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ON A PERIODIC TYPE BOUNDARY-VALUE PROBLEM FOR FIRST ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS*

ПРО ГРАНИЧНУ ЗАДАЧУ ПЕРІОДИЧНОГО ТИПУ ДЛЯ ЛІНІЙНИХ ФУНКЦІОНАЛЬНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ПЕРШОГО ПОРЯДКУ

R. Hakl

Math. Inst., Czech Acad. Sci.
Žižkova 22, 616 62 Brno, Czech Republic
e-mail: hakl@ipm.cz

A. Lomtatidze

Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
e-mail: bacho@math.muni.cz

J. Šremr

Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
e-mail: sremr@math.muni.c

Nonimprovable sufficient conditions are established for unique solvability of the boundary-value problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = \lambda u(b) + c,$$

as well as for nonnegativeness of its solution, where $\ell : C([a,b];R) \to L([a,b];R)$ *is a linear bounded operator,* $q \in L([a,b];R)$*,* $\lambda \in R_+$*, and* $c \in R$*.*

Знайдено достатні умови, що не можуть бути поліпшені, для однозначної розв'язності граничної задачі $u'(t) = \ell(u)(t) + q(t), \quad u(a) = \lambda u(b) + c$*, та невід'ємності її розв'язку, де* $\ell : C([a,b];R) \to L([a,b];R)$ *— неперервний лінійний оператор,* $q \in L([a,b];R)$*,* $\lambda \in R_+$ *та* $c \in R$*.*

Introduction. The following notation is used throughout the paper.

R is the set of all real numbers, $R_+ = [0, +\infty[$, $R_- =]-\infty, 0]$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R) : u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}([a,b];R)$ is the set of absolutely continuous functions $u : [a,b] \to R$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|ds$.

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$L([a,b];D) = \{p \in L([a,b];R) : p : [a,b] \to D\}$, where $D \subseteq R$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];R) \to L([a,b];R)$.

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

$[x]_+ = \frac{1}{2}(|x|+x)$, $[x]_- = \frac{1}{2}(|x|-x)$.

By a solution of the equation

$$u'(t) = \ell(u)(t) + q(t), \tag{0.1}$$

where $\ell \in \mathcal{L}_{ab}$ and $q \in L([a,b];R)$, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equation (0.1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of (0.1) satisfying the boundary condition

$$u(a) = \lambda u(b) + c, \tag{0.2}$$

where $\lambda \in R_+$, $c \in R$.

The general boundary-value problems for functional differential equations have been studied very intensively. There are a lot of general results (see, e.g., [1–27]), but still only a few effective criteria for the solvability of special boundary-value problems for functional differential equations are known even in the linear case. In the present paper, we try to fill to some extent the existing gap in a certain way. More precisely, in Section 1 we give nonimprovable effective sufficient conditions for the unique solvability of the problem (0.1), (0.2) as well as for the nonnegativeness of a solution of that problem. Sections 2 and 3 are devoted respectively to the proofs of the main results and the examples verifying their optimality.

All results will be concretized for the differential equation with deviating arguments, i.e., for the case where the equation (0.1) has the form

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q(t), \tag{0.3}$$

where $p, g \in L([a,b];R_+)$, $q \in L([a,b];R)$, and $\tau, \mu \in \mathcal{M}_{ab}$.

The special cases of the discussed boundary-value problem are the Cauchy problem (for $\lambda = 0$) and the periodic boundary-value problem (for $\lambda = 1$). In these cases, the below theorems coincide with the results obtained in [4] and [10].

Along with the problem (0.1), (0.2) we consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{0.1_0}$$

$$u(a) = \lambda u(b). \tag{0.2_0}$$

From the general theory of linear boundary-value problem for functional differential equations, the following result is known (see, e.g., [3, 19, 27]).

Theorem 0.1. *The problem (0.1), (0.2) is uniquely solvable if and only if the corresponding homogeneous problem* (0.1_0), (0.2_0) *has only the trivial solution.*

1. Main results.

Theorem 1.1. *Let* $\lambda \in]0,1]$*, the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$*, where*

$$\ell_0, \ell_1 \in \mathcal{P}_{ab}, \tag{1.1}$$

and let either

$$\|\ell_0(1)\|_L < 1, \tag{1.2}$$

$$\frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{1-\lambda}{\lambda} < \|\ell_1(1)\|_L < 1 + \lambda + 2\sqrt{1 - \|\ell_0(1)\|_L}, \tag{1.3}$$

or

$$\|\ell_1(1)\|_L < \lambda, \tag{1.4}$$

$$\frac{1}{\lambda - \|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L < 2 + 2\sqrt{\lambda - \|\ell_1(1)\|_L}. \tag{1.5}$$

Then the problem (0.1), (0.2) has a unique solution.

Remark 1.1. For $\lambda = 0$, the first inequality in (1.3) becomes unimportant. Consequently, Theorem 1.3 in [3] can be understoond as a limit case of Theorem 1.3 as λ tends to zero.

