UDC 517.9

# OSCILLATION AND ASYMPTOTIC BEHAVIOR OF ODD ORDER DELAY DIFFERENTIAL EQUATIONS WITH IMPULSES*

# КОЛИВНА ТА АСИМПТОТИЧНА ПОВЕДІНКА ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ НЕПАРНОГО ПОРЯДКУ З ІМПУЛЬСНОЮ ДІЄЮ ТА ЗАПІЗНЕННЯМ

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*The oscillation of solutions for a class of odd order nonlinear differential equation with impulses is considered and some sufficient conditions for oscillation of solutions are obtained, which improve and popularize some results in parts of the relative references. Two examples are given to illustrate the obtained results and one of them shows that not only delay but also impulses play a very important role in giving rise to the oscillations of equations.*

*Розглянуто коливання розв'язків класу нелінійних диференціальних рівнянь з імпульсною дією та отримано певні достатні умови коливання розв'язків. Отримані результати покращують та роблять доступнішими деякі результати, що отримані у відповідних цитованих роботах. Розглянуто два приклади, що ілюструють отримані результати. Один із них показує, що імпульсна дія відіграє велику роль у виникненні коливань розв'язків.*

**1. Introduction.** Oscillation and asymptotic behavior of solutions for impulsive differential equations are of very important realistic significance, which has been focused on and deeply investigated by many authors. There are some good results on oscillation of solutions of second order differential equations with impulses [1–7]. Furthermore, some general results are obtained in [8–10] for even order linear or nonlinear differential equations with impulses. However, there are few references relating oscillation and asymptotic behavior of solutions for odd order impulsive differential equations [9–12]. Chen and Wen [9] investigated oscillation and nonoscillation of solutions for $n$-order linear impulsive differential equation. Mao and Wan [11] discussed oscillation and asymptotic behavior of solutions for third order nonlinear impulsive delay differential equations. Li et al. [12] obtained oscillatory criteria for third order difference equation with impulses. There is no literature on generalizing oscillation and asymptotic behavior of solutions for odd order nonlinear impulsive differential equations. In this paper, we investigate odd order nonlinear impulsive differential equations with delay, and obtain some sufficient conditions of oscillation and asymptotic behavior of solutions for these equations. In addition, two examples are given to illustrate our results and one of them shows that though an ordinary differential equation without impulses is nonoscillatory, it may become oscillatory if some impulses are added to it.

* This work was partially supported by the Scientific Research Foundation of Hunan Provincial Education Department and the Construct Program of the Key Discipline in Hunan Province.

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Considering the following odd order nonlinear differential equations with impulses and delay:

$$x^{(n)}(t) + f(t, x(t-\tau)) = 0, \quad t \geq t_0, \quad t \neq t_k,$$

$$x^{(i)}(t_k^+) = g_k^{(i)}(x^{(i)}(t_k)), \quad i = 0, 1, \ldots, n-1, \quad k = 1, 2, \ldots, \tag{1}$$

$$x(t) = \phi_0, \quad x^{(i)}(t_0^+) = x_0^{(i)}, \quad i = 1, 2, \ldots, n-1, \quad t \in [t_0 - \tau, t_0],$$

where $x^{(0)}(t) = x(t)$, $n$ is odd and $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$, $\lim_{k\to+\infty} t_k = +\infty$, $2\tau < t_{k+1} - t_k < +\infty$,

$$x^{(i)}(t_k^+) = \lim_{h\to+0} \frac{x^{(i-1)}(t_k+h) - x^{(i-1)}(t_k^+)}{h}, \quad x^{(i)}(t_k) = \lim_{h\to-0} \frac{x^{(i-1)}(t_k+h) - x^{(i-1)}(t_k)}{h}.$$

Here, we will establish oscillatory and asymptotic stability results based on combinations of the following conditions:

$(H_1)$ $f(t,x)$ is continuous in $[t_0, +\infty) \times (-\infty, +\infty)$, $xf(t,x) > 0\,(x \neq 0)$ and $f(t,x)/\varphi(x) \geq$ $\geq p(t)\,(x \neq 0)$, where $p(t)$ is continuous in $[t_0, +\infty)$, $p(t) \geq 0$ and $x\varphi(x) > 0$ $(x \neq 0)$, $\varphi'(x) \geq 0$ for $x \in (-\infty, \infty)$, $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$;

$(H_2)$ $g_k^{(i)}(x)$ are continuous in $(-\infty, +\infty)$, and there exist positive numbers $a_k^{(i)}$, $b_k^{(i)}$ such that $a_k^i \leq g_k^{(i)}(x)/x \leq b_k^i$, $i = 0, 1, \ldots, n-1$;

$(H_3)$ for $i = 1, 2, \ldots, n-1$,

$$\int_{t_0}^{\infty} \prod_{t_0<t_k<s} \frac{a_k^{(i)}}{b_k^{(i-1)}}\, ds = \infty;$$

$(H_4)$ $\displaystyle\sum_{i=2}^{\infty} \frac{\prod_{1\leq k<i-1} \varphi(a_k^{(0)})}{\prod_{1<k\leq i-1} b_k^{(n-1)}} \left[ \int_{t_{i-1}}^{t_{i-1}+\tau} p(s)\,ds + \varphi(a_{i-1}^{(0)}) \int_{t_{i-1}+\tau}^{t_i} p(s)\,ds \right] = \infty;$

$(H_5)$ partial product sequences $\displaystyle\prod_{i=1}^{m} b_i^{(0)}$ are bounded for $m \in N$, series $\sum_{i=1}^{\infty} |a_i^{(0)} - 1|$ and $\sum_{i=1}^{\infty} |b_i^{(0)} - 1|$ converge;

$(H_6)$ partial product sequences $\prod_{i=1}^{m} b_i^{(n-1)}$ are bounded for $m \in N$, series $\displaystyle\sum_{i=1}^{\infty} |a_i^{(n-1)} - 1|$ and $\displaystyle\sum_{i=1}^{\infty} |b_i^{(n-1)} - 1|$ converge;

$(H_7)$ $\displaystyle\int_{t_0}^{\infty} p(t)dt = \infty;$

$(H_8)$ $\varphi(a_k^{(0)}) \geq 1$ and

$$\sum_{i=1}^{\infty} \frac{1}{\prod\limits_{1<k\leq i} b_k^{(n-1)}} \int_{t_i}^{t_{i+1}} p(t)\,dt = \infty;$$

$(H_9)$ $b_k^{(i)} \le 1$, $i = 0, 1, \ldots, n-1$, and

$$\int\limits_{t_0}^{\infty} t^{n-1} p(t)\, dt = \infty;$$

$(H_{10})$ $\varphi(a_k^{(0)}) \le 1$, $b_k^{(i)} \le 1$, $i = 0, 1, \ldots, n-1$, and

$$\frac{\varphi(a_k^{(0)})}{b_{k+1}^{(n-1)}} \le \left(\frac{t_{k+1}}{t_k}\right)^{n-2};$$

$(H_{11})$ $\varphi(a_k^{(0)}) \ge 1$, $b_k^{(i)} \le 1$, $i = 0, 1, \ldots, n-1$,

$$\frac{\varphi(a_k^{(0)})}{b_{k+1}^{(n-1)}} \le \left(\frac{t_{k+2}}{t_{k+1}}\right)^{\alpha},$$

and

$$\int\limits_{t_0}^{\infty} t^{\alpha} p(t)\, dt = \infty,$$

where $0 \le \alpha \le n-2$.

