ${y_*}'(+0) = z_0 = Z(y_*(+0))$, one can assert that there exists a maximal number $x_2 \in (0, x_1]$ such that the inequality (23) holds for all $x \in (0, x_2)$. It turns out that $x_2 = x_1$. In fact, if we suppose that $x_2 < x_1$, we would get ${y_*}'(x_2) = Z(y_*(x_2))$ and, in view of $p_0(y, Z(y))Z(y) + q_0(y) = 0$,

$$
\begin{aligned}
{y_*}''(x_2) &= \lambda_* y_*(x_2) - p_1\big(x_2, y_*(x_2), {y_*}'(x_2)\big){y_*}'(x_2) - q_1\big(x_2, y_*(x_2)\big) \geq \\
&\geq \lambda_* y_0/2 - \max_{[0,x_1]\times\mathcal{F}} |p_1(x,y,z)z + q_1(x,y)| > \\
&> \max_{y_0-\delta\leq y\leq y_0} Z'(y)Z(y) \geq \frac{dZ(y_*(x_2))}{dx}
\end{aligned}
$$

provided that λ_* was chosen sufficiently large. On the other hand, by the definition of x_2, ${y_*}''(x_2) \leq \dfrac{dZ(y_*(x_2))}{dx}$. This contradiction proves that $x_2 = x_1$.

Now it is not hard to see that if $y_*(x_1) \in (y_0 - \delta, y_0)$ then ${y_*}'(x_1) = 0$; otherwise, $y_*(x_1) = y_0 - \delta$. In the first case we have $y''_*(x_1) \geq 0$ and thus $q(x_1, y_*(x_1)) - \lambda_* y_*(x_1) \leq 0$. Together with H_4, this implies $Q(x_1, y_*(x_1)) - \lambda_* y_*^2(x_1)/2 < 0$ and $V[y_*(x_1)] < 0$. Now the required property of $y_*(x)$ follows from Proposition 3 . In the second case, by means of the inequality

$$
\frac{y'_*(x)}{Z(y_*(x))} \leq 1, \quad x \in [0, x_1],
$$

we get for x_1 the lower bound which is independent of λ_*,

$$
x_1 \geq \int\limits_{y_0}^{y_0-\delta} \frac{dy}{Z(y)} \geq \frac{\delta}{\max\limits_{y_0-\delta\leq y\leq y_0} |Z(y)|}.
$$

Now

$$
V[y_*(x_1)] \leq \left(\max_{y_0-\delta\leq y\leq y_0} |Z(y)|\right)^2 /2 + \max_{y_0-\delta\leq y\leq y_0} Q(x_1, y) - \frac{\lambda_* y_0^2}{8} < 0
$$

provided that λ_* was chosen sufficiently large. Again we can apply Proposition 3.

Proposition 5. *For any $m = 0, \ldots, l$ the number of zeroes of each function $y_m(x)$ does not exceed m.*

Proof. In fact, otherwise $y_m(x)$ would have at least $m+1$ simple zeroes. Then $y(x, \lambda)$ would have the same property for λ belonging to some neighborhood of λ_m. This contradicts the definition of λ_m as an exact upper bound.

Proposition 6. *Let $\lambda \in \mathcal{L}_m$ and $x_1(\lambda) < \cdots < x_{m+1}(\lambda)$ be successive zeroes of $y(x, \lambda)$. Then* $x_{m+1}(\lambda) \to \infty, \ \lambda \to \lambda_m - 0$.

Proof. Should this assertion be false, there would exist a sequence $\lambda^{(k)} \to \lambda_m - 0$, $k \to \infty$, such that each sequence $\{x_j(\lambda^{(k)})\}_{k=1,2,\ldots}$, $j = 1, \ldots, m+1$, be convergent to a finite limit x_j^* which is a zero of $y_m(x)$. It is not hard to see that in such a case

$$
x_1^* < \cdots < x_{m+1}^*,
$$

otherwise $y_m(x)$ would have at least one multiple zero. Hence, $y_m(x)$ has $m+1$ different simple zeroes. This, however, contradicts Proposition 5.