Remark 1.2. Let $\lambda \in [1, +\infty[$ and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. Define an operator $\psi : L([a,b]; R) \to L([a,b]; R)$ by

$$\psi(w)(t) \stackrel{\text{df}}{=} w(a+b-t) \qquad \text{for } t \in [a,b].$$

Let φ be a restriction of ψ to the space $C([a,b]; R)$. Put $\mu = \dfrac{1}{\lambda}$, and

$$\widehat{\ell}_0(w)(t) \stackrel{\text{df}}{=} \psi(\ell_0(\varphi(w)))(t), \qquad \widehat{\ell}_1(w)(t) \stackrel{\text{df}}{=} \psi(\ell_1(\varphi(w)))(t) \qquad \text{for } t \in [a,b].$$

It is clear that if u is a solution of the problem (0.1_0), (0.2_0), then the function $v \stackrel{\text{df}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{\ell}_1(v)(t) - \widehat{\ell}_0(v)(t), \qquad v(a) = \mu v(b), \tag{1.6}$$

and vice versa, if v is a solution of the problem (1.6), then the function $u \stackrel{\text{df}}{=} \varphi(v)$ is a solution of the problem (0.1_0), (0.2_0),.

It is evident also that

$$\|\widehat{\ell}_0(1)\|_L = \|\ell_0(1)\|_L, \qquad \|\widehat{\ell}_1(1)\|_L = \|\ell_1(1)\|_L.$$

Therefore, Theorem 1.1 immediately yields the following theorem.

Theorem 1.2. *Let* $\lambda \in [1, +\infty[$*, the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$*, where* ℓ_0 *and* ℓ_1 *satisfy the condition (1.1), and let either*

$$\|\ell_1(1)\|_L < 1,$$

$$\frac{\|\ell_1(1)\|_L}{1-\|\ell_1(1)\|_L}+1-\lambda < \|\ell_0(1)\|_L < 1+\frac{1}{\lambda}+2\sqrt{1-\|\ell_1(1)\|_L},$$

or

$$\|\ell_0(1)\|_L < \frac{1}{\lambda},$$

$$\frac{1}{\frac{1}{\lambda}-\|\ell_0(1)\|_L}-1 < \|\ell_1(1)\|_L < 2+2\sqrt{\frac{1}{\lambda}-\|\ell_0(1)\|_L}.$$

Then the problem (0.1), (0.2) has a unique solution.

Remark 1.3. In Section 3 we give examples (see Examples 3.1 – 3.6) showing that neither one of the strict inequalities (1.2) – (1.5) can be replaced by the nonstrict ones. According to Remark 1.2 and the above-said, the conditions of Theorem 1.2 are also nonimprovable.

Theorem 1.3. *Let $\lambda \in]0,1]$, $q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$, and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the condition (1.1). Let, moreover,*

$$\|\ell_0(1)\|_L < 1, \qquad \|\ell_1(1)\|_L < \lambda \quad (resp.\ \|\ell_1(1)\|_L \leq \lambda) \tag{1.7}$$

and

$$\frac{\|\ell_0(1)\|_L}{1-\|\ell_0(1)\|_L}-\frac{1-\lambda}{\lambda} < \|\ell_1(1)\|_L. \tag{1.8}$$

Then the problem (0.1), (0.2) has a unique solution, and this solution is positive (resp. nonnegative).

Theorem 1.4. *Let $\lambda \in]0,1]$, $q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$, and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the condition (1.1). Let, moreover,*

$$\|\ell_0(1)\|_L < 1 \quad (resp.\ \|\ell_0(1)\|_L \leq 1), \qquad \|\ell_1(1)\|_L < \lambda \tag{1.9}$$

and

$$\frac{1}{\lambda-\|\ell_1(1)\|_L}-1 < \|\ell_0(1)\|_L. \tag{1.10}$$

Then the problem (0.1), (0.2) has a unique solution, and this solution is negative (resp. nonpositive).

According to Remark 1.2, from Theorems 1.3 and 1.4 we have the following.

Theorem 1.5. *Let $\lambda \in [1,+\infty[$, $q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$, and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the condition (1.1). If, moreover,*

$$\|\ell_1(1)\|_L < 1, \qquad \|\ell_0(1)\|_L < \frac{1}{\lambda} \quad \left(resp.\ \|\ell_0(1)\|_L \leq \frac{1}{\lambda}\right)$$

and

$$\frac{\|\ell_1(1)\|_L}{1-\|\ell_1(1)\|_L}+1-\lambda < \|\ell_0(1)\|_L,$$

then the problem (0.1), (0.2) has a unique solution, and this solution is negative (resp. nonpositive).

Theorem 1.6. *Let $\lambda \in [1, +\infty[$, $q \in L([a,b]; R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$, and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the condition (1.1). If, moreover,*

$$\|\ell_1(1)\|_L < 1 \quad (resp.\ \|\ell_1(1)\|_L \leq 1), \qquad \|\ell_0(1)\|_L < \frac{1}{\lambda}$$

and

$$\frac{1}{\frac{1}{\lambda} - \|\ell_0(1)\|_L} - 1 < \|\ell_1(1)\|_L,$$

then the problem (0.1), (0.2) has a unique solution, and this solution is positive (resp. nonnegative).

Remark 1.4. In Section 3 we give examples (see Examples 3.7 and 3.8) showing that neither one of the inequalities (1.7) – (1.10) can be weakened. According to Remark 1.2 and the abovesaid, the conditions of Theorems 1.5 and 1.6 are also nonimprovable.

For an equation of the type (0.3), from Theorems 1.1 – 1.6 we get the following assertions.