In this paper, we always assume that conditions $(H_1)$, $(H_2)$ and $(H_3)$ hold.

**Definition 1.1.** *A function $x(t) : [t_0 - \tau, t_0 + a) \to R$, $t_0 > 0$, $a > 0$ is said to be a solution of equation (1), if*

*(a)* $x(t) = \phi_0$, $x^{(i)}(t_0^+) = x_0^{(i)}$, $i = 1, 2, \ldots, n-1$, $t \in [t_0 - \tau, t_0]$;

*(b)* $x(t)$ *satisfies* $x^{(n)}(t) + f(t, x(t-\tau)) = 0$ *when* $t \in [t_0, t_0 + a)$, $t \ne t_k$ *and* $t \ne t_k + \tau$; $x(t)$, $x'(t), \ldots, x^{(n-1)}(t)$ *are continuous for* $t \ne t_k$;

*(c)* $x^{(i)}(t_k^+) = g_k^{(i)}(x^{(i)}(t_k))$, $x^{(i)}(t)$ *are left continuous on* $t = t_k$, $i = 0, 1, \ldots, n-1$.

The uniqueness of solutions and the existence of global solutions of differential equations with impulses can be seen in [13]. In the following, we always assume the solutions of equation (1) exist on $[t_0 - \tau, +\infty)$.

**Definition 1.2.** *A solution of equation (1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.*

**2. Main results.** Similar to Lemma 2 and Lemma 3 in [10], for $n$ be odd we also have the following lemmas.

**Lemma 2.1.** *Let $x = x(t)$ be a solution of equation (1). Suppose that there are $i \in \{1, 2, \ldots, n-1\}$ and $T \ge t_0$ such that $x^{(i)}(t) \ge 0$ and $x^{(i+1)}(t) \ge 0$ for $t \ge T$. Then there exists $T_1 \ge T$ such that $x^{(i-1)}(t) \ge 0$ for $t \ge T_1$.*

**Lemma 2.2.** *Let $x = x(t)$ be a solution of equation (1). Suppose that there are $i \in \{1, 2, \ldots, n\}$ and $T \ge t_0$ such that $x(t) \ge 0$ and $x^{(i)}(t) \le 0$ for $t \ge T$. Then $x^{(i-1)}(t) \ge 0$ for all sufficiently large $t$.*

**Lemma 2.3.** *Let $x(t)$ be a solution of equation (1). Suppose that there exists some $T \geq t_0$ such that $x(t-\tau) > 0$ for $t \geq T$. If condition $(H_4)$ holds, then*

$$(-1)^i x^{(i)}(t) \geq 0, \quad i = 1, 2, \ldots, n,$$

*for all sufficiently large $t$.*

***Proof.*** Let $x(t-\tau) > 0$ for $t \in (t_k, t_{k+1}]$, where $t_k \geq T$. By equation (1) and condition $(H_1)$, we obtain

$$x^{(n)}(t) = -f(t, x(t-\tau)) \leq -p(t)\varphi(x(t-\tau)) \leq 0.$$

Hence, by Lemma 2.2, there exists a $T_1 \geq T$ such that $x^{(n-1)}(t) \geq 0$ on $t \in (t_k, t_{k+1}]$ for $t_k \geq T_1$.

For $x^{(n-2)}(t)$, $x^{(n-1)}(t) \geq 0$ deduces that $x^{(n-2)}(t)$ is monotonically nondecreasing on $t \in \in (t_k, t_{k+1}]$ for $t_k \geq T_1$. There are two cases for the sign of $x^{(n-2)}(t)$.

One is that there is a $t_j$ such that $x^{(n-2)}(t_j) \geq 0$, then $x^{(n-2)}(t) \geq 0$ for all $t \geq t_j$. By Lemma 2.1, $x^{(n-1)}(t) \geq 0$ and $x^{(n-2)}(t) \geq 0$ imply that $x^{(n-3)}(t) \geq 0$ for all sufficiently large $t$. Applying Lemma 2.1 repeatedly, we have $x^{(i)}(t) \geq 0$, $i = 1, 2, \ldots, n-1$, for all sufficiently large $t$.

Another is that for any $t_k$, $x^{(n-2)}(t_k) \leq 0$, then $x^{(n-2)}(t) \leq 0$ for $t \in (t_k, t_{k+1}]$, $t_k \geq T_1$. By Lemma 2.2, $x^{(n-1)}(t) \geq 0$ and $x^{(n-2)}(t_k) \leq 0$ yield that there exists a $T_2 > T_1$ such that $x^{(n-3)}(t) \geq 0$ for $t \in (t_k, t_{k+1}]$, $t_k \geq T_2$. Thus we have a need to continuously define the sign of $x^{(n-4)}(t)$.

For $x^{(n-4)}(t)$, $x^{(n-3)}(t) \geq 0$ deduces that $x^{(n-4)}(t)$ is monotonically nondecreasing on $t \in \in (t_k, t_{k+1}]$ for $t_k \geq T_2$. Similar to the argument of the sign of $x^{(n-2)}(t)$, we also obtain two statements from $x^{(n-4)}(t) \geq 0$. One is $x^{(i)}(t) \geq 0$, $i = 1, 2, \ldots, n-3$, for all sufficiently large $t$, an other is $x^{(n-4)}(t) \leq 0$ and $x^{(n-5)}(t) \geq 0$ for all sufficiently large $t$.

By induction, eventually there are a $T' \geq T$ and an odd integer $l \in \{1, 3, \ldots, n\}$ such that $(-1)^i x^{(i)}(t) \geq 0$, $l \leq i \leq n$, and $x^{(i)}(t) \geq 0$, $0 \leq i \leq l-1$, for $t \in (t_k, t_{k+1}]$, $t_k \geq T'$.

In the following, we prove that $l$ must be equal to 1, that is, $(-1)^i x^{(i)}(t) \geq 0$, $i = 1, 2, \ldots, n$, for $t \in (t_k, t_{k+1}]$, $t_k \geq T'$. For the sake of contradiction, assume that there exists $x'(t-\tau) > 0$ for $t \in (t_k, t_{k+1}]$, $t_k \geq T'$.