Proposition 7. *Let the hypotheses H_3, H_4, $H_6 - H_8$ be true. If for some fixed $\lambda > 0$ the equation (1) has a solution $y(x)$ which possesses only a finite number of zeroes and satisfies the condition $V[y(x)] > 0\ \forall x > 0$, then $y(x)y'(x) < 0$ for all sufficiently large positive x and*

$$|y(x)| + |y'(x)| \to 0, \quad x \to +\infty.$$

Proof. Without loss of generality we analyze the case where there exists such a positive x_0 that $y(x) > 0\ \forall x > x_0$. First we suppose that $y'(x_0) > 0$ and show that there is $x_1 > x_0$ such that $y'(x_1) = 0$. Let, on the contrary, $y'(x)$ do not vanish on $[x_0, \infty)$. Put $\nu := \inf_{x \geq x_0} y'(x)$ and consider each of the following possible cases: (a) $\nu > 0$; (b) $\nu = 0$.

In the case (a) we would have $y(x) \geq y(x_0) + \nu x > \nu x$ and, under an appropriate choice of $\varepsilon = \varepsilon(\nu, x_0)$,

$$V[y(x)] = V[y(x_0)] + \int\limits_{x_0}^{x} V'[y(s)]\, ds \leq V[y(x_0)] - \int\limits_{x_0}^{x} \alpha_{\pm}(s, \varepsilon)\, ds \to -\infty, \quad x \to \infty.$$

Thus the case (a) is impossible.

In the case (b), for any given $\varepsilon > 0$, there exists $\tilde{x} > \max(x_0, R)$ such that $0 < y'(\tilde{x}) < \varepsilon$. From H_4 it follows that for any sufficiently small $\varepsilon > 0$ the equation

$$Q(x, y) - \frac{\lambda y^2}{2} = -\frac{\varepsilon^2}{2}$$

has a pair of positive roots $\xi_1(x, \lambda, \varepsilon)$, $\xi_2(x, \lambda, \varepsilon)$ satisfying the inequalities

$$0 < \xi_1(x, \lambda, \varepsilon) < \eta_+(x, \lambda) < \xi_2(x, \lambda, \varepsilon) < \xi_+(x, \lambda).$$

It is not hard to see that the function $\xi_1(x, \lambda, \varepsilon)$ is nonincreasing while the functions $\xi_2(x, \lambda, \varepsilon)$, $\eta_+(x, \lambda)$, $\xi_+(x, \lambda)$ are nondecreasing with respect to x. Moreover, $\xi_1(x, \lambda, \varepsilon) \to 0$, $\varepsilon \to 0$, and $\xi_2(x, \lambda, \varepsilon) \to \xi_+(x, \lambda)$, $\varepsilon \to 0$. We may think that $\xi_1(x, \lambda, \varepsilon) < y(x_0)$ for all $x \geq \tilde{x}$. Now it is clear that until $x \geq \tilde{x}$ and $0 < y'(x) < \varepsilon$ the following inequalities hold:

$$Q(x, y(x)) - \frac{\lambda y^2(x)}{2} > -\frac{\varepsilon^2}{2}, \tag{24}$$

$$y(x) > \xi_2(x, \lambda, \varepsilon), \quad q(x, y(x)) - \lambda y(x) > 0. \tag{25}$$

Hence,

$$y''(x) \leq \lambda y(x) - q(x, y(x)) < 0$$

and $y'(x)$ is monotone decreasing. Obviously, in case (b) we have $y'(x) \to 0$, $x \to \infty$. Thus, (25) must be realized for all $x \geq \tilde{x}$.

To obtain a contradiction from this fact, let us estimate $y''(x)$ more accurately. Rewriting the equality

$$Q(x, \xi_2) - \frac{\lambda \xi_2^2}{2} = -\frac{\varepsilon^2}{2}$$

in the form

$$\mu_+(x, \lambda) + \int\limits_{\eta_+(x,\lambda)}^{\xi_2(x,\lambda,\varepsilon)} (q(x, y) - \lambda y)\, dy = -\frac{\varepsilon^2}{2}$$

and assuming that ε is so small that $\varepsilon < |\mu_+(R, \lambda)|/2$ we get

$$\int\limits_{\eta_+(x,\lambda)}^{\xi_2(x,\lambda,\varepsilon)} (q(x, y) - \lambda y)\, dy \geq |\mu_+(x, \lambda)|/2.$$