Corollary 1.1. *Let $\lambda \in]0,1]$, $p, g \in L([a,b]; R_+)$, and let either*

$$\int_a^b p(s)ds < 1, \qquad \frac{\int_a^b p(s)ds}{1 - \int_a^b p(s)ds} - \frac{1-\lambda}{\lambda} < \int_a^b g(s)ds < 1 + \lambda + 2\sqrt{1 - \int_a^b p(s)ds},$$

or

$$\int_a^b g(s)ds < \lambda, \qquad \frac{1}{\lambda - \int_a^b g(s)ds} - 1 < \int_a^b p(s)ds < 2 + 2\sqrt{\lambda - \int_a^b g(s)ds}.$$

Then the problem (0.3), (0.2) has a unique solution.

Corollary 1.2. *Let $\lambda \in [1, +\infty[$, $p, g \in L([a,b]; R_+)$, and let either*

$$\int_a^b g(s)ds < 1, \qquad \frac{\int_a^b g(s)ds}{1 - \int_a^b g(s)ds} + 1 - \lambda < \int_a^b p(s)ds < 1 + \frac{1}{\lambda} + 2\sqrt{1 - \int_a^b g(s)ds},$$

or

$$\int_a^b p(s)ds < \frac{1}{\lambda}, \qquad \frac{1}{\frac{1}{\lambda} - \int_a^b p(s)ds} - 1 < \int_a^b g(s)ds < 2 + 2\sqrt{\frac{1}{\lambda} - \int_a^b p(s)ds}.$$

Then the problem (0.3), (0.2) has a unique solution.

Corollary 1.3. *Let* $\lambda \in]0,1]$, $p,g,q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$,

$$\int_a^b p(s)ds < 1, \qquad \int_a^b g(s)ds < \lambda \qquad \left(resp.\ \int_a^b g(s)ds \leq \lambda \right),$$

and

$$\frac{\int_a^b p(s)ds}{1 - \int_a^b p(s)ds} - \frac{1-\lambda}{\lambda} < \int_a^b g(s)ds.$$

Then the problem (0.3), (0.2) has a unique solution, and this solution is positive (resp. nonnegative).

Corollary 1.4. *Let* $\lambda \in]0,1]$, $p,g,q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$,

$$\int_a^b p(s)ds < 1 \qquad \left(resp.\ \int_a^b p(s)ds \leq 1 \right), \qquad \int_a^b g(s)ds < \lambda,$$

and

$$\frac{1}{\lambda - \int_a^b g(s)ds} - 1 < \int_a^b p(s)ds.$$

Then the problem (0.3), (0.2) has a unique solution, and this solution is negative (resp. nonpositive).

Corollary 1.5. *Let* $\lambda \in [1,+\infty[$, $p,g,q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$,

$$\int_a^b p(s)ds < \frac{1}{\lambda} \qquad \left(resp.\ \int_a^b p(s)ds \leq \frac{1}{\lambda} \right), \qquad \int_a^b g(s)ds < 1,$$

and

$$\frac{\int_a^b g(s)ds}{1 - \int_a^b g(s)ds} + 1 - \lambda < \int_a^b p(s)ds.$$

Then the problem (0.3), (0.2) has a unique solution, and this solution is negative (resp. nonpositive).

Corollary 1.6. *Let* $\lambda \in [1,+\infty[$, $p,g,q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$,

$$\int_a^b p(s)ds < \frac{1}{\lambda}, \qquad \int_a^b g(s)ds < 1 \qquad \left(resp.\ \int_a^b g(s)ds \leq 1 \right),$$

and

$$\frac{1}{\dfrac{1}{\lambda} - \displaystyle\int_a^b p(s)ds} - 1 < \int_a^b g(s)ds.$$

Then the problem (0.3), (0.2) has a unique solution, and this solution is positive (resp. nonnegative).

2. Proofs.

To prove Theorems 1.1, 1.3, and 1.4, we need the following lemmas.

Lemma 2.1. *Let* $\lambda \in]0,1]$, $q \in L([a,b];R_-)$, $c \in R_-$, *the operator* ℓ *admit the representation* $\ell = \ell_0 - \ell_1$, *where* ℓ_0 *and* ℓ_1 *satisfy the condition (1.1), and let*

$$\|\ell_0(1)\|_L < 1, \qquad \frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{1-\lambda}{\lambda} < \|\ell_1(1)\|_L. \tag{2.1}$$

Then the problem (0.1), (0.2) has no nontrivial solution u *satisfying the inequality*

$$u(t) \geq 0 \qquad \text{for } t \in [a,b]. \tag{2.2}$$

Proof. Assume the contrary that the problem (0.1), (0.2) has a nontrivial solution u satisfying the condition (2.2). Put

$$M = \max\{u(t) : t \in [a,b]\}, \qquad m = \min\{u(t) : t \in [a,b]\} \tag{2.3}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = m. \tag{2.4}$$

Obviously, $M > 0$, $m \geq 0$ and either

$$t_M > t_m, \tag{2.5}$$

or

$$t_M < t_m. \tag{2.6}$$

First suppose that (2.5) holds. The integration of (0.1) from t_m to t_M, on account of (1.1), (2.3), (2.4), and the assumption $q \in L([a,b];R_-)$, results in

$$M - m = \int_{t_m}^{t_M} [\ell_0(u)(s) - \ell_1(u)(s) + q(s)]ds \leq M \int_{t_m}^{t_M} \ell_0(1)(s)ds \leq M\|\ell_0(1)\|_L.$$