Let $u(t) = \dfrac{x^{(n-1)}(t)}{\varphi(x(t-\tau))}$, then $u(t) \geq 0$. By (1) and condition $(H_1)$, we have

$$u'(t) = \frac{x^{(n)}(t)}{\varphi(x(t-\tau))} - \frac{x^{(n-1)}(t)\varphi'(x(t-\tau))x'(t-\tau)}{\varphi^2(x(t-\tau))} \leq -p(t) \leq 0 \ \text{ for } \ t \neq t_k, \ \ t_k + \tau, \tag{2}$$

$$u(t_k^+) = \frac{x^{(n-1)}(t_k^+)}{\varphi(x(t_k-\tau))} \leq \frac{b_k^{(n-1)} x^{(n-1)}(t_k)}{\varphi(x(t_k-\tau))} = b_k^{(n-1)} u(t_k),$$

$$u((t_k+\tau)^+) = \frac{x^{(n-1)}(t_k+\tau)}{\varphi(x(t_k^+))} \leq \frac{x^{(n-1)}(t_k+\tau)}{\varphi(a_k^{(0)} x(t_k))} \leq \frac{x^{(n-1)}(t_k+\tau)}{\varphi(a_k^{(0)})\varphi(x(t_k))} = \frac{1}{\varphi(a_k^{(0)})}\, u(t_k+\tau).$$

Integrating (2) from $t_k$ to $t_k + \tau$, we have

$$u(t_k+\tau) \leq u(t_k^+) - \int\limits_{t_k}^{t_k+\tau} p(s)\, ds. \tag{3}$$

Meanwhile, integrating (2) from $t_k+\tau$ to $t_{k+1}$, we have

$$u(t_{k+1}) \le u((t_k+\tau)^+) - \int\limits_{t_k+\tau}^{t_{k+1}} p(s)\,ds. \tag{4}$$

By (3) and (4), we get

$$u(t_{k+1}) \le \frac{1}{\varphi(a_k^{(0)})}\left(u(t_k^+) - \int\limits_{t_k}^{t_k+\tau} p(s)\,ds - \varphi(a_k^{(0)})\int\limits_{t_k+\tau}^{t_{k+1}} p(s)\,ds\right).$$

Similarly, the following inequalities hold:

$$u(t_{k+1}+\tau) \le \frac{b_{k+1}^{(n-1)}}{\varphi(a_k^{(0)})}\left(u(t_k^+) - \int\limits_{t_k}^{t_k+\tau} p(s)\,ds - \varphi(a_k^{(0)})\int\limits_{t_k+\tau}^{t_{k+1}} p(s)\,ds - \frac{\varphi(a_k^{(0)})}{b_{k+1}^{(n-1)}}\int\limits_{t_{k+1}}^{t_{k+1}+\tau} p(s)\,ds\right),$$

$$u(t_{k+2}) \le \frac{b_{k+1}^{(n-1)}}{\varphi(a_{k+1}^{(0)})\varphi(a_k^{(0)})}\left(u(t_k^+) - \int\limits_{t_k}^{t_k+\tau} p(s)\,ds - \varphi(a_k^{(0)})\int\limits_{t_k+\tau}^{t_{k+1}} p(s)\,ds - \right.$$

$$\left. - \frac{\varphi(a_k^{(0)})}{b_{k+1}^{(n-1)}}\int\limits_{t_{k+1}}^{t_{k+1}+\tau} p(s)\,ds - \frac{\varphi(a_{k+1}^{(0)})\varphi(a_k^{(0)})}{b_{k+1}^{(n-1)}}\int\limits_{t_{k+1}+\tau}^{t_{k+2}} p(s)\,ds\right).$$

By induction, for any natural number $j \ge k$, we have

$$u(t_{j+1}) \le \frac{b_j^{(n-1)}b_{j-1}^{(n-1)}\ldots b_{k+1}^{(n-1)}}{\varphi(a_j^{(0)})\varphi(a_{j-1}^{(0)})\ldots\varphi(a_k^{(0)})}\left(u(t_k^+) - \int\limits_{t_k}^{t_k+\tau} p(s)ds - \varphi(a_k^{(0)})\int\limits_{t_k+\tau}^{t_{k+1}} p(s)\,ds - \ldots\right.$$

$$\ldots - \frac{\varphi(a_{j-1}^{(0)})\varphi(a_{j-2}^{(0)})\ldots\varphi(a_k^{(0)})}{b_j^{(n-1)}b_{j-1}^{(n-1)}\ldots b_{k+1}^{(n-1)}}\int\limits_{t_j}^{t_j+\tau} p(s)\,ds -$$

$$\left. - \frac{\varphi(a_j^{(0)})\varphi(a_{j-1}^{(0)})\ldots\varphi(a_k^{(0)})}{b_j^{(n-1)}b_{j-1}^{(n-1)}\ldots b_{k+1}^{(n-1)}}\int\limits_{t_j+\tau}^{t_{j+1}} p(s)ds\right). \tag{5}$$

By (5) and condition $(H_4)$, we get $u(t_{j+1}) \to -\infty$ as $t_{j+1} \to \infty$. This is a contradiction with $u(t) \ge 0$ for $t > T'$. Hence, $x'(t-\tau) \le 0$ for $t \in (t_k, t_{k+1}]$, $t_k \ge T'$, which means that $l = 1$.

Lemma 2.3 is proved.

***Remark 2.1.*** In Lemma 2.3, if we replace the condition "$x(t-\tau) > 0$" with "$x(t-\tau) < 0$", then under condition $(H_4)$, we can prove similarly that

$$(-1)^i x^{(i)}(t) \le 0, \quad i = 1, 2, \ldots, n,$$

for all sufficiently large $t$.

**Lemma 2.4** [14]. *Set $x(t)$ is continuous for $t \ne t_k$, and left continuous for $t_k$, and exists right limit. If*

$(i)$ *there exists a $\bar{t} \in R^+$ such that $x(t) > 0$ (or $< 0$), for $t \ge \bar{t}$;*

$(ii)$ *there exists a $m \in N$ such that $x(t)$ is nonincreasing (or nondecreasing) on the interval $(t_k, t_{k+1}]$, where $k \ge m$;*

$(iii)$ *series $\sum_{k=1}^{\infty}[x(t_k^+) - x(t_k)]$ converges.*

*Then $\lim_{t\to\infty} x(t) = \alpha$ exists and $\alpha \ge 0$ (or $\le 0$).*

**Theorem 2.1.** *Suppose that conditions $(H_4)$, $(H_5)$, $(H_6)$ and $(H_7)$ hold, then every solution of (1) is either oscillatory or tends eventually to zero.*

***Proof.*** Assume that (1) has an eventually positive solution $x(t)$. Then there exists a $T \ge t_0$ such that $x(t-\tau) > 0$ for $t \ge T$, and Lemma 2.3 holds by condition $(H_4)$.