Observe that $[q(x, y) - \lambda y]_{y=\eta_+(x,\lambda)} = 0$ and for any fixed x the derivative $q'_y(x, y) - \lambda$ is increasing in y. Taking into account that $q(x, y) - \lambda y > 0$ for $y > \eta_+(x, \lambda)$ we see that $q'_y(x, y) - \lambda > 0$ once $y > \eta_+(x, \lambda)$. Hence, $q(x, y) - \lambda y$ is monotone increasing in y on $(\eta_+(x, \lambda), \infty)$. Now by the mean value theorem we get

$$(q(x, y(x)) - \lambda y(x))(\xi_2 - \eta_+) \geq (q(x, y) - y)\big|_{y=\xi_2}(\xi_2 - \eta_+) \geq \frac{|\mu_+|}{2}$$

and thus

$$q(x, y(x)) - \lambda y(x) \geq \frac{|\mu_+(x, \lambda)|}{2(\xi_2(x, \lambda, \varepsilon) - \eta_+(x, \lambda))} \geq \frac{|\mu_+(x, \lambda)|}{2(\xi_+(x, \lambda) - \eta_+(x, \lambda))}. \tag{26}$$

Solving for $x > \tilde{x}$ the differential inequality

$$y''(x) \leq -\beta_+(x, \lambda) y'(x) - \frac{|\mu_+(x, \lambda)|}{2(\xi_+(x, \lambda) - \eta_+(x, \lambda))}$$

we obtain the estimate

$$y'(x) \leq e^{-\int_{\tilde{x}}^{x} \beta_+(s,\lambda)\, ds} \left(y'(\tilde{x}) - \int\limits_{\tilde{x}}^{x} e^{\int_{\tilde{x}}^{\tau} \beta_+(s,\lambda)\, ds} \frac{|\mu_+(\tau, \lambda)|}{2(\xi_+(\tau, \lambda) - \eta_+(\tau, \lambda))}\, d\tau \right).$$

Now if (11) holds, then $y'(x) \to -\infty$ as $x \to \infty$ and we arrive at a contradiction. Otherwise, (12) holds, and hence, $V[y(x)] \to -\infty,\ x \to \infty$. This is again impossible.

Thus we have proved that there exists $x_1 > x_0$ such that $y'(x_1) = 0,\ y(x_1) > \xi_+(x_1, \lambda)$ and

$$y''(x_1) \leq \lambda y(x_1) - q(x_1, y(x_1)) < 0.$$

From this it follows that $y'(x) < 0$ for all $x > x_1$. In fact, should the first moment $x_2 > x_1$ existed for which $y'(x_2) = 0$ we would have $y''(x_2) \geq 0$ and hence $q(x_2, y(x_2)) - \lambda y(x_2) \leq 0$. This would imply the contradiction: $V[y(x_2)] \leq 0$.

Let $y_* := \lim_{x\to\infty} y(x)$. In view of H_3 the function $Q(x, y(x))$ is nonincreasing,

$$Q(x'', y(x'')) \le Q(x'', y(x')) \le Q(x', y(x')) \quad \forall x'' > x'.$$

As $Q(x, y(x)),\ y(x)$ and $V[y(x)]$ have finite limits when $x \to \infty$, $y'(x)$ has the same property, and obviously $\lim_{x\to\infty} y'(x) = 0$.

Let us now show that $y_* = 0$. Suppose on the contrary that $y_* > 0$. We may think that ε was chosen so small that $\xi_1(R, \lambda, \varepsilon) < y_*$ and $|y'(x)| < \varepsilon$ for all sufficiently large x. Then $y(x) \ge y_* \ge \xi_1(x, \lambda, \varepsilon),\ V[y(x)] > 0$, and hence, the inequality (24) holds. This yields (25) and boundedness of the difference $\xi_2(x, \lambda, \varepsilon) - \eta_+(x, \lambda)$. For this reason in view of (26),

$$\liminf_{x\to\infty} (q(x, y(x)) - \lambda y(x)) > 0$$

and hence, taking into account H_6,

$$\limsup_{x\to\infty} y''(x) < 0.$$

This inequality contradicts existence of zero limit for $y'(x)$. The proposition is proved.

Lemma 2. *Let the coefficients $a(x),\ b(x) \in C[x_0, \infty)$ of the equation*

$$y'' + a(x)y' + b(x)y = 0 \tag{27}$$

satisfy the inequalities

$$|a(x) - \alpha|^2 \le 4\beta - \varepsilon, \quad -B \le b(x) \le -\beta \quad \forall x \ge x_0$$

with some constants $\alpha \ne 0$, $B > \beta > 0$ and $\varepsilon \in (0, 4\beta)$.

