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TRIPLE POSITIVE SOLUTIONS FOR A CLASS OF TWO-POINT BOUNDARY-VALUE PROBLEMS. A FUNDAMENTAL APPROACH

ТРІЙКА ДОДАТНИХ РОЗВ'ЯЗКІВ ДЛЯ КЛАСУ ДВОТОЧКОВИХ ГРАНИЧНИХ ЗАДАЧ. ЗАГАЛЬНИЙ ПІДХІД

P. K. Palamides

Naval Academy Greece
Piraeus, 451 10, Greece
e-mail: ppalam@otenet.gr

A. P. Palamides

Technol. Educat. Inst. Piraeus
Athens, Greece
e-mail: palamid@uop.gr
palamid@gmail.com

In this paper, we prove the existence of three positive and concave solutions, by means of an elementary simple approach, to the 2^{th} order two-point boundary-value problem

$$x''(t) = \alpha(t)f(t, x(t), x'(t)), \quad 0 < t < 1,$$

$$x(0) = x(1) = 0.$$

We rely on a combination of the analysis of the corresponding vector field on the phase-space along with Kneser's type properties of the solutions funnel and the Schauder's fixed point theorem. The obtained results justify the simplicity and efficiency (one could study the problem with more general boundary conditions) of our new approach compared to the commonly used ones, like the Leggett – Williams Fixed Point Theorem and its generalizations.

З допомогою елементарного підходу до двоточкової граничної задачі другого порядку

$$x''(t) = \alpha(t)f(t, x(t), x'(t)), \quad 0 < t < 1,$$

$$x(0) = x(1) = 0,$$

доведено існування трьох додатних та вгнутих розв'язків. При цьому використано аналіз відповідного векторного поля на фазовому просторі, кнессеровські властивості множини розв'язків та теорему Шаудера про нерухому точку. Отримані результати пояснюють простоту та ефективність розробленого нового підходу (можливість вивчати задачу з більш загальними граничними значеннями) в порівнянні з методами, що використовувалися раніше, наприклад теоремою Логгетт та Вільямса про нерухому точку та її узагальнення.

1. Introduction. In the past 20 years, there has been much attention focused on questions of positive solutions for diverse nonlinear ordinary differential equation, difference equation, and functional differential equation boundary-value problems without dependence on the first-order derivative. To identify a few, we refer the reader to [15, 16] and references therein.

The problem of existence of positive solutions for boundary-value problems generated by applications in applied mathematics, physics, mechanics, chemistry, biology, etc. was extensively studied in the literature, for details see the bibliography of this article. The main tools used are fixed-point theorems, such as the well known Guo – Krasnosel'skii fixed-point theorem in a cone. For example in [10, 14], this theorem plays an extremely important role. Fixed-point theorems and their applications to nonlinear problems have a long history some of which is documented in the recent book by Agarwal, O'Regan and Wong [1] which contains an excellent summary of the current results and applications.

Concerning the question of multiplicity of solutions, most of the work done is based on the well known Leggett – Williams Fixed Point Theorem [15]. Especially, an interest in triple solutions evolved from that theorem. Lately, several triple fixed-point theorems have been established. See for example, Avery [2], Ren et al. [19] and Avery and Peterson [5] where such triple fixed-point theorems were applied in order to obtain triple solutions of certain boundary-value problems for ordinary differential equations as well as for their discrete analogues. Also recently, some new fixed-point theorems of cone expansion and compression of functional type, due to Avery and Anderson (see [3]) have been proved. A different extension was given in [12]. Karakostas in [12], using the Leggett – Williams fixed point theorem in a cone proved the existence of triple positive solutions for a boundary-value problem governed by the ϕ-Laplacian when the boundary conditions include nonlinear expressions at the end points. Also we refer to the work in [13], where existence results for a countable set of solutions of a nonlocal boundary-value problem were obtained.