We first prove that $\lim_{x\to\infty} x(t) = a$ $(a \ge 0)$ by Lemma 2.4. Obviously, condition $(i)$ of Lemma 2.4 holds. By Lemma 2.3, there exists a $T' \ge T$ such that $x'(t) \ge 0$ for $t \ge T'$, which means that $x(t)$ is monotonically nonincreasing on $(t_k, t_{k+1}]$ for $t_k \ge T'$, then condition $(ii)$ of Lemma 2.4 also holds. In the following, we prove that condition $(iii)$ of Lemma 2.4 holds, that is, $\sum_{i=1}^{\infty}[x(t_k^+) - x(t_k)]$ converges. Since $x(t)$ is monotonically nonincreasing on $(t_k, t_{k+1}]$ for $t_k \ge$ $\ge T'$, we see that $x(t_{k+1}) \le x(t_k^+)$, $x(t_{k+1}^+) \le b_{k+1}^{(0)} x(t_k^+)$. By induction, we obtain that $x(t_{k+m}) \le$ $\le b_{k+m-1}^{(0)} b_{k+m-2}^{(0)} \ldots b_{k+1}^{(0)} x(t_k^+)$, $x(t_{k+m}^+) \le b_{k+m}^{(0)} b_{k+m-1}^{(0)} \ldots b_{k+1}^{(0)} x(t_k^+)$ for any $m \in N$. Note that partial product sequences $\prod_{i=1}^{m} b_{k+i}^{(0)}$ are bounded, thus sequence $\{x(t_i)\}$ is also bounded, there exists a constant $M > 0$ such that $x(t_i) < M$. Since $a_i^{(0)} x(t_i) \le x(t_i^+) \le b_i^{(0)} x(t_i)$, we have

$$(a_i^{(0)} - 1)x(t_i) \le x(t_i^+) - x(t_i) \le (b_i^{(0)} - 1)x(t_i),$$

then

$$|x(t_i^+) - x(t_i)| \le |(a_i^{(0)} - 1)x(t_i)| + |(b_i^{(0)} - 1)x(t_i)| \le M\left(|a_i^{(0)} - 1| + |b_i^{(0)} - 1|\right),$$

$$\sum_{i=1}^{\infty} |x(t_i^+) - x(t_i)| \le M \sum_{i=1}^{\infty} \left(|a_i^{(0)} - 1| + |b_i^{(0)} - 1|\right).$$

Thus $\sum_{i=1}^{\infty} |x(t_i^+) - x(t_i)|$ converges. From $\left|\sum_{i=1}^{\infty}[x(t_i^+) - x(t_i)]\right| \le \sum_{i=1}^{\infty} |x(t_i^+) - x(t_i)|$, we have $\sum_{i=1}^{\infty}[x(t_i^+) - x(t_i)]$ also converges. So condition $(iii)$ of Lemma 2.4 holds. By Lemma 2.4, we have $\lim_{x\to\infty} x(t) = a \ge 0$.

Next we prove that $a = 0$. If it is not true, then $a > 0$, $\varphi(a) > 0$ and $\lim_{t\to\infty} \varphi(x(t-\tau)) =$ $= \varphi(a)$. Note that $\varphi'(x) > 0$, there exists a $T'' \ge T'$ such that $\varphi(x(t-\tau)) \ge \varphi(a)/2 > 0$.

From (1) we have

$$x^{(n)}(s) = -f(s, x(s-\tau)) \le -p(s)\varphi(x(s-\tau)) \le -\frac{\varphi(a)}{2}\, p(s). \tag{6}$$

Integrating (6) from $t_j$ to $t$, we have

$$x^{(n-1)}(t) - x^{(n-1)}(t_j^+) = -\int\limits_{t_j}^{t} f(s, x(s-\tau))\,ds + \sum_{t_j<t_i\le t}\Big(x^{(n-1)}(t_i^+) - x^{(n-1)}(t_i)\Big) \le$$

$$\le -\frac{\varphi(a)}{2}\int\limits_{t_j}^{t} p(s)\,ds + \sum_{t_j<t_i\le t}\Big(x^{(n-1)}(t_i^+) - x^{(n-1)}(t_i)\Big), \qquad (7)$$

where $t_j \ge T''$. Similar to the argument of $\sum_{i=1}^{\infty}[x(t_i^+) - x(t_i)]$, condition $(H_6)$ implies that $\sum_{i=1}^{\infty}[x^{(n-1)}(t_i^+) - x^{(n-1)}(t_i)]$ converges. By (7) and condition $(H_7)$, we have $x^{(n-1)}(t) \to -\infty$ as $t \to \infty$, which is a contradiction with $x^{(n-1)}(t) \ge 0$ for all sufficiently large $t$. Then $a$ must be equal to 0. Therefore, every solution of (1) is either oscillatory or tends eventually to zero.

Theorem 2.1 is proved.

**Theorem 2.2.** *Suppose that conditions $(H_5)$ and $(H_8)$ hold, then every solution of (1) is either oscillatory or tends eventually to zero.*

***Proof.*** Assume that (1) has an eventually positive solution $x(t)$. Then there exists a $T \ge t_0$ such that $x(t-\tau) > 0$ for $t \ge T$.

By condition $(H_8)$, we have that condition $(H_4)$ holds, then Lemma 2.3 holds. Similar to the proof of Theorem 2.1, condition $(H_5)$ yields that $\lim_{t\to\infty} x(t) = a$ $(a \ge 0)$. If $a > 0$, then $\varphi(a) > 0$ and $\lim_{t\to\infty}\varphi(x(t-\tau)) = \varphi(a) > 0$. $\varphi'(x) > 0$ yields that there exists $T' \ge T$ such that $\varphi(x(t-\tau)) \ge \varphi(a)/2 > 0$.

Integrating (6) from $t_j$ to $t_{j+1}$ $(t_j \ge T')$, we have

$$x^{(n-1)}(t_{j+1}) = x^{(n-1)}(t_j^+) - \int\limits_{t_j}^{t_{j+1}} f(s, x(s-\tau))\,ds \le x^{(n-1)}(t_j^+) - \frac{\varphi(a)}{2}\int\limits_{t_j}^{t_{j+1}} p(s)\,ds.$$

Similarly, we have

$$x^{(n-1)}(t_{j+2}) \le x^{(n-1)}(t_{j+1}^+) - \frac{\varphi(a)}{2}\int\limits_{t_{j+1}}^{t_{j+2}} p(s)\,ds \le$$

$$\le b_{j+1}^{(n-1)}\left[x^{(n-1)}(t_j^+) - \frac{\varphi(a)}{2}\left(\int\limits_{t_j}^{t_{j+1}} p(s)\,ds + \frac{1}{b_{j+1}^{(n-1)}}\int\limits_{t_{j+1}}^{t_{j+2}} p(s)\,ds\right)\right].$$