Then there exist positive numbers $K = K(x_0, \alpha, \beta, B, \varepsilon)$, $\gamma = \gamma(\alpha, \beta, B, \varepsilon)$ and linearly independent solutions $y_1(x),\ y_2(x)$ of (27) for which the following estimates hold

$$|y_1(x)| + |y_1'(x)| \le Ke^{-\gamma x}, \quad |y_2(x)| \ge \frac{e^{\gamma x}}{K}, \quad x \ge x_0. \tag{28}$$

If $y(x)$ is a nontrivial solution of (27) vanishing at a point $x_1 > x_0$, then

$$|y'(x_1)| \le K|y(x_0)y_2'(x_0) - y'(x_0)y_2(x_0)|e^{-\gamma x_1 - \int_{x_0}^{x_1} a(s)\,ds}. \tag{29}$$

Proof. The derivative of the quadratic form $W(y, z) := -\alpha y^2 - 2yz$, in virtue of the system

$$y' = z, \quad z' = -b(x)y - a(x)z \tag{30}$$

being equivalent to the equation (27), equals $W'(y, z) = -2(z^2 - b(x)y^2 + \big(\alpha - a(x)\big)yz)$. For eigenvalues $\lambda_1(x),\ \lambda_2(x)$ of the corresponding matrix

$$\begin{pmatrix} 2b(x) & a(x) - \alpha \\ a(x) - \alpha & -2 \end{pmatrix}$$

the following inequalities holds:

$$\lambda_1(x) + \lambda_2(x) \le -2(1+\beta), \quad \lambda_1(x)\lambda_2(x) \ge \varepsilon.$$

Hence, these eigenvalues are negative and separated from zero. Since the matrix of $W(y,z)$ is nondegenerate and indefinite, the system (30) is dichotomic (see, e.g. [18]). Hence, it has a fundamental system $(y_1(x), z_1(x))$, $(y_2(x), z_2(x))$ such that

$$W(y_1(x), z_1(x)) > 0, \quad W(y_2(x), z_2(x)) < 0 \quad \forall x \in [x_0, \infty), \tag{31}$$

and for certain $K_1 > 0$ and $\gamma > 0$ the following inequalities holds:

$$|y_1(x)| + |z_1(x)| \le K_1 e^{-\gamma x}, \quad |y_2(x)| + |z_2(x)| \ge e^{\gamma x}/K_1 \quad \forall x \ge x_0.$$

Observe that we may choose as $(y_2(x), z_2(x))$ an arbitrary solution of the system (30) with initial values satisfying the condition $W(y_2(x_0), z_2(x_0)) < 0$. Set, e.g., $z_2(x_0) = (1+|\alpha|)y_2(x_0)$ and $y_2(x_0) > 0$. Now to establish (28) it remains only to prove that for some $K > 0$ the function $y_2(x)$ obeys $|y_2(x)| \ge e^{\gamma x}/K$ for all $x \ge x_0$.

The second inequality in (31) implies $y_2(x) > 0$ and $y_2'(x) = z_2(x) > -\dfrac{\alpha}{2} y_2(x)$ for all $x \ge x_0$. Let us show that $y_2'(x) < A y_2(x)$, where A is a positive number such that

$$A > y_2'(x_0)/y_2(x_0) \quad \text{and} \quad A^2 > A \sup_{x \ge x_0} |a(x)| + \sup_{x \ge x_0} |b(x)|.$$

In fact, should there exist such a first “moment” $\bar{x} > x_0$ that $y_2'(\bar{x}) = A y_2(\bar{x})$, then $y_2''(\bar{x}) \ge$ $\ge A y_2'(\bar{x})$ and hence, in virtue of (27),

$$-a(\bar{x}) A y_2(\bar{x}) - b(\bar{x}) y_2(\bar{x}) \ge A^2 y_2(\bar{x}).$$

This, however, is impossible for the choice of A.

Thus, we have proved that $|y_2'(x)| = |z_2(x)| \le \max\left(\left|\dfrac{\alpha}{2}\right|, A\right) |y_2(x)|$. Consequently, we may set $K = K_1(1 + \max(|\alpha|, A))$.

Now we will deduce (29). Let w_0 stand for the value of the Wronskian of $y_1(x)$, $y_2(x)$ at $x = x_0$. Then $y(x) = c_1 y_1(x) + c_2 y_2(x)$ where

$$c_1 = \frac{y(x_0) y_2'(x_0) - y'(x_0) y_2(x_0)}{w_0}, \quad c_2 = -\frac{c_1 y_1(x_1)}{y_2(x_1)},$$

and hence,

$$y'(x_1) = c_1 \frac{y_1'(x_1) y_2(x_1) - y_1(x_1) y_2'(x_1)}{y_2(x_1)} =$$

$$= [y(x_0) y_2'(x_0) - y'(x_0) y_2(x_0)] e^{-\int_{x_0}^{x_1} a(s)\, ds} / y_2(x_1).$$

This, obviously, implies (29).