All of them can be regarded as extensions of the Leggett – Williams fixed-point theorem and Guo – Krasnosel'skii fixed-point theorem. We notice however that, the most of the works on positive solutions was done under the assumption that the first order derivative is not involved explicitly in the nonlinear term. On the other hand the multiplicity of positive solutions with dependence on derivatives is considered in very few cases, see [11]. For other literature regarding the existence of triple solutions, that are not necessary positive, we refer reader to [5].

In [6] Bai, Wang and Ge using a fixed-point theorem of Avery and Peterson [5], obtained sufficient conditions for the existence of at least three positive solutions for the equation

$$x''(t) + q(t)f(t, x(t), x'(t)) = 0$$

subject to some boundary conditions. In [7] Bai and Ge generalized the Leggett – Williams fixed-point theorem and proved the existence of triple positive solutions for the second-order two point boundary-value problem

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \quad x(0) = x(1) = 0. \tag{E}$$

Bai and Ge assumed except the positivity of the nonlinearity, that there exist constants $r_2 \geq \geq 4b > b > r_1 > 0$ and $L_2 \geq L_1 > 0$ such that $8b \leq \min\{r_2, L_2\}$, and

$$f(t, u, v) < \min\{8r_1, 2L_1\}, \quad (t, u, v) \in [0, 1] \times [0, r_1] \times [-L_1, L_1],$$

$$f(t, u, v) > 16b, \quad (t, u, v) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [b, 4b] \times [-L_2, L_2], \tag{A$_0$}$$

$$f(t, u, v) \leq \min\{8r_2, 2L_2\}, \ (t, u, v) \in [0, 1] \times [0, r_2] \times [-L_2, L_2].$$

Motivated, by the above work and especially by the last two papers, we obtain sufficient conditions for the existence of triple positive solutions of the simple boundary-value problem (E). Although the conditions are very similar (actually a little bit more strict) than those in [7], the proposed approach is quite different and very simple as it is of geometrical nature. Actually, we use the flow generated by (E) (see Remark 1) and reduces boundary-value problem to the algebraic problem of determining the initial values of solutions from the boundary conditions. It turns out that such an approach, applied to a broad class of boundary-value problem, considerably simplifies the proofs. We use a combination of the Schauder's fixed point theorem, the associating Green function and the Kneser's property (the cross-section of the solutions funnel is a continuum) under the light of the associated vector field.

We assume that the nonlinear function f is continuous and

$$f(t,u,v) \geq 0, \quad \text{for almost all} \quad t \in [0,1], \quad \text{and any} \quad u \geq 0 \quad \text{and} \quad v \in \mathbb{R}. \tag{A$_1$}$$

As we mentioned above, the presence of v in $f(t,u,v)$ causes some difficulties. We overcome this predicament, by modifying below suitably the assumption (A_0).

Remark 1. We notice here that the differential equation (E) defines a vector field, the properties of which will be crucial for our study. More specifically, let's look at the (x, x') face semi-plane $(x > 0)$. By the sign condition on f, we immediately see that $x'' \leq 0$. Thus any trajectory $(x(t), x'(t))$, $t \geq 0$, emanating from the semi-line

$$E_0 := \{(x, x') : x' > 0, x = 0\}$$

"evolves"in a natural way (when $x'(t) > 0$), toward the positive x-semi-axis and then trends toward the negative x'-semi-axis. Lastly, by setting a certain growth rate on f (say superlinearity) we can control the vector field, so that some trajectory satisfies the given boundary condition

$$x(1) = 0$$

at the time $t = 1$. These properties will be referred as "The nature of the vector field" throughout the rest of paper.

So, the technique presented here is different to that given in the above mentioned papers, but very close to the one given in [18]. Actually, we rely on the above "nature of the vector field"and on the simple shooting method. Finally, we refer for completeness the well-known Kneser's theorem (see for example the Copel's text-book [8]) as well as the Schauder's fixed point theorem.

