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STABILIZATION OF OPTICALLY COUPLED LASERS WITH PERIODIC PUMPING*
СТАБІЛІЗАЦІЯ ОПТИЧНО ЗВ'ЯЗАНИХ ЛАЗЕРІВ З ПЕРІОДИЧНОЮ НАКАЧКОЮ

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We study a periodically forced system modeling the synchronization of two optically coupled lasers pumped by an alternating current. A necessary and sufficient condition for existence of a periodic solution is given, as well as a sufficient condition for uniqueness and asymptotic stability.

Вивчаються періодично збурювані системи, що моделюють синхронізацію двох оптично зв'язаних лазерів, що накачуються за допомогою змінного струму. Наведено необхідні і достатні умови існування періодичного розв'язку, а також достатню умову для його єдиності та асимптотичної стійкості.

1. Introduction. In Nonlinear Optics, the synchronization of two optically coupled lasers pumped by an alternating current is a known fact that has deserved the attention of the specialists from a theoretical and experimental perspective [1–4]. In [3], it is shown that in its simplest formulation, two coupled lasers with periodic pumping behave like an "equivalent" single laser whose dynamical behavior is described by the system

$$\begin{aligned} \tau\dot{g} &= g_0(t) - g\left(1+E^2\right), \\ \dot{E} &= \frac{1}{2}\,(g-\tilde{g}_{th})E. \end{aligned} \tag{1}$$

Here, g is the amplification factor and E is the amplitude of the locked field. The parameter $\tau > 0$ is the effective relaxation time of the active medium, $g_0(t) = A(1+\sin\omega t)$ is the $\frac{2\pi}{\omega}$-periodic pumping, and $\tilde{g}_{th} > 0$ is the renormalised threshold gain defined by

$$\tilde{g}_{th} = g_{th} + 2M\left(1-\sqrt{1-\left(\frac{\Delta}{M}\right)^2}\right),$$

where $g_{th} > 0$ is the original threshold gain, $M > 0$ is the coupling coefficient, and $\Delta > 0$ is proportional to the off-tuning of the natural frequencies of the cavities.

On the basis of the mentioned references, the theoretical objective is to find an asymptotically stable periodic solution of system (1). To this purpose, the very recent paper [4] presents

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a numerical-analytic scheme of investigation of periodic solutions of system (1) by means of a technique of construction of matrix-valued Lyapunov functions [5]. The aim of this paper is to contribute to the literature from a different point of view, in such a way that we are able to identify explicit regions of parameters where there exists a unique periodic solution which is asymptotically stable.

From now on, the minimal period of the periodic pumping is denoted by $T = \frac{2\pi}{\omega}$. Our main results are stated below.

Theorem 1. *The condition*

$$\tilde{g}_{th} < A \tag{2}$$

is necessary and sufficient for the existence of a T-periodic solution of system (1).

Theorem 2. *Assume that (2) holds. Let us fix the constants*

$$m_1 := \frac{1}{2}\ln\left(\frac{A}{\tilde{g}_{th}} - 1\right) - \frac{\sqrt{3}}{2\sqrt{2}}AT, \quad m_2 := \frac{1}{2}\ln\left(\frac{A}{\tilde{g}_{th}} - 1\right) + \frac{\sqrt{3}}{2\sqrt{2}}AT.$$

Then, under the assumptions

(i) $\tau \geq \frac{T^2}{4}\left(\frac{AT}{\tau} + \tilde{g}_{th}\right)e^{2m_2} + \frac{T}{2}(e^{2m_2} + 1)$,

(ii) $\tau\tilde{g}_{th}e^{2m_1} \geq \frac{1}{4}(e^{2m_2} + 1)^2 + ATe^{2m_2}$,

the T-periodic solution given by Theorem 1 is unique and asymptotically stable.

From this latter result, the following consequence is direct.

Corollary 1. *There exists an explicitly computable τ_0 (depending on the rest of parameters A, ω, g_{th}, M, Δ) such that for any $\tau > \tau_0$, system (1) has a unique T-periodic solution which is asymptotically stable.*

The paper is structured as follows: after this introduction, the existence result is proved in Section 2, by using a transformation to a Liénard equation and a classical topological degree argument. Section 3 is devoted to the uniqueness and asymptotic stability. We use a classical stability criterium by Erbe [6]. Finally, in Section 4 some final remarks are given.