Lemma 3. *Let* $f(y) \in C^2[a,b] \cap C^3(a,b)$, $f(0) = f'(0) = 0$, *and* $f(y) > 0$, $f'''(y) < 0$ *for all* $y \in (a,b)$. *Then*

$$\frac{d^2\sqrt{f(y)}}{dy^2} < 0 \quad \forall y \in (a,b).$$

Proof. We have

$$\frac{d^2\sqrt{f(y)}}{dy^2} = \frac{1}{2}(f(y))^{-3/2}\left[f(y)f''(y) - \frac{1}{2}f'^2(y)\right].$$

Now let us show that the function $F(y) = f(y)f''(y) - \dfrac{1}{2}f'^2(y)$ takes only negative values on (a,b). In fact, $F(0) = 0$ and $F'(y) = f(y)f'''(y) < 0$ for all $y \in (a,b)$.

Proposition 8. *Let the hypotheses* H_3, H_4, H_6, H_{10} *be true and for some* $\lambda > 0$ *the equation (1) have a solution* $y(x)$ *for which there exist* $x_0 > R$ *and* $x^* > x_0$ *such that*

$$y(x_0) = 0, \quad 0 < y'(x_0) < \frac{\eta_+(R,\lambda)}{2}, \quad y(x^*) = \eta(x^*,\lambda),$$

$$V[y(x)] > 0, \quad 0 < y(x) < \eta_+(x,\lambda) \quad \forall x \in (x_0, x^*).$$

Then

$$x^* - M(\lambda)|\ln \eta_+(x^*,\lambda)| \le x_0 + e^{P(\lambda)M(\lambda)} + M(\lambda)|\ln y'(x_0)|$$

where

$$M(\lambda) := \sup_{x \ge R} \frac{\eta_+(x,\lambda)}{\sqrt{2|\mu_+(x,\lambda)|}}, \quad P(\lambda) := \sup_{\substack{x \ge R \\ 0 \le y \le \eta_+(R,\lambda)M(\lambda) \\ 0 \le z \le \eta_+(R,\lambda)}} p(x,y,z).$$

Proof. Let x_1 be a point at which the function $y'(x)$ attains its minimal value on the segment $[x_0, x^*]$. In virtue of Lemma 3, for any fixed x and λ, the function $y \mapsto \mapsto \sqrt{\lambda y^2 - 2Q(x,y)}$ is convex on the interval $(0, \xi_+(x,\lambda))$. For this reason

$$y'(x) \ge \sqrt{\lambda y^2(x) - 2Q(x,y(x))} \ge \frac{\sqrt{2|\mu_+(x,\lambda)|}}{\eta_+(x,\lambda)} y(x) \quad \forall x \in [x_0, x^*].$$

Thus,

$$y'(x_1) \ge \frac{\sqrt{2|\mu_+(x,\lambda)|}}{\eta_+(x,\lambda)} y'(x_1)(x_1 - x_0)$$

and

$$x_1 - x_0 \le M(\lambda), \quad y(x_1) \le \frac{\eta_+(R,\lambda)M(\lambda)}{2}. \tag{32}$$

Let $x_* \in (x_0, x^*)$ be a point such that

$$y(x_*) = \frac{\eta_+(x_*,\lambda)}{2}, \quad \frac{\eta_+(x,\lambda)}{2} < y(x) < \eta_+(x,\lambda) \quad \forall x \in (x_*, x^*).$$

Since $y(x_*) \geq \dfrac{\eta_+(R,\lambda)}{2} \geq y'(x_0)$, there exists $x_2 \in (x_0, x_*]$ such that $y(x_2) = y'(x_0)$. But $y(x_2) \geq y'(x_1)(x_2 - x_0)$. Hence, $x_2 - x_0 \leq y'(x_0)/y'(x_1)$. To estimate $y'(x_1)$ observe that in view of (32)

$$y''(x) \geq -p(x, y(x), y'(x))y'(x) \geq -P(\lambda)y'(x) \quad \forall x \in [x_0, x_1].$$