By induction we get, for any natural number $m \geq 2$,

$$x^{(n-1)}(t_{j+m}) \leq b_{j+m-1}^{(n-1)} b_{j+m-2}^{(n-1)} \ldots b_{j+1}^{(n-1)} \left[ x^{(n-1)}(t_j^+) - \frac{\varphi(a)}{2} \left( \int\limits_{t_j}^{t_{j+1}} p(s)\, ds + \right.\right.$$

$$\left.\left. + \frac{1}{b_{j+1}^{(n-1)}} \int\limits_{t_{j+1}}^{t_{j+2}} p(s)\, ds + \ldots + \frac{1}{b_{j+m-1}^{(n-1)} b_{j+m-2}^{(n-1)} \ldots b_{j+1}^{(n-1)}} \int\limits_{t_{j+m-1}}^{t_{j+m}} p(s)\, ds \right) \right]. \quad (8)$$

By (8) and condition $(H_8)$, we have $x^{(n-1)}(t_i) \to -\infty$ for $t_i \to \infty$, which is a contradiction with $x^{(n-1)}(t) \geq 0$ for all sufficient large $t$. Then $a = 0$, which means that every solution of (1) is either oscillatory or tends eventually to zero.

**Theorem 2.3.** *Suppose that conditions $(H_4)$, $(H_5)$ and $(H_9)$ hold, then every solution of (1) is either oscillatory or tends eventually to zero.*

***Proof.*** Assume that (1) has an eventually positive solution $x(t)$. Then there exists a $T \geq t_0$ such that $x(t-\tau) > 0$ for $t \geq T$.

According to the proof of Theorem 2.1, we can see that $\lim_{t\to\infty} x(t-\tau) = a \geq 0$. If $a > 0$, then $\varphi(a) > 0$ and $\lim_{t\to\infty} \varphi(x(t-\tau)) = \varphi(a)$. $\varphi(x)$ is monotonically nondecreasing implies that there exists a $T' \geq T$ such that $\varphi(x(t-\tau)) > \varphi(a)/2$.

Multiplying (6) by $s^{n-1}$, and integrating it from $t_k$ to $t$, we have

$$\int\limits_{t_k}^{t} s^{n-1} x^{(n)}(s)\, ds < -\frac{\varphi(a)}{2} \int\limits_{t_k}^{t} s^{n-1} p(s)\, ds, \quad (9)$$

where $t_k \geq T'$. Let $t$ be equal to $t_{k+1}$ at (9) and applying integration by parts repeatedly to the left side of (9), we obtain

$$\int\limits_{t_k}^{t_{k+1}} s^{n-1} x^{(n)}(s)\, ds = \int\limits_{t_k}^{t_{k+1}} s^{n-1}\, dx^{(n-1)}(s) =$$

$$= t_{k+1}^{n-1} x^{(n-1)}(t_{k+1}) - t_k^{n-1} x^{(n-1)}(t_k^+) - (n-1) \int\limits_{t_k}^{t_{k+1}} s^{n-2} x^{(n-1)}(s) ds = \ldots$$

$$\ldots = t_{k+1}^{n-1} x^{(n-1)}(t_{k+1}) - t_k^{n-1} x^{(n-1)}(t_k^+) +$$

$$+ \sum_{i=1}^{n-1} (-1)^i \prod_{j=1}^{i} (n-j) \left( t_{k+1}^{n-i-1} x^{(n-i-1)}(t_{k+1}) - t_k^{n-i-1} x^{(n-i-1)}(t_k^+) \right).$$

By induction, for any natural number $m$, we have

$$\int\limits_{t_k}^{t_{k+m}} s^{n-1}x^{(n)}(s)\,ds = \sum_{l=0}^{m-1}\int\limits_{t_{k+l}}^{t_{k+l+1}} s^{n-1}x^{(n)}(s)\,ds =$$

$$= \sum_{l=0}^{m-1}\Bigg[t_{k+l+1}^{n-1}x^{(n-1)}(t_{k+l+1}) - t_{k+l}^{n-1}x^{(n-1)}(t_{k+l}^{+} +$$

$$+ \sum_{i=1}^{n-1}(-1)^i\prod_{j=1}^{i}(n-j)\left(t_{k+l+1}^{n-i-1}x^{(n-i-1)}(t_{k+l+1}) - t_{k+l}^{n-i-1}x^{(n-i-1)}(t_{k+l}^{+})\right)\Bigg] =$$

$$= t_{k+m}^{n-1}x^{(n-1)}(t_{k+m}) - \sum_{l=1}^{m-1} t_{k+l}^{n-1}\left(x^{(n-1)}(t_{k+l}^{+}) - x^{(n-1)}(t_{k+l})\right) -$$

$$- t_k^{n-1}x^{(n-1)}(t_k^{+}) + \sum_{i=1}^{n-1}(-1)^i\prod_{j=1}^{i}(n-j)t_{k+m}^{n-i-1}x^{(n-i-1)}(t_{k+m}) -$$

$$- \sum_{l=1}^{m-1}\sum_{i=1}^{n-1}(-1)^i\prod_{j=1}^{i}(n-j)t_{k+l}^{n-i-1}\left(x^{(n-i-1)}(t_{k+l}^{+}) - x^{(n-i-1)}(t_{k+l})\right) -$$

$$- \sum_{i=1}^{n-1}(-1)^i\prod_{j=1}^{i}(n-j)t_k^{n-i-1}x^{(n-i-1)}(t_k^{+}). \tag{10}$$

Noting that $(-1)^i x^{(i)}(t) \geq 0$ for sufficiently large $t$, (10) yields

$$\int\limits_{t_k}^{t_{k+m}} s^{n-1}x^{(n)}(s)\,ds \geq t_{k+m}^{n-1}x^{(n-1)}(t_{k+m}) - \sum_{l=1}^{m-1} t_{k+l}^{n-1}\left(b_{k+l}^{(n-1)} - 1\right)x^{(n-1)}(t_{k+l}) -$$

$$- t_k^{n-1}x^{(n-1)}(t_k^{+}) + \sum_{i=1}^{n-1}(-1)^i\prod_{j=1}^{i}(n-j)t_{k+m}^{n-i-1}x^{(n-i-1)}(t_{k+m}) -$$

$$- \sum_{l=1}^{m-1}\sum_{i=1}^{n-1}(-1)^i\prod_{j=1}^{i}(n-j)t_{k+l}^{n-i-1}\left(b_{k+l}^{(n-i-1)} - 1\right)x^{(n-i-1)}(t_{k+l}) -$$

$$- \sum_{i=1}^{n-1}(-1)^i\prod_{j=1}^{i}(n-j)t_k^{n-i-1}x^{(n-i-1)}(t_k^{+}) \geq$$

$$\geq - \sum_{l=1}^{m-1} t_{k+l}^{n-1}\left(b_{k+l}^{(n-1)} - 1\right)x^{(n-1)}(t_{k+l}) -$$

$$- t_k^{n-1} x^{(n-1)}(t_k^+) + \prod_{j=1}^{n-1}(n-j) t_{k+m} x(t_{k+m}) -$$

$$- \sum_{l=1}^{m-1} \sum_{i=1}^{n-1} (-1)^i \prod_{j=1}^{i} (n-j) t_{k+l}^{n-i-1} \left( b_{k+l}^{(n-i-1)} - 1 \right) x^{(n-i-1)}(t_{k+l}) -$$

$$- \sum_{i=1}^{n-1} (-1)^i \prod_{j=1}^{i} (n-j) t_k^{n-i-1} x^{(n-i-1)}(t_k^+). \tag{11}$$