2. A priori bounds and existence. The first step is to write system (1) as an equivalent Liénard equation. Since E is the amplitude of the locked field, it is always positive. By introducing the change of variable $E = e^x$, the second equation of system (1) is written as

$$\dot{x} = \frac{1}{2}(g(t) - \tilde{g}_{th}).$$

Deriving this equation and substituting into the first equation of system (1), one gets the Liénard equation

$$\ddot{x} + f(x)\dot{x} + h(x) = \frac{1}{2\tau}g_0(t), \tag{3}$$

where $f(x) = \frac{1}{\tau}(1+e^{2x})$, $h(x) = \frac{1}{2\tau}\tilde{g}_{th}(1+e^{2x})$. As it was indicated in the Introduction, $g_0(t) = A\,(1+\sin\omega t)$. Needless to say, Eq. (3) is equivalent to system (1), and from a given solution x of (3) one can recover the original solution

$$E = e^x, \quad g = 2\dot{x} + \tilde{g}_{th}. \tag{4}$$

We begin the study of Eq. (3) with a result on a priori bounds.

Lemma 1. *Any eventual T-periodic solution of (3) verifies the bounds*

$$m_1 < x(t) < m_2 \tag{5}$$

and

$$\|\dot{x}(t)\| < m_3 := \frac{AT}{2\tau} \tag{6}$$

for every t, with m_1, m_2 as defined in Theorem 2.

Proof. Let us assume that $x(t)$ is a given T-periodic solution. By integrating the equation over $[0,T]$ one gets

$$\tilde{g}_{th}\int\limits_0^T (1+e^{2x(t)})\,dt = AT. \tag{7}$$

By the integral Mean Value Theorem, there exists $t_0 \in]0,T[$ such that

$$x(t_0) = \frac{1}{2}\ln\left(\frac{A}{\tilde{g}_{th}} - 1\right). \tag{8}$$

On the other hand, multiplying (3) by $\dot{x}$ and integrating over a period,

$$\frac{1}{\tau}\|\dot{x}\|_2^2 < \frac{1}{\tau}\int\limits_0^T (1+e^{2x})\dot{x}^2 dt = \frac{1}{2\tau}\int\limits_0^T g_0(t)\dot{x}dt \le \frac{1}{2\tau}\|g_0(t)\|_2\,\|\dot{x}\|_2, \tag{9}$$

after a basic application of Cauchy – Bunyakowskii – Schwarz inequality. Hence,

$$\|\dot{x}\|_2 < \frac{1}{2}\|g_0(t)\|_2 = \frac{A}{2}\sqrt{\frac{3T}{2}}. \tag{10}$$

Now, for every $t \in [0,T]$ we have

$$|x(t) - x(t_0)| = \left|\int\limits_{t_0}^t \dot{x}(s)ds\right| \le \|\dot{x}\|_1 \le \|\dot{x}\|_2\sqrt{T} < \frac{\sqrt{3}}{2\sqrt{2}}AT,$$

as a result of Cauchy–Bunyakowskii–Schwarz inequality and (10). From this inequality and (8), (5) is easily obtained.

The next aim is to prove (6). Let us take $t_* \in [0,T]$ such that $x(t_*) = \min_t x(t)$, then for any $t \in]t_*, t_* + T[$ one has

$$\dot{x}(t) = \int_{t_*}^{t} \ddot{x}(s)\,ds = -\int_{t_*}^{t} f(x(s))\dot{x}(s)\,ds - \int_{t_*}^{t} h(x(s))\,ds + \frac{1}{2\tau}\int_{t_*}^{t} g_0(s)\,ds <$$

$$< -\int_{x(t_*)}^{x(t)} f(s)\,ds + \frac{1}{2\tau}\,\|g_0\|_1 \leq \frac{1}{2\tau}\,\|g_0\|_1 = \frac{AT}{2\tau}, \tag{11}$$

where we have used that f, h are positive functions and $g_0(t)$ is non-negative. In a similar way, let us take $t^* \in [0,T]$ such that $x(t^*) = \max_t x(t)$, then for any $t \in]t^*, t^* + T[$ one has

$$\dot{x}(t) = \int_{t^*}^{t} \ddot{x}(s)\,ds = -\int_{t^*}^{t} f(x(s))\dot{x}(s)\,ds - \int_{t^*}^{t} h(x(s))\,ds + \frac{1}{2\tau}\int_{t^*}^{t} g_0(s)ds >$$

$$> \int_{x(t)}^{x(t^*)} f(s)\,ds - \int_{t^*}^{t^*+T} h(x(s))\,ds \geq -\frac{AT}{2\tau}, \tag{12}$$

where (7) has been used in the last inequality. From (11) and (12), one gets (6).