This implies $y'(x_1) \geq y'(x_0)e^{-P(\lambda)M(\lambda)}$ and thus

$$x_2 - x_0 \leq e^{P(\lambda)M(\lambda)}.$$

Since the inequality $y'(x) \geq y(x)/M(\lambda)$ holds on $[x_0, x^*]$ we get

$$\eta(x^*, \lambda) = y(x^*) \geq e^{(x^* - x_2)/M(\lambda)} y(x_2) = e^{(x^* - x_2)/M(\lambda)} y'(x_0)$$

and finally

$$x^* \leq x_2 + M(\lambda)\big[|\ln \eta_+(x^*, \lambda)| + |\ln y'(x_0)|\big] \leq$$
$$\leq x_0 + e^{P(\lambda)M(\lambda)} + M(\lambda)\big[|\ln \eta_+(x^*, \lambda)| + |\ln y'(x_0)|\big].$$

1. *Kiguradze I. T., Shekhter B. L.* Singular boundary-value problems for ordinary second-order differential equations // J. Sov. Math. – 1988. – **43**, № 2. – P. 2340 – 2417.
2. *Berestycki H., Lions P. L., Peletier L. A.* An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^n$ // Indiana Math. J. – 1981. – **30**, № 1. – P. 141 – 157.
3. *Zhidkov P. E., Sakbaev V. Zh.* On a nonlinear ordinary differential equation // Math. Notes. – 1994. – **55**, № 4. – P. 351 – 357.
4. *Cheng Sh., Zhang Y.* Singular boundary value problems on a half-line // J. Math. Anal. and Appl. – 1995. – **195**, № 2. – P. 449 – 468.
5. *O'Regan D.* Singular nonlinear differential equations on the half line // Topol. Methods Nonlinear Anal. – 1996 . – **8**, № 1. – P. 137 – 159.
6. *Agarwal R. P., O'Regan D.* Singular initial and boundary value problems with sign changing nonlinearities // IMA J. Appl. Math. – 2000. – **65**, № 2. – P. 173 – 198.
7. *Pozur S. V., Protsak L. V.* Singular eigenvalue problem for the Emden – Fowler equation. Existence of solution with a given number of zeroes // Bull. Kiev Univ. Ser. Math., Mech. – 2000. – № 5. – P. 34 – 39.
8. *Fushchich W. I., Serov N. I.* On some exact solutions of the three-dimensional nonlinear Schrödinger equation // J. Phys. A: Math. Gen. – 1987. – **20**. – P. 929 – 933.
9. *Grace S. R., Lalli B. S.* Oscillation theorem for certain second order perturbed nonlinear fifferential equations // J. Math. Anal. and Appl. – 1980. – **77**, № 1. – P. 205 – 214.
10. *Yeh Chen-Chin.* Oscillation criteria for second order nonlinear perturbed differential equations // Ibid. – 1989. – **138**, № 1. – P. 157 – 165.
11. *Grace S. R.* Oscillation criteria for second order differential equations with damping // J. Austral. Math. Soc. Ser. A. – 1990. – **49**, № 1. – P. 43 – 54.
12. *Knezhevich-Milyanovich Yu.* On asymptotic properties of the solutions of a second order nonlinear equation // Usp. Mat. Nauk. – 1992. – **47**, № 3. – P. 163 – 164.
13. *Grace S. R.* Oscillation theorems for nonlinear differential equations of second order // J. Math. Anal. and Appl. – 1992. – **171**, № 1. – P. 220 – 241.

14. *Naito Y.* Bounded solutions with prescribed number of zeros for the Emden – Fowler differential equations // Hiroshima Math. J. – 1994. – **24**. – P. 177 – 220.

15. *Wong P. J. Y., Agarwal R. P.* Oscillation theorems and existence criteria of asymptotically monotone solutions for second order differential equations // Dyn. Syst. Appl. – 1995. – **4**, № 4. – P. 477 – 496.

16. *Rogovchenko Y. V.* Oscillation criteria for second order nonlinear perturbed differential equations // J. Math. Anal. and Appl. – 1997. – **215**, № 2. – P. 334 – 357.

17. *Cecchi M., Marini M., Villari G.* Comparison results for oscillation of nonlinear differential equations // Nonlinear Different. Equat. and Appl. – 1999. – **6**. – P. 173 – 190.

18. *Samoilenko A. M.* On exponential dichotomy on $\mathbb{R}$ of linear differential equations in $\mathbb{R}^n$ // Ukr. Mat. Zh. – 2001. – **53**, No 3. – P. 356 – 371.

Received 02.04.2002