By (11) and $b_k^{(i)} \leq 1$, we obtain

$$\int_{t_k}^{t_{k+m}} s^{n-1} x^{(n)}(s)\, ds \geq \prod_{j=1}^{n-1}(n-j) t_{k+m} x(t_{k+m}) - t_k^{n-1} x^{(n-1)}(t_k^+) -$$

$$- \sum_{i=1}^{n-1} (-1)^i \prod_{j=1}^{i} (n-j) t_k^{n-i-1} x^{(n-i-1)}(t_k^+). \tag{12}$$

By (9), (12) and $\int_{t_0}^{\infty} t^{n-1} p(t)\, dt = \infty$, we get that $x(t_i) < 0$ as $t_i \to \infty$, which is a contradiction with $x(t)$ being eventually positive.

Theorem 2.3 is proved.

**Theorem 2.4.** *Suppose that conditions $(H_4)$, $(H_5)$ and $(H_{10})$ hold, then every solution of (1) is either oscillatory or tends eventually to zero.*

***Proof.*** Assume that (1) has an eventually positive solution $x(t)$. Then there exists a $T \geq t_0$ such that $x(t-\tau) > 0$ for $t \geq T$. Similar to the proof Theorem 2.1, let $\lim_{t\to\infty} x(t-\tau) = a$ $(a \geq 0)$, we have $\lim_{t\to\infty} \varphi(x(t-\tau)) = \varphi(a) > 0$ and $\varphi(x(t-\tau)) > \varphi(a)/2$ for $t \geq T' \geq T$.

Multiplying (6) by $s^{n-2}$, and integrating it from $t_k$ to $t$, we have

$$\int_{t_k}^{t} s^{n-2} x^{(n)}(s) ds < -\frac{\varphi(a)}{2} \int_{t_k}^{t} s^{n-2} p(s)\, ds, \tag{13}$$

where $t_k \geq T'$. Similar to the argument of Theorem 2.3, for any natural number $m$,

$$-\frac{\varphi(a)}{2} \int_{t_k}^{t_{k+m}} s^{n-2} p(s)\, ds > \int_{t_k}^{t_{k+m}} s^{n-2} x^{(n)}(s)\, ds \geq - \sum_{l=1}^{m-1} t_{k+l}^{n-2} \left( b_{k+l}^{(n-1)} - 1 \right) x^{(n-1)}(t_{k+l}) -$$

$$- t_k^{n-2} x^{(n-1)}(t_k^+) + \prod_{j=1}^{n-1}(n-j) t_{k+m} x(t_{k+m}) -$$

$$-\sum_{l=1}^{m-1}\sum_{i=1}^{n-2}(-1)^i\prod_{j=1}^{i}(n-j)t_{k+l}^{n-i-2}\left(b_{k+l}^{(n-i-1)}-1\right)x^{(n-i-1)}(t_{k+l})-$$

$$-\sum_{i=1}^{n-2}(-1)^i\prod_{j=1}^{i}(n-j)t_k^{n-i-2}x^{(n-i-1)}(t_k^+)\geq$$

$$\geq -t_k^{n-2}x^{(n-1)}(t_k^+)-\sum_{i=1}^{n-2}(-1)^i\prod_{j=1}^{i}(n-j)t_k^{n-i-2}x^{(n-i-1)}(t_k^+). \tag{14}$$

On the other hand, by condition $(H_{10})$, it follows that

$$\frac{\varphi(a)}{2}t_k^{n-2}\left(\int\limits_{t_k}^{t_k+\tau}p(s)ds+\varphi(a_k^{(0)})\int\limits_{t_k+\tau}^{t_{k+1}}p(s)\,ds+\ldots+\frac{\prod_{k\leq i<k+m-1}\varphi(a_i^{(0)})}{\prod_{k<i\leq k+m-1}b_i^{(n-1)}}\int\limits_{t_{k+m-1}}^{t_{k+m-1}+\tau}p(s)\,ds+\right.$$

$$\left.+\frac{\prod_{k\leq i\leq k+m-1}\varphi(a_i^{(0)})}{\prod_{k<i\leq k+m-1}b_i^{(n-1)}}\int\limits_{t_{k+m-1}+\tau}^{t_{k+m}}p(s)\,ds\right)\leq$$

$$\leq\frac{\varphi(a)}{2}t_k^{n-2}\left(\int\limits_{t_k}^{t_{k+1}}p(s)\,ds+\frac{\varphi(a_k^{(0)})}{b_{k+1}^{(n-1)}}\int\limits_{t_{k+1}}^{t_{k+2}}p(s)\,ds+\ldots+\frac{\prod_{k\leq i\leq k+m-1}\varphi(a_i^{(0)})}{\prod_{k\leq i\leq k+m-1}b_i^{(n-1)}}\int\limits_{t_{k+m-1}}^{t_{k+m}}p(s)\,ds\right)\leq$$

$$\leq\frac{\varphi(a)}{2}t_k^{n-2}\left[\int\limits_{t_k}^{t_{k+1}}p(s)\,ds+\left(\frac{t_{k+1}}{t_k}\right)^{n-2}\int\limits_{t_{k+1}}^{t_{k+2}}p(s)\,ds+\ldots+\left(\frac{t_{k+m-1}}{t_k}\right)^{n-2}\int\limits_{t_{k+m-1}}^{t_{k+m}}p(s)\,ds\right]\leq$$

$$\leq\frac{\varphi(a)}{2}t_k^{n-2}\left[\int\limits_{t_k}^{t_{k+1}}p(s)\,ds+\left(\frac{1}{t_k}\right)^{n-2}\int\limits_{t_{k+1}}^{t_{k+2}}s^{n-2}p(s)\,ds+\ldots+\left(\frac{1}{t_k}\right)^{n-2}\int\limits_{t_{k+m-1}}^{t_{k+m}}s^{n-2}p(s)\,ds\right]\leq$$

$$\leq\frac{\varphi(a)}{2}\left(\int\limits_{t_k}^{t_{k+1}}(t_k^{n-2}-s^{n-2})p(s)\,ds+\int\limits_{t_k}^{t_{k+m}}s^{n-2}p(s)\,ds\right). \tag{15}$$