Theorem 1. *Consider a system*

$$x' = f(t, x, x'), \quad (t, x, x') \in \Omega := [\alpha, \beta] \times \mathbb{R}^{2n}, \tag{1.1}$$

where the function f is continuous. Let $\hat{E}_0$ be a continuum (compact and connected) set in $\Omega_0 :=$ $:= \{(t, x, x') \in \Omega : t = \alpha\}$ and let $\mathcal{X}(\hat{E}_0)$ be the family of all solutions of (1.1) emanating from $\hat{E}_0$. If any solution $x \in \mathcal{X}(\hat{E}_0)$ is defined on the interval $[\alpha, \tau]$, then the set (cross-section at the point τ)

$$\mathcal{X}(\tau; \hat{E}_0) := \Big\{(x(\tau), x'(\tau)) : x \in \mathcal{X}(\hat{E}_0)\Big\}$$

is a continuum in $\mathbb{R}^{2n}$.

Theorem 2 (Schauder's FPT). *If X is a Banach space, $C \subseteq X$ is nonempty, bounded, closed and convex and $T : C \to C$ is completely continuous, then T has a fixed point.*

2. Main results. We are concerned with the existence of triple positive solutions for the second-order two point boundary-value problem (E), where $f : [0,1] \times [0,\infty) \times R \to [0,\infty)$ is a continuous function.

Let $X = C^1\{0,1]$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$, for all $t \in [0,1]$, and the maximum norm, $\|x\| = \max\{\max_{0\leq t\leq 1}|x(t)|, \max_{0\leq t\leq 1}|x'(t)|\}$. From the fact $x''(t) = -f(t,x,x') \leq 0$, we know that x is concave on $[0,1]$. So, we define the cone $P \subseteq X$ as

$$P = \{x \in X |\, x(t) \geq 0,\ \ x \text{ is concave on } [0,1]\} \subseteq X.$$

Denote by $G(t,s)$ the Green's function for the boundary-value problem

$$x''(t) = 0, \quad 0 < t < 1,$$
$$x(0) = x(1) = 0,$$

then $G(t,s) \geq 0$ for $0 \leq t,\, s \leq 1$ and

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Let

$$\delta = \min\left\{ \int\limits_{1/4}^{3/4} G\left(\frac{1}{4}, s\right)\, ds, \int\limits_{1/4}^{3/4} G\left(\frac{3}{4}, s\right)\, ds \right\} = \frac{1}{16}.$$

We set now

$$u^* = \begin{cases} 0, & u \leq 0, \\ u, & 0 < u \leq r_2, \\ r_2, & r_2 < u, \end{cases} \qquad \text{and} \quad v^* = \begin{cases} -L_2, & v \leq -L_2, \\ v, & -L_2 < v \leq L_2, \\ L_2, & L_2 < v, \end{cases}$$

and consider the modified boundary-value problem

$$x''(t) + g(t,x(t),x'(t)) = 0, \quad 0 < t < 1, \quad x(0) = x(1) = 0, \tag{2.1}$$

where

$$g(t,u,v) = f(t,u^*,v^*), \quad (t,u,v) \in [0,1] \times \mathbb{R}^2.$$

Remark 2. It is obvious that the map g is nonnegative and satisfies the conditions (A_1) – (A_3) below, on the entire region $[0,1] \times \mathbb{R}^2$.

Theorem 3. *Suppose that there exist constants L_1, L_2 and $r_2 > b \geq r_1 > L_1 > 0$, such that $\dfrac{352}{5}\, b \geq L_2 \geq \dfrac{32}{15}\, b$, $8b \geq 1$, and the following assumptions hold:*

(A_1) $f(t,u,v) < \min\{8r_1, 2L_1\}$, $(t,u,v) \in [0,1] \times [0,r_1] \times [-L_1, L_1]$;
(A_2) $f(t,u,v) > 24b + \dfrac{3L_2}{4}$, $(t,u,v) \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right] \times [b, 4b] \times [-L_2, L_2]$;
(A_3) $f(t,u,v) \le \min\{8r_2, L_2\}$, $(t,u,v) \in [0,1] \times [0,r_2] \times [-L_2, L_2]$.