Now suppose that (2.6) is fulfilled. The integration of (0.1) from a to t_M and from t_m to b, in view of (1.1), (2.3), (2.4) and the assumption $q \in L([a,b];R_-)$, yields

$$M - u(a) \leq M \int_a^{t_M} \ell_0(1)(s)ds, \qquad u(b) - m \leq M \int_{t_m}^{b} \ell_0(1)(s)ds.$$

Summing the last two inequalities and taking into account the condition

$$u(b) - u(a) \geq \lambda u(b) - u(a) = -c \geq 0,$$

we obtain

$$M(1 - \|\ell_0(1)\|_L) \leq m. \tag{2.7}$$

Therefore, in both cases (2.5) and (2.6), the inequality (2.7) is valid.

On the other hand, the integration of (0.1) from a to b, in view of (1.1), (2.3), and the assumption $q \in L([a, b]; R_-)$, implies

$$u(b) - u(a) = \int_a^b [\ell_0(u)(s) - \ell_1(u)(s) + q(s)]ds \leq M\|\ell_0(1)\|_L - m\|\ell_1(1)\|_L.$$

Hence, by (2.3), (0.2) and the assumptions $\lambda \in]0, 1]$, $c \in R_-$, we have

$$m\|\ell_1(1)\|_L \leq M\|\ell_0(1)\|_L + u(a)\left(1 - \frac{1}{\lambda}\right) + \frac{1}{\lambda}c \leq M\|\ell_0(1)\|_L + m\left(1 - \frac{1}{\lambda}\right).$$

Thus

$$m\left(\|\ell_1(1)\|_L + \frac{1-\lambda}{\lambda}\right) \leq M\|\ell_0(1)\|_L.$$

This inequality together with (2.7) results in

$$\|\ell_1(1)\|_L \leq \frac{\|\ell_0(1)\|_L}{1 - \|\ell_0(1)\|_L} - \frac{1-\lambda}{\lambda},$$

which contradicts the second inequality in (2.1).

Lemma 2.2. *Let $\lambda \in]0, 1]$, $q \in L([a, b]; R_+)$, $c \in R_+$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the condition (1.1), and let*

$$\|\ell_1(1)\|_L < \lambda, \qquad \frac{1}{\lambda - \|\ell_1(1)\|_L} - 1 < \|\ell_0(1)\|_L. \tag{2.8}$$

Then the problem (0.1), (0.2) has no nontrivial solution u satisfying the inequality (2.2).

Proof. Assume the contrary that the problem (0.1), (0.2) has a nontrivial solution u satisfying the condition (2.2). Define the numbers M and m by (2.3) and choose $t_M, t_m \in [a, b]$ such that (2.4) is fulfilled. Obviously, $M > 0, m \geq 0$ and either (2.5) or (2.6) is valid.

First suppose that (2.6) holds. The integration of (0.1) from t_M to t_m, on account of (1.1), (2.3), (2.4) and the assumptions $\lambda \in]0, 1]$ and $q \in L([a, b]; R_+)$, results in

$$\lambda M - m \leq M - m = \int_{t_M}^{t_m} [\ell_1(u)(s) - \ell_0(u)(s) - q(s)]ds \leq$$

$$\leq M \int_{t_M}^{t_m} \ell_1(1)(s)ds \leq M\|\ell_1(1)\|_L.$$

Now suppose that (2.5) is fulfilled. The integration of (0.1) from a to t_m and from t_M to b, in view of (1.1), (2.3), (2.4) and the assumptions $\lambda \in]0,1]$, $q \in L([a,b];R_+)$, yields

$$u(a) - m \leq M \int_a^{t_m} \ell_1(1)(s)ds, \qquad \lambda(M - u(b)) \leq M - u(b) \leq M \int_{t_M}^{b} \ell_1(1)(s)ds.$$

Summing the last two inequalities and taking into account the condition

$$u(a) - \lambda u(b) = c \geq 0,$$

we obtain

$$M(\lambda - \|\ell_1(1)\|_L) \leq m. \tag{2.9}$$

Therefore, in both cases (2.5) and (2.6), the inequality (2.9) is valid.

On the other hand, the integration of (0.1) from a to b, in view of (1.1), (2.3), and the assumption $q \in L([a,b];R_+)$, results in

$$u(a) - u(b) = \int_a^b [\ell_1(u)(s) - \ell_0(u)(s) - q(s)]ds \leq M\|\ell_1(1)\|_L - m\|\ell_0(1)\|_L.$$

Hence, by (2.3), (0.2) and the assumptions $\lambda \in]0,1]$, $c \in R_+$, we have

$$m\|\ell_0(1)\|_L \leq M\|\ell_1(1)\|_L + u(b)\,(1-\lambda) - c \leq M\|\ell_1(1)\|_L + M\,(1-\lambda)\,.$$

Thus

$$m\|\ell_0(1)\|_L \leq M\,(\|\ell_1(1)\|_L - \lambda + 1)\,.$$

This inequality together with (2.9) yields

$$\|\ell_0(1)\|_L \leq \frac{1}{\lambda - \|\ell_1(1)\|_L} - 1,$$

which contradicts the second inequality in (2.8).