By (14) and (15), we have

$$\frac{\varphi(a)}{2}t_k^{n-2}\left(\int\limits_{t_k}^{t_k+\tau}p(s)\,ds+\varphi(a_k^{(0)})\int\limits_{t_k+\tau}^{t_{k+1}}p(s)\,ds+\ldots\right.$$

$$\left.\ldots+\frac{\prod_{k\leq i<k+m-1}\varphi(a_i^{(0)})}{\prod_{k<i\leq k+m-1}b_i^{(n-1)}}\int\limits_{t_{k+m-1}}^{t_{k+m-1}+\tau}p(s)\,ds+\frac{\prod_{k\leq i\leq k+m-1}\varphi(a_i^{(0)})}{\prod_{k<i\leq k+m-1}b_i^{(n-1)}}\int\limits_{t_{k+m-1}+\tau}^{t_{k+m}}p(s)\,ds\right)\leq$$

$$\leq \frac{\varphi(a)}{2} \int\limits_{t_k}^{t_{k+1}} (t_k^{n-2} - s^{n-2}) p(s)\, ds + t_k^{n-2} x^{(n-1)}(t_k^+) +$$

$$+ \sum_{i=1}^{n-2} (-1)^i \prod_{j=1}^{i} (n-j) t_k^{n-i-2} x^{(n-i-1)}(t_k^+). \tag{16}$$

By condition $(H_4)$ and (16), a contradiction is reached if we let $m \to \infty$, that is, the left of (16) tends to infinity but its right is limited. Then $a$ must be zero.

Theorem 2.4 is proved.

**Theorem 2.5.** *Suppose that conditions $(H_5)$ and $(H_{11})$ hold, then every solution of (1) is either oscillatory or tends eventually to zero.*

***Proof.*** We have

$$\int\limits_{t_1}^{t_1+\tau} p(s)\, ds + \varphi(a_1^{(0)}) \int\limits_{t_1+\tau}^{t_2} p(s)\, ds + \ldots + \frac{\prod_{1 \leq i < m-1} \varphi(a_i^{(0)})}{\prod_{1 < i \leq m-1} b_i^{(n-1)}} \int\limits_{t_{m-1}}^{t_{m-1}+\tau} p(s)\, ds +$$

$$+ \frac{\prod_{1 \leq i \leq m-1} \varphi(a_i^{(0)})}{\prod_{1 < i \leq m-1} b_i^{(n-1)}} \int\limits_{t_{m-1}+\tau}^{t_m} p(s)\, ds \geq$$

$$\geq \int\limits_{t_1}^{t_2} p(s)\, ds + \frac{\varphi(a_1^{(0)})}{b_2^{(n-1)}} \int\limits_{t_2}^{t_3} p(s)\, ds + \ldots + \frac{\prod_{1 \leq i < m-1} \varphi(a_i^{(0)})}{\prod_{1 < i \leq m-1} b_i^{(n-1)}} \int\limits_{t_{m-1}}^{t_m} p(s)\, ds \geq$$

$$\geq \int\limits_{t_1}^{t_2} p(s)\, ds + \left(\frac{t_3}{t_2}\right)^{\alpha} \int\limits_{t_2}^{t_3} p(s)\, ds + \ldots + \left(\frac{t_m}{t_2}\right)^{\alpha} \int\limits_{t_{m-1}}^{t_m} p(s)\, ds \geq$$

$$\geq \int\limits_{t_1}^{t_2} (1 - s^{\alpha}) p(s) ds + \left(\frac{1}{t_2}\right)^{\alpha} \int\limits_{t_1}^{t_m} s^{\alpha} p(s)\, ds. \tag{17}$$

If we let $t_m \to \infty$, (17) and $\displaystyle\int_{t_0}^{\infty} s^{\alpha} p(s)\, ds = \infty$ imply that condition $(H_4)$ holds. Then Lemma 2.3 holds. Similar to the proof of Theorem 2.1, condition $(H_5)$ yields that $\lim_{t\to\infty} x(t) = a$ $(a \geq 0)$. If $a > 0$, $0 \leq \alpha \leq n-2$ yields that

$$\int\limits_{t_0}^{\infty} s^{n-2} p(s)\, ds \geq \int\limits_{t_0}^{\infty} s^{\alpha} p(s)\, ds = \infty, \tag{18}$$

and (14) and (18) will lead to a contradiction. Then $a = 0$.

Theorem 2.5 is proved.

## 3. Examples.

***Example 3.1.*** Consider

$$x'''(t) + t^{-1}x^3\left(t - \frac{1}{2}\right) = 0, \quad t \geq 1, \quad t \neq 2^k,$$

$$x((2^k)^+) = \left(1 + \frac{1}{2^k}\right)x(2^k), \ \ x^{(i)}((2^k)^+) = \left(1 - \frac{1}{2^k}\right)x^{(i)}(2^k), \ \ i = 1, 2, \ \ k = 1, 2, \ldots, \tag{19}$$

$$x(t) = \phi_0, \quad x^{(i)}(1) = x_0^{(i)}, \quad i = 1, 2, \quad t \in \left[\frac{1}{2}, 1\right],$$

where $a_k^{(0)} = b_k^{(0)} = 1 + \dfrac{1}{2^k}$, $a_k^{(i)} = b_k^{(i)} = 1 - \dfrac{1}{2^k}$, $i = 1, 2$, $p(t) = t^{-1}$, $t_k = 2^k$ and $\varphi(x) = x^3$. It is easy to see that conditions $(H_1)$, $(H_2)$, $(H_5)$, $(H_6)$ and $(H_7)$ are satisfied. For condition $(H_3)$,

$$\int_{t_0}^{\infty} \prod_{t_0 < t_k < s} \frac{a_k^{(1)}}{b_k^{(0)}}\, ds = 1 + \frac{2-1}{2+1}(2^2 - 2) + \ldots + \frac{(2^m - 1)\ldots(2-1)}{(2^m+1)\ldots(2+1)}(2^{m+1} - 2^m) + \ldots >$$

$$> 1 + \frac{2(2-1)}{2+1} + \ldots + \frac{2^m(2^m - 1)\ldots(2-1)}{(2^m+1)\ldots(2+1)} + \ldots >$$

$$> 1 + \frac{2}{3} + \frac{4}{5} + 1 + \ldots + 1 + \ldots = \infty,$$

and $\int_{t_0}^{\infty} \prod_{t_0 < t_k < s} \dfrac{a_k^{(2)}}{b_k^{(1)}}\, ds = \int_1^{\infty} ds = \infty$, then condition $(H_3)$ is satisfied. Meanwhile,

$$\int_{t_1}^{t_1+\tau} p(s)\, ds + \varphi(a_1^{(0)}) \int_{t_1+\tau}^{t_2} p(s) ds + \ldots + \frac{\prod_{1 \leq i < m-1} \varphi(a_i^{(0)})}{\prod_{1 < i \leq m-1} b_i^{(2)}} \int_{t_{m-1}}^{t_{m-1}+\tau} p(s)\, ds +$$