Then, the boundary-value problem (E) has at least three positive and concave solutions x_1, x_2 *and* x_3 *satisfying*

$$\max_{0\le t\le 1} x_1(t) \le r_1, \quad \max_{0\le t\le 1} x_1'(t) \le L_1;$$

$$b < \min_{1/4\le t\le 3/4} x_2(t) < \max_{0\le t\le 1} x_2(t) < r_2, \quad \max_{0\le t\le 1} |x_2'(t)| < L_2;$$

$$\max_{0\le t\le 1} x_3(t) \le 4b, \quad \max_{0\le t\le 1} |x_3'(t)| \le L_2.$$

Proof. It is well known that the Problem (E) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = Tx(t) = \int\limits_0^1 G(t,s) f(s, x(s), x'(s))\, ds$$

and that $T : P \to X$ is completely continuous.

Consider the set

$$\Omega = \left\{ x \in P : \max_{0\le t\le 1} x(t) \le r_1 \quad \text{and} \quad 0 \le \max_{0\le t\le 1} |x'(t)| \le L_1 \right\}.$$

By the positivity of the functions $G(t,s)$ and $f(t,u,v)$, we immediately get that $T : P \to P$. Furthermore we will show that $T(\Omega) \subseteq \Omega$. Indeed, for any $x \in \Omega$, by the assumption (A_1), we get

$$\max_{0\le t\le 1} Tx(t) = \max_{0\le t\le 1} \int\limits_0^1 G(t,s) f(s, x(s), x'(s))\, ds \le 8r_1 \max_{0\le t\le 1} \int\limits_0^1 G(t,s)\, ds = r_1.$$

In view of Remark 1 and the inequality

$$(Tx)''(t) = -f(t, x(t), x'(t)) \le 0, \quad 0 \le t \le 1,$$

the map $(Tx)'(t)$ is decreasing and thus $\max_{0\le t\le 1} (Tx)'(t) = \max\{|(Tx)'(0)|, |(Tx)'(1)|\}$. Consequently,

$$\max_{0\le t\le 1}|(Tx)'(t)| = \max_{0\le t\le 1}\left|-\int_0^t sf(s,x(s),x'(s))\,ds + \int_t^1 (1-s)f(s,x(s),x'(s))\,ds\right| =$$

$$= \max\left\{\int_0^1 sf(s,x(s),x'(s))\,ds, \int_0^1 (1-s)f(s,x(s),x'(s))\,ds\right\} \le$$

$$\le 2L_1 \max\left\{\int_0^1 s\,ds, \int_0^1 (1-s)\,ds\right\} \le 2L_1\frac{1}{2} = L_1.$$

Thus $T(\Omega) \subseteq \Omega$. In addition, the set Ω is nonempty, bounded, closed and convex. Thus by the Schauder's fixed point Theorem 2, we conclude the existence of a solution $x_1 \in \Omega$ of the boundary-value problem (E).

Consider now the boundary-value problem (2.1), as well as the initial value problem

$$x''(t) + g(t,x(t),x'(t)) = 0, \quad 0 < t < 1, \quad x(0) = 0, \quad x'(0) = \xi \ge 0. \tag{2.2}$$

For $\xi = L_2$, via the Taylor's theorem, the assumption (A_3) and the definition of the modification g, we get a $\tau \in [0,1]$ such that

$$x(1) = x'(0) - \frac{1}{2}\, g\left(\tau, x(\tau), x'(\tau)\right) \ge L_2 - \frac{L_2}{2} \ge 0. \tag{2.3}$$