Proof of Theorem 1.1. According to Theorem 0.1, it is sufficient to show that the homogeneous problem (0.1_0), (0.2_0) has no nontrivial solution.

First suppose that (1.2) and (1.3) hold. Assume the contrary that the problem (0.1_0), (0.2_0) has a nontrivial solution u. According to Lemma 2.1, u has to change its sign. Put

$$M = \max\{u(t) : t \in [a,b]\}, \qquad m = -\min\{u(t) : t \in [a,b]\} \tag{2.10}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{2.11}$$

Obviously, $M > 0$, $m > 0$. Without loss of generality we can assume that $t_m < t_M$. The integration of (0.1_0) from a to t_m, from t_m to t_M and from t_M to b, by (2.10), (2.11) and (1.1), results in

$$u(a) + m = \int_a^{t_m} [\ell_1(u)(s) - \ell_0(u)(s)]ds \le M \int_a^{t_m} \ell_1(1)(s)ds + m \int_a^{t_m} \ell_0(1)(s)ds, \qquad (2.12)$$

$$M + m = \int_{t_m}^{t_M} [\ell_0(u)(s) - \ell_1(u)(s)]ds \le M \int_{t_m}^{t_M} \ell_0(1)(s)ds + m \int_{t_m}^{t_M} \ell_1(1)(s)ds, \qquad (2.13)$$

$$M - u(b) = \int_{t_M}^{b} [\ell_1(u)(s) - \ell_0(u)(s)]ds \le M \int_{t_M}^{b} \ell_1(1)(s)ds + m \int_{t_M}^{b} \ell_0(1)(s)ds. \qquad (2.14)$$

Multiplying the both sides of (2.14) by λ and taking into account (2.10) and the assumption $\lambda \in]0, 1]$, we get

$$\lambda M - \lambda u(b) \le M \int_{t_M}^{b} \ell_1(1)(s)ds + m \int_{t_M}^{b} \ell_0(1)(s)ds.$$

Summing the last inequality and (2.13), by (0.2_0) we obtain

$$\lambda M + m \le M \int_J \ell_1(1)(s)ds + m \int_J \ell_0(1)(s)ds, \qquad (2.15)$$

where $J = [a, t_m] \cup [t_M, b]$. From (2.13) and (2.15) it follows that

$$M(1 - D) \le m(B - 1), \qquad m(1 - C) \le M(A - \lambda), \qquad (2.16)$$

where

$$A = \int_J \ell_1(1)(s)ds, \qquad B = \int_{t_m}^{t_M} \ell_1(1)(s)ds,$$
$$C = \int_J \ell_0(1)(s)ds, \qquad D = \int_{t_m}^{t_M} \ell_0(1)(s)ds. \qquad (2.17)$$

Due to (1.2), $C < 1$ and $D < 1$. Consequently, (2.16) implies $A > \lambda$, $B > 1$, and

$$0 < (1 - C)(1 - D) \le (A - \lambda)(B - 1). \qquad (2.18)$$

Obviously,

$$(1-C)(1-D) \geq 1-(C+D) = 1-\|\ell_0(1)\|_L > 0,$$

$$4(A-\lambda)(B-1) \leq [A+B-(1+\lambda)]^2 = [\|\ell_1(1)\|_L-(1+\lambda)]^2.$$

By the last inequalities, (2.18) results in

$$0 < 4(1-\|\ell_0(1)\|_L) \leq [\|\ell_1(1)\|_L-(1+\lambda)]^2,$$

which contradicts the second inequality in (1.3).

Now suppose that (1.4) and (1.5) are fulfilled. Assume the contrary that the problem (0.1_0), (0.2_0) has a nontrivial solution u. According to Lemma 2.2, u has to change its sign. Define M and m by (2.10) and choose $t_M, t_m \in [a,b]$ such that (2.11) is fulfilled. Without loss of generality we can assume that $t_m < t_M$. Analogously to the above, one can show that the inequalities (2.12) – (2.15) hold, where $J = [a,t_m] \cup [t_M,b]$. From (2.13) and (2.15) it follows that

$$m(1-B) \leq M(D-1), \qquad M(\lambda-A) \leq m(C-1), \tag{2.19}$$

where A, B, C, D are defined by (2.17). According to (1.4), $A < \lambda$ and $B < \lambda \leq 1$. Consequently, (2.19) implies $C > 1$, $D > 1$ and

$$0 < (\lambda-A)(1-B) \leq (C-1)(D-1). \tag{2.20}$$

Obviously,

$$(\lambda-A)(1-B) \geq \lambda-(A+B) = \lambda-\|\ell_1(1)\|_L > 0,$$

$$4(C-1)(D-1) \leq (C+D-2)^2 = (\|\ell_0(1)\|_L-2)^2.$$

By the last inequalities, from (2.20) we get

$$0 < 4(\lambda-\|\ell_1(1)\|_L) \leq (\|\ell_0(1)\|_L-2)^2,$$

which contradicts the second inequality in (1.5).

Proof of Theorem 1.3. According to Theorem 1.1 and the conditions (1.7), (1.8), the problem (0.1), (0.2) has a unique solution u.