$$+ \frac{\prod_{1 \leq i \leq m-1} \varphi(a_i^{(0)})}{\prod_{1 < i \leq m-1} b_i^{(2)}} \int_{t_{m-1}+\tau}^{t_m} p(s)\, ds + \ldots = \int_2^{\frac{5}{2}} s^{-1}\, ds + \left(1 + \frac{1}{2}\right)^3 \int_{\frac{5}{2}}^{4} s^{-1}\, ds + \ldots$$

$$\ldots + \frac{\left[\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right)\ldots\left(1 + \frac{1}{2^{m-2}}\right)\right]^3}{\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\ldots\left(1 - \frac{1}{2^{m-1}}\right)} \int_{2^{m-1}}^{2^{m-1}+\frac{1}{2}} s^{-1}\, ds +$$

$$+\frac{\left[\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{2^{m-1}}\right)\right]^3}{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ldots\left(1-\frac{1}{2^{m-1}}\right)}\int\limits_{2^{m-1}+\frac{1}{2}}^{2^m} s^{-1}\,ds+\ldots >$$

$$>\int\limits_{2}^{\frac{5}{2}} s^{-1}\,ds+\int\limits_{\frac{5}{2}}^{4} s^{-1}\,ds+\ldots+\int\limits_{2^{m-1}}^{2^{m-1}+\frac{1}{2}} s^{-1}\,ds+$$

$$+\int\limits_{2^{m-1}+\frac{1}{2}}^{2^m} s^{-1}\,ds+\ldots=\int\limits_{2}^{\infty} s^{-1}\,ds=\infty,$$

which means that condition $(H_4)$ is satisfied. By Theorem 2.1, every solution of (19) is either oscillatory or tends eventually to zero.

***Example 3.2.*** Consider

$$x'''(t)+\frac{3}{8}t^{-2}x^{\frac{1}{3}}(t-\tau)=0,\quad \tau=\frac{1}{4},\quad t\geq 1,\quad t\neq k,$$

$$x(k^+)=\left(1-\frac{1}{k^2}\right)^3 x(k),\quad x^{(i)}(k^+)=\left(1-\frac{1}{k}\right)x^{(i)}(k),\quad i=1,2,\quad k=2,3,\ldots,\qquad (20)$$

$$x(t)=\phi_0,\quad x^{(i)}(1)=x_0^{(i)},\quad i=1,2,\quad t\in\left[\frac{3}{4},1\right],$$

where $a_k^{(0)}=b_k^{(0)}=\left[1-\dfrac{1}{(k+1)^2}\right]^3$, $a_k^{(i)}=b_k^{(i)}=1-\dfrac{1}{k+1}$, $i=1,2$, $k=1,2,\ldots$, $p(t)=$ $=\dfrac{3}{8}t^{-2}$, $t_k=k+1$ and $\varphi(x)=x^{\frac{1}{3}}$. It is easy to see that conditions $(H_1)$, $(H_2)$, $(H_5)$ and $(H_9)$ are satisfied. For condition $(H_3)$,

$$\int\limits_{t_0}^{\infty}\prod_{t_0<t_k<s}\frac{a_k^{(1)}}{b_k^{(0)}}\,ds=1+\frac{2^3\times 2^2}{3^3}+\frac{2^3\times 3^2}{4^3}+\ldots+\frac{2^3\times k^2}{(k+1)^3}+\ldots=$$

$$=1+2^3\left(\frac{2^2}{3^3}+\frac{3^2}{4^3}+\ldots+\frac{k^2}{(k+1)^3}+\ldots\right)=\infty,$$

and $\displaystyle\int_{t_0}^{\infty}\prod_{t_0<t_k<s}\frac{a_k^{(2)}}{b_k^{(1)}}\,ds=\int_1^{\infty}ds=\infty$, then condition $(H_3)$ is satisfied. For condition $(H_4)$, by $\varphi(a_k^{(0)})=1-\dfrac{1}{(k+1)^2}<1$, we have

$$\int\limits_{t_1}^{t_1+\tau}p(s)\,ds+\varphi(a_1^{(0)})\int\limits_{t_1+\tau}^{t_2}p(s)\,ds+\ldots+\frac{\prod_{1\leq i<m-1}\varphi(a_i^{(0)})}{\prod_{1<i\leq m-1}b_i^{(2)}}\int\limits_{t_{m-1}}^{t_{m-1}+\tau}p(s)\,ds+$$

$$+\frac{\prod_{1\le i\le m-1}\varphi(a_i^{(0)})}{\prod_{1<i\le m-1}b_i^{(2)}}\int\limits_{t_{m-1}+\tau}^{t_m} p(s)\,ds+\ldots>$$

$$>\varphi(a_1^{(0)})\int\limits_{t_1}^{t_2} p(s)\,ds+\frac{\varphi(a_2^{(0)})\varphi(a_1^{(0)})}{b_1^{(2)}}\int\limits_{t_2}^{t_3} p(s)\,ds+\ldots$$

$$\ldots+\frac{\prod_{1\le i\le m-1}\varphi(a_i^{(0)})}{\prod_{1<i\le m-1}b_i^{(2)}}\int\limits_{t_{m-1}}^{t_m} p(s)\,ds+\ldots=$$

$$=\left(1-\frac{1}{2^2}\right)\int\limits_2^3\frac{3}{8}\,s^{-2}\,ds+\frac{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)}{\left(1-\frac{1}{3}\right)}\int\limits_3^4\frac{3}{8}\,s^{-2}\,ds+\ldots$$

$$\ldots+\frac{\prod_{k=2}^{m-1}\left(1-\frac{1}{k^2}\right)}{\prod_{k=3}^{m-1}\left(1-\frac{1}{k}\right)}\int\limits_{m-1}^{m}\frac{3}{8}\,s^{-2}\,ds+\ldots=$$

$$=\frac{3}{8}\left(1-\frac{1}{2^2}\right)\left[\frac{1}{2\times 3}+\left(1+\frac{1}{3}\right)\times\frac{1}{3\times 4}+\ldots+\prod_{k=3}^{m-1}\left(1+\frac{1}{k}\right)\times\frac{1}{(m-1)\times m}+\ldots\right]=$$

$$=\frac{3}{32}\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m-1}+\ldots\right)=\infty,$$

which means that condition $(H_4)$ is satisfied. By Theorem 2.3, every solution of (20) is either oscillatory or tends eventually to zero. However, if we let $\tau=0$ in (20), the classical ordinary differential equation $x'''(t)+\dfrac{3}{8}\,t^{-2}x^{\frac{1}{3}}(t)=0$ has a nonnegative solution $x=t^{\frac{3}{2}}$. This example shows that, in some cases, not only delay but also impulses play a critical role in giving rise to the oscillations of equations.


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*Received 07.06.10,*
*after revision — 06.12.11*