Similarly for $\xi = 4b + \dfrac{L_2}{8}$ and any $x \in \mathcal{X}([\xi, L_2])$ we have

$$x\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)x'(0) - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^2 g\left(\tau, x(\tau), x'(\tau)\right) \ge$$

$$\ge \left(\frac{1}{4}\right)\left(4b + \frac{L_2}{8}\right) - \frac{1}{2.16}\, L_2 = b. \tag{2.4}$$

Let $x \in X(\xi)$. Then we obtain an $\alpha \in \left[0, \dfrac{1}{4}\right)$ such that $x(\alpha) = b$. We assert that

$$x\left(\frac{3}{4}\right) < b. \tag{2.5}$$

Indeed, supposing on the contrary that $x\left(\dfrac{3}{4}\right) \ge b$, we get via the assumption (A_2), the contradiction

$$x\left(\frac{3}{4}\right) = x(\alpha) + \left(\frac{3}{4} - \alpha\right)x'(\alpha) - \left(\frac{1}{2}\right)\left(\frac{3}{4} - \alpha\right)^2 g\left(\tau, x(\tau), x'(\tau)\right) <$$

$$< b + \frac{3}{4}\left(4b + \frac{L_2}{8}\right) - \frac{1}{2}\left(\frac{1}{2}\right)^2\left(24b + \frac{3L_2}{4}\right) = b.$$

On the other hand, we have for any $x \in \mathcal{X}(L_2)$,

$$x\left(\frac{3}{4}\right) = \frac{3}{4}\,x'(0) - \frac{1}{2}\left(\frac{3}{4}\right)^2 g\left(\tau, x(\tau), x'(\tau)\right) \geq \frac{3}{4}\,L_2 - \frac{1}{2}\left(\frac{3}{4}\right)^2 2L_2 \geq b,$$

given that $L_2 \geq \dfrac{32}{15}\,b$.

Thus, by the Kneser's property and (2.4), we obtain a point $\xi_0 \in \left(4b + \dfrac{L_2}{8}, L_2\right)$ and $x \in \mathcal{X}(\xi_0)$ such that

$$x(t) \geq b, \quad t \in \left(\frac{1}{4}, \frac{3}{4}\right) \quad \text{and} \quad x\left(\frac{3}{4}\right) = b.$$

Moreover, by the monotonicity of $x'(t)$, we conclude that

$$x'\left(\frac{3}{4}\right) = x'\left(\frac{1}{4}\right) - \frac{1}{2}\,g\left(\tau, x(\tau), x'(\tau)\right) \leq L_2 - \frac{1}{2}\left(24b + \frac{3L_2}{2}\right) = \frac{1}{4}\,L_2 - 12b.$$

Consequently, since $L_2 \leq \dfrac{352}{5}\,b$, the above solution $x \in \mathcal{X}(\xi_0)$ satisfies in addition

$$x(1) = x\left(\frac{3}{4}\right) + \frac{1}{4}\,x'\left(\frac{3}{4}\right) - \frac{1}{2}\left(\frac{1}{4}\right)^2 g\left(\tau, x(\tau), x'(\tau)\right) \leq$$

$$\leq b + \frac{1}{4}\left(\frac{1}{4}L_2 - 12b\right) - \frac{1}{2}\left(\frac{1}{4}\right)^2\left(24b + \frac{3L_2}{4}\right) \leq 0. \tag{2.6}$$

Applying now the Kneser's Theorem 1, in view of (2.3) and (2.6), we obtain another solution $x_2 \in \mathcal{X}((\xi_0, L_2)) \subset \mathcal{X}\left(\left[4b + \dfrac{L_2}{8}, L_2\right]\right)$ of the boundary-value problem (2.1). Following similar resonance, as for the previous solution $x_1(t)$, we may easily prove that

$$0 \leq x_2(t) \leq r_2 \quad \text{and} \quad -L_2 \leq x_2'(t) \leq L_2, \quad 0 \leq t \leq 1.$$

Thus $x_2(t)$, $0 \leq t \leq 1$, is actually a solution of the initial boundary-value problem (E).