Show that u has no zero (resp. does not change its sign). Assume the contrary that there exists $t_1 \in [a,b]$ (resp. $t_2, t_3 \in [a,b]$) such that

$$u(t_1) = 0 \qquad (\text{resp.} \quad u(t_2)u(t_3) < 0). \tag{2.21}$$

Define numbers M and m by (2.10) and choose $t_M, t_m \in [a,b]$ such that (2.11) is fulfilled. Obviously,

$$M \geq 0, \qquad m \geq 0, \qquad M+m > 0 \tag{2.22}$$

$$\big(\text{resp.}\quad M > 0, \qquad m > 0\big), \tag{2.23}$$

and either (2.5) or (2.6) is valid.

First suppose that (2.5) holds. The integration of (0.1) from a to t_m and from t_M to b, on account of (1.1), (2.10), (2.11) and the assumptions $\lambda \in]0,1]$ and $q \in L([a,b]; R_+)$, results in

$$m + u(a) \leq M \int_a^{t_m} \ell_1(1)(s)ds + m \int_a^{t_m} \ell_0(1)(s)ds, \tag{2.24}$$

$$\lambda(M - u(b)) \leq M - u(b) \leq M \int_{t_M}^{b} \ell_1(1)(s)ds + m \int_{t_M}^{b} \ell_0(1)(s)ds. \tag{2.25}$$

Summing the last two inequalities and taking into account the condition

$$u(a) - \lambda u(b) = c \geq 0,$$

we obtain

$$\lambda M + m \leq M\|\ell_1(1)\|_L + m\|\ell_0(1)\|_L, \tag{2.26}$$

which by (1.7) yields the contradiction $\lambda M + m < \lambda M + m$ (resp. $m < m$).

Now suppose that (2.6) is fulfilled. The integration of (0.1) from t_M to t_m, on account of (1.1), (2.10), (2.11) and the assumption $q \in L([a,b]; R_+)$, results in

$$M + m = \int_{t_M}^{t_m} [\ell_1(u)(s) - \ell_0(u)(s) - q(s)]ds \leq M\|\ell_1(1)\|_L + m\|\ell_0(1)\|_L. \tag{2.27}$$

Hence, by (1.7) and the assumption $\lambda \in]0,1]$, we get the contradiction $M + m < M + m$. Thus u has no zero (resp. does not change its sign), and so according to Lemma 2.1, u is positive (resp. nonnegative).

Proof of Theorem 1.4. According to Theorem 1.1 and the conditions (1.9), (1.10), the problem (0.1), (0.2) has a unique solution u.

Show that u has no zero (resp. does not change its sign). Assume the contrary that there exists $t_1 \in [a,b]$ (resp. $t_2, t_3 \in [a,b]$) such that (2.21) is fulfilled. Define numbers M and m by (2.10) and choose $t_M, t_m \in [a,b]$ such that (2.11) is fulfilled. Obviously, (2.22) (resp. (2.23)) is satisfied, and either (2.5) or (2.6) is valid.

By the same arguments as in the proof of Theorem 1.3 one can show that the assumption (2.5) yields the contradiction $\lambda M + m < \lambda M + m$ (resp. $M < M$), and the assumption (2.6) yields the contradiction $M + m < M + m$. Thus u has no zero (resp. does not change its sign), and so according to Lemma 2.2, u is negative (resp. nonpositive).

3. On Remarks 1.3 and 1.4. ***On Remark 1.3.*** Let $\lambda \in]0,1[$ (the case $\lambda = 0$, resp. $\lambda = 1$, is studied in [4], resp. [10], where the examples are also given verifying the optimality of the obtained results). Denote by H^+, resp. H^-, the set of pairs $(x,y) \in R_+ \times R_+$ such that

$$x < 1, \qquad \frac{x}{1-x} - \frac{1-\lambda}{\lambda} < y < 1 + \lambda + 2\sqrt{1-x},$$

resp.

$$y < \lambda, \qquad \frac{1}{\lambda - y} - 1 < x < 2 + 2\sqrt{\lambda - y}.$$

By Theorem 1.1, if $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in H^+ \cup H^-$, then the problem (0.1), (0.2) has a unique solution. (Note also that for $\lambda \le \frac{1}{4}$, $H^- = \emptyset$.)

Below we give the examples which show that for any pair $(x_0, y_0) \notin H^+ \cup H^-$, $x_0 \ge 0$, $y_0 \ge 0$ there exist functions $h \in L([a,b];R)$ and $\tau \in \mathcal{M}_{ab}$ such that

$$\int_a^b [h(s)]_+ ds = x_0, \qquad \int_a^b [h(s)]_- ds = y_0, \tag{3.1}$$

and the problem

$$u'(t) = h(t)u(\tau(t)), \qquad u(a) = \lambda u(b) \tag{3.2}$$

has a nontrivial solution. Then by Theorem 0.1, there exist $q \in L([a,b];R)$ and $c \in R$ such that the problem (0.1), (0.2), where $\ell = \ell_0 - \ell_1$,

$$\ell_0(w)(t) \overset{\mathrm{df}}{=} [h(t)]_+ w(\tau(t)), \qquad \ell_1(w)(t) \overset{\mathrm{df}}{=} [h(t)]_- w(\tau(t)), \tag{3.3}$$

either has no solution or has an infinite set of solutions.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin H^+ \cup H^-$, then (x_0, y_0) belongs at least to one of the following sets:

$$H_1 = \{(x,y) \in R \times R \;:\; 1 \le x,\ \lambda \le y\},$$

$$H_2 = \left\{(x,y) \in R \times R \;:\; 0 \le x < 1,\ 1 + \lambda + 2\sqrt{1 - x} \le y\right\},$$

$$H_3 = \left\{(x,y) \in R \times R \;:\; 0 \le y < \lambda,\ 2 + 2\sqrt{\lambda - y} \le x\right\},$$

$$H_4 = \left\{(x,y) \in R \times R \;:\; 0 \le y < \lambda,\ y + 1 - \lambda \le x \le \frac{y + 1 - \lambda}{\lambda - y}\right\},$$

$$H_5 = \left\{(x,y) \in R \times R \;:\; 1 - \lambda < x < 1,\ \frac{x}{\lambda} + 1 - \frac{1}{\lambda} \le y \le \frac{x + \lambda - 1}{\lambda(1 - x)}\right\},$$

$$H_6 = \left\{(x,y) \in R \times R \;:\; 1 - \lambda < x < 1,\ x - 1 + \lambda \le y \le \frac{x}{\lambda} + 1 - \frac{1}{\lambda}\right\}.$$

Example 3.1. Let $(x_0, y_0) \in H_1$. Put $a = 0, b = 4$,

$$h(t) = \begin{cases} -\lambda & \text{for } t \in [0,1[; \\ x_0 - 1 & \text{for } t \in [1,2[; \\ \lambda - y_0 & \text{for } t \in [2,3[; \\ 1 & \text{for } t \in [3,4], \end{cases} \qquad \tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[\cup[3,4]; \\ 1 & \text{for } t \in [1,3[. \end{cases}$$

Then (3.1) holds, and the problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} \lambda(1-t) & \text{for } t \in [0,1[; \\ 0 & \text{for } t \in [1,3[; \\ t-3 & \text{for } t \in [3,4]. \end{cases}$$

Example 3.2. Let $(x_0, y_0) \in H_2$. Put $a = 0, b = 6, \alpha = \sqrt{1 - x_0}, \beta = y_0 - 1 - \lambda - 2\alpha$,

$$h(t) = \begin{cases} -\lambda & \text{for } t \in [0,1[; \\ -\beta & \text{for } t \in [1,2[; \\ -\alpha & \text{for } t \in [2,4[; \\ -1 & \text{for } t \in [4,5[; \\ x_0 & \text{for } t \in [5,6], \end{cases} \qquad \tau(t) = \begin{cases} 6 & \text{for } t \in [0,1[\cup[2,3[\cup[5,6]; \\ 1 & \text{for } t \in [1,2[; \\ 3 & \text{for } t \in [3,5[. \end{cases}$$

Then (3.1) holds, and the problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} \lambda(1-t) & \text{for } t \in [0,1[; \\ 0 & \text{for } t \in [1,2[; \\ \alpha(2-t) & \text{for } t \in [2,3[; \\ \alpha^2(t-3) - \alpha & \text{for } t \in [3,4[; \\ \alpha(t-5) + \alpha^2 & \text{for } t \in [4,5[; \\ x_0(t-6) + 1 & \text{for } t \in [5,6]. \end{cases}$$

Example 3.3. Let $(x_0, y_0) \in H_3$. Put $a = 0, b = 6, \alpha = \sqrt{\lambda - y_0}, \beta = x_0 - 2 - 2\alpha$,

$$h(t) = \begin{cases} \alpha & \text{for } t \in [0,1[; \\ -y_0 & \text{for } t \in [1,2[; \\ \beta & \text{for } t \in [2,3[; \\ 1 & \text{for } t \in [3,4[; \\ \alpha & \text{for } t \in [4,5[; \\ 1 & \text{for } t \in [5,6], \end{cases} \qquad \tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[\cup[3,4[; \\ 6 & \text{for } t \in [1,2[\cup[4,6]; \\ 2 & \text{for } t \in [2,3[. \end{cases}$$

Then (3.1) holds, and the problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} -\alpha^2 t + \lambda & \text{for } t \in [0,1[; \\ y_0(2-t) & \text{for } t \in [1,2[; \\ 0 & \text{for } t \in [2,3[; \\ \alpha(3-t) & \text{for } t \in [3,4[; \\ \alpha(t-5) & \text{for } t \in [4,5[; \\ t-5 & \text{for } t \in [5,6]. \end{cases}$$

Example 3.4. Let $(x_0, y_0) \in H_4$. Put $a = 0$, $b = 2$, $\alpha = 1 - \lambda + y_0$, $t_0 = \dfrac{1}{x_0} - \dfrac{1}{\alpha} + 2$,

$$h(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[; \\ x_0 & \text{for } t \in [1,2], \end{cases} \qquad \tau(t) = \begin{cases} 2 & \text{for } t \in [0,1[; \\ t_0 & \text{for } t \in [1,2]. \end{cases}$$

Then (3.1) holds, and the problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} -y_0 t + \lambda & \text{for } t \in [0,1[; \\ \alpha(t-2) + 1 & \text{for } t \in [1,2]. \end{cases}$$