Furthermore, we assert that

$$\min_{1/4 \leq t \leq 3/4} x_2(t) \geq b. \tag{2.7}$$

Indeed, since (2.4) is true for any $x \in \mathcal{X}\left(\left[4b + \dfrac{L_2}{8}, L_2\right]\right)$, we obviously have $x_2\left(\dfrac{1}{4}\right) > b$ and assuming on the contrary that there exists $\beta \in \left(\dfrac{1}{4}, \dfrac{3}{4}\right)$ such that $x_2(\beta) = b$ and of course

$x_2\left(\dfrac{3}{4}\right) < b$, we get

$$x_2(\beta) = \int\limits_0^1 G(\beta,s)f(s,x_2(s),x_2'(s))ds == \int\limits_0^\beta G(\beta,s)f(s,x_2(s),x_2'(s))\,ds +$$

$$+ \int\limits_\beta^1 G(\beta,s)f(s,x_2(s),x_2'(s))\,ds \ge \int\limits_{1/4}^\beta G(\beta,s)f(s,x_2(s),x_2'(s))\,ds >$$

$$> \int\limits_{1/4}^\beta G\left(\frac{1}{4},s\right)f(s,x_2(s),x_2'(s))\,ds \ge \frac{1}{16}\left(24b + \frac{3L_2}{4}\right) > b,$$

a contradiction.

We seek now for the third solution $x_3(t)$. The existence of $x_3(t)$ follows directly by (2.6) and since, by Remark 2, for any $x \in \mathcal{X}(L_1)$,

$$x(1) = x'(0) - \frac{1}{2}g\left(\tau, x(\tau), x'(\tau)\right) \ge 4r_1 - \frac{1}{2}8r_1 = 0.$$

The application of the condition (A_1) is possible, since for any $x \in \mathcal{X}(L_1)$,

$$\left(x(t), x'(t)\right) \in [0, r_1] \times [-L_1, L_1].$$

In fact, let's assume that there exists a $\tau \in [0,1]$ such that

$$0 \le x(t) \le x(\tau) = r_1, \quad 0 \le t \le \tau,$$

and then, in view of the vector field, we know that

$$0 \le x'(t) \le L_1, \quad 0 \le t \le \tau.$$

Consequently, we get the contradiction

$$r_1 = x(\tau) = \tau L_1 - \frac{1}{2}\tau^2 g\left(\tau, x(\tau), x'(\tau)\right) < \tau L_1 < L_1.$$

Then the Kneser's theorem may be applied once again to get a solution $x_3 \in \mathcal{X}((L_1, \xi_0))$. We notice that since $L_1 < 4b + \dfrac{L_2}{8}$, clearly x_3 is different than the other two obtained solutions x_1 and x_2.

We replace now the set Ω by the next one:

$$\Omega^* = \left\{x \in P : \max_{0\le t\le 1} x(t) \le r_2 \text{ and } -L_2 \le \max_{0\le t\le 1} x'(t) \le L_2\right\}.$$

Then obviously $T(\Omega^*) \subseteq \Omega^*$, and by the formula

$$x(t) = \int_0^1 G(t,s) f(s, x(s), x'(s))\, ds,$$

we conclude that x_3 is actually a solution of the original boundary-value problem (E).

Theorem 3 is proved.

Remark 3. In view of (2.5), we may choose the initial value ξ_0 as a minimal one, in the sense that any solution $x_3 \in \mathcal{X}((L_1, \xi_0))$ of the boundary-value problem (E) satisfies the inequalities

$$\max_{t\in[0,1]} x_3(t) \le L_2 \text{ and } \min_{x\in\left[\frac{1}{4},\frac{3}{4}\right]} x_3(t) \le b.$$

Because of the additional relation

$$\min_{x\in\left[\frac{1}{4},\frac{3}{4}\right]} x_3(t) \ge \frac{1}{4} \max_{t\in[0,1]} x_3(t)$$

we immediately conclude that $\max_{t\in[0,1]} x_3(t) \le 4b$.

Finally we present an example to illustrate our results.

Example 1. Consider the boundary-value problem (E), where

$$f(t,u,v) = \begin{cases} \dfrac{u3}{4} + \left(\dfrac{v}{3200}\right)^2, & \text{for} \quad u \le \dfrac{1}{32}, \quad |v| \le 70, \\[2ex] 5114.6u - 159.81 + \left(\dfrac{v}{3200}\right)^2, & \text{for} \quad \dfrac{1}{32} \le u \le \dfrac{1}{24}, \quad |v| \le 70, \\[2ex] 53.338 + \left(\dfrac{v}{3200}\right)^2, & \text{for} \quad \dfrac{1}{24} \le u \le \dfrac{4}{24}, \quad |v| \le 70, \\[2ex] 53.29 + 0.06u + \left(\dfrac{v}{3200}\right)^2, & \text{for} \quad \dfrac{4}{24} \le u \le 120, \quad |v| \le 70. \end{cases}$$

Choose $L_1 = \dfrac{1}{64}$, $r_1 = \dfrac{1}{32}$, $b = \dfrac{1}{24}$, $r_2 = 120$ and $L_2 = 70 \le \dfrac{352}{5}$. Then the nonlinearity satisfies the assumptions $(A_1)-(A_3)$, namely

$$f(t,u,v) \le \frac{1}{4}\left(\frac{1}{32}\right)^3 \frac{1}{32} + \left(\frac{640}{3200}\right)^2 < \min\{8r_1, 2L_1\} \le \frac{1}{4},$$

$$(t,u,v) \in [0,1] \times \left[0, \frac{1}{32}\right] \times \left[-\frac{1}{64}, \frac{1}{64}\right];$$

$$f(t,u,v) \ge 53.338 > 24b + \frac{3L_2}{4} = 53.25,$$

$$(t,u,v) \in \left[\frac{1}{4},\frac{3}{4}\right] \times \left[\frac{1}{24},\frac{1}{6}\right] \times [-70,70];$$

$$f(t,u,v) \le 53.338 \le \min\{8r_2, L_2\} = 70,$$

$$(t,u,v) \in [0,1] \times [0,120] \times [-70,70].$$

An application of Theorem 3, guarantees the existence of three positive solutions x_1, x_2, x_3 such that

$$0 \le x_1(t) \le \frac{1}{32}, \quad -\frac{1}{64} \le x_1'(t) \le \frac{1}{64}, \quad 0 \le t \le 1,$$

$$\frac{1}{24} \le \min_{\frac{1}{4}\le t\le\frac{3}{4}} x_2(t) \le \max_{0\le t\le 1} x_2(t) \le 120, \quad -70 \le x_1'(t) \le 70, \quad 0 \le t \le 1,$$

$$0 \le x_3(t) \le \frac{4}{24}, \quad -70 \le x_1'(t) \le 70, \quad 0 \le t \le 1.$$

Remark 4. The proposed method except its simplicity, has several advantages. For example the well-known upper-lower solutions method requires a Nagumo type growth rate in the third variable of the nonlinearity [21]. In order to avoid such a restriction, in several previous papers [1–5, 17, 20] the nonlinearity does not depends on it. Mainly, we may study more general boundary value problems, as we indicate in the next remark (see also [9]).

Remark 5. By the above analysis, it is obvious that we are able to replace the boundary conditions in (E), for example, by the next ones

$$\alpha x(0) - \beta x'(0) = 0, \quad \gamma x(1) + \delta x'(1) = 0,$$

under the assumption

$$\alpha b - \beta L_2 \ge 0 \ge \gamma b - \delta L_2.$$

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