Example 3.5. Let $(x_0, y_0) \in H_5$. Put $a = 0$, $b = 2$, $\alpha = \dfrac{\lambda + x_0 - 1}{1 - x_0}$, $\beta = \dfrac{\lambda x_0}{1 - x_0}$, $t_0 = \left(\dfrac{\alpha}{y_0} - \lambda\right)\dfrac{1}{\beta}$,

$$h(t) = \begin{cases} x_0 & \text{for } t \in [0,1[; \\ -y_0 & \text{for } t \in [1,2], \end{cases} \qquad \tau(t) = \begin{cases} 1 & \text{for } t \in [0,1[; \\ t_0 & \text{for } t \in [1,2]. \end{cases}$$

Then (3.1) holds, and the problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} \beta t + \lambda & \text{for } t \in [0,1[; \\ \alpha(2-t) + 1 & \text{for } t \in [1,2]. \end{cases}$$

Example 3.6. Let $(x_0, y_0) \in H_6$. Put $a = 0$, $b = 2$, $\alpha = \lambda + x_0 - 1$, $t_0 = \dfrac{\alpha - y_0}{x_0 y_0} + 2$,

$$h(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[; \\ x_0 & \text{for } t \in [1,2], \end{cases} \qquad \tau(t) = \begin{cases} t_0 & \text{for } t \in [0,1[; \\ 2 & \text{for } t \in [1,2]. \end{cases}$$

Then (3.1) holds, and the problem (3.2) has the nontrivial solution

$$u(t) = \begin{cases} -\alpha t + \lambda & \text{for } t \in [0,1[; \\ x_0(t-2) + 1 & \text{for } t \in [1,2]. \end{cases}$$

On Remark 1.4. Let $\lambda \in]0,1]$ (the case $\lambda = 0$ is studied in [4]). Denote by G^+, resp. G^-, the set of pairs $(x,y) \in R_+ \times R_+$ such that

$$x < 1, \qquad \frac{x}{1-x} - \frac{1-\lambda}{\lambda} < y < \lambda,$$

resp.

$$y < \lambda, \qquad \frac{1}{\lambda - y} - 1 < x < 1.$$

It is clear that $G^+ \subset H^+$ and $G^- \subset H^-$. (Note also that for $\lambda \le \frac{1}{2}$, $G^- = \emptyset$.)

By Theorem 1.3, resp. Theorem 1.4, if

$$(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in G^+, \quad \text{resp.} \quad (\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in G^-,$$

then the problem (0.1), (0.2) with $q \in L([a,b];R_+)$, $c \in R_+$, $\|q\|_L + c \neq 0$ has a unique solution and this solution is positive, resp. negative.

Below we give the examples which show that for any pair $(x_0,y_0) \in H^+ \setminus G^+$, resp. $(x_0,y_0) \in H^- \setminus G^-$, there exist functions $h \in L([a,b];R)$, $q \in L([a,b];R_+)$ and $\tau \in \mathcal{M}_{ab}$ such that $q \not\equiv 0$, (3.1) is fulfilled, and the problem

$$u'(t) = h(t)u(\tau(t)) + q(t), \qquad u(a) = \lambda u(b), \tag{3.4}$$

or equivalently, the problem (0.1), (0.2_0) where $\ell = \ell_0 - \ell_1$, and ℓ_0, ℓ_1 are defined by (3.3), has a solution which is not positive, resp. negative.

From Example 3.7, resp. Example 3.8, it also follows that in Theorem 1.3, resp. Theorem 1.4, the inequality $\|\ell_1(1)\|_L \le \lambda$, resp. $\|\ell_0(1)\|_L \le 1$, in the condition (1.7), resp. (1.9), cannot be replaced by the inequality $\|\ell_1(1)\|_L \le \lambda + \varepsilon$, resp. $\|\ell_0(1)\|_L \le 1 + \varepsilon$, no matter how small $\varepsilon > 0$ would be.

Example 3.7. Let $(x_0,y_0) \in H^+ \setminus G^+$. Put $a = 0$, $b = 2$, $\alpha = y_0 - x_0 - \lambda + 1$, $\beta = 1 + y_0 - \lambda$, $\tau \equiv 2$,

$$h(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[; \\ x_0 & \text{for } t \in [1,2], \end{cases} \qquad q(t) = \begin{cases} 0 & \text{for } t \in [0,1[; \\ \alpha & \text{for } t \in [1,2]. \end{cases}$$

Then (3.1) holds, and the problem (3.4) has the solution

$$u(t) = \begin{cases} -y_0 t + \lambda & \text{for } t \in [0,1[; \\ \beta(t-2) + 1 & \text{for } t \in [1,2] \end{cases}$$

with $u(1) = \lambda - y_0 \le 0$.

Example 3.8. Let $(x_0,y_0) \in H^- \setminus G^-$. Put $a = 0$, $b = 2$, $\alpha = x_0 - y_0 + \lambda - 1$, $\beta = x_0 + \lambda - 1$, $\tau \equiv 2$,

$$h(t) = \begin{cases} -y_0 & \text{for } t \in [0,1[; \\ x_0 & \text{for } t \in [1,2], \end{cases} \qquad q(t) = \begin{cases} \alpha & \text{for } t \in [0,1[; \\ 0 & \text{for } t \in [1,2]. \end{cases}$$

Then (3.1) holds, and the problem (3.4) has the solution

$$u(t) = \begin{cases} \beta t - \lambda & \text{for } t \in [0,1[; \\ x_0(2-t) - 1 & \text{for } t \in [1,2] \end{cases}$$

with $u(1) = x_0 - 1 \geq 0$.

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