Lemma 1 is proved.

Obviously, the previous result gives explicit bounds for the eventual T-periodic solutions of the original system (1), which are specified in the lemma below since they may be of independent interest for the physical model.

Lemma 2. *Any eventual T-periodic solution (g, E) of system (1) verifies the bounds*

$$\begin{gathered} \left(\frac{A}{\tilde{g}_{th}} - 1\right)^{\frac{1}{2}} e^{-\frac{\sqrt{3}}{2\sqrt{2}}AT} < E(t) < \left(\frac{A}{\tilde{g}_{th}} - 1\right)^{\frac{1}{2}} e^{\frac{\sqrt{3}}{2\sqrt{2}}AT}, \\ -\frac{AT}{\tau} + \tilde{g}_{th} < g(t) < \frac{AT}{\tau} + \tilde{g}_{th} \end{gathered} \tag{13}$$

for every t.

Proof. Just use (5), (6) into (4).

Proof of Theorem 1. Let us prove that (2) is necessary and sufficient for the existence of a T-periodic solution of system (1). The necessity comes from an integration of Eq. (3) over a whole period, then (7) is obtained, and from there is it evident that $\tilde{g}_{th} < A$.

The sufficient condition follows from classical results on topological degree theory. In fact, if $\tilde{g}_{th} < A$ holds the equation verifies the well-known Landesman–Lazer conditions and for instance [7] (Theorem 2), can be directly applied. For completeness, we will give here a sketch of a different proof. Let us consider the homotopic equation

$$\ddot{x} + f(x)\dot{x} + h(x) = \frac{1}{2\tau}\left[(1-\lambda)A + \lambda g_0(t)\right], \tag{14}$$

with $\lambda \in [0,1]$. For $\lambda = 1$, it corresponds to Eq. (3). By using the same arguments as in Lemma 1, one can find uniform bounds (not depending on λ) on the possible T-periodic solutions of (14) and their derivatives. For $\lambda = 0$, we get the autonomous equation

$$\ddot{x} + f(x)\dot{x} + h(x) = \frac{1}{2\tau}A,$$

which is equivalent to the planar vectorial field

$$F(u,v) = \left(v, -f(u)v - h(u) + \frac{1}{2\tau}A\right).$$

By [8] (Theorem 2), the existence of a T-periodic solution of (3) is proved if the Brouwer degree of F over a large ball is different from zero. The unique fixed point of F is $(u_0, v_0) = \left(\frac{1}{2}\ln\left(\frac{A}{\tilde{g}_{th}} - 1\right), 0\right)$, and after some elementary computations one can see that the Jacobian matrix $JF(u_0, v_0)$ has positive determinant. Then the Brouwer degree of F over a large ball is 1 and therefore the proof is finished.

3. Uniqueness and asymptotic stability. In this section, it is assumed that (2) holds. It was proved in the last section that Eq. (3) has at least one T-periodic solution. The objective is to prove that, under the assumptions of Theorem 2, such a solution is unique and asymptotically stable. The following stability result of Erbe [6] (Section 3) will be useful.

Proposition 1. *Let us assume that $p, q \in C(\mathbb{R}/T\mathbb{Z})$ are continuous and T-periodic functions verifying*

(1) $\int_0^T p(t)dt > 0,$

(2) $\int_0^T q(t)dt + 2\|p\|_\infty \leq \frac{4}{T},$

(3) $4q(t) \geq p(t)^2$ *for every* t.

Then, the linear differential equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{15}$$

is asymptotically stable.

For convenience, let us remember that by Lemma 1, any T-periodic solution of (3) verifies $m_1 < x(t) < m_2$, $|\dot{x}(t)| < m_3$ for every t.

Proof of Theorem 2. As it was noted before, system (1) is equivalent to Eq. (3), hence along this proof we will work directly with this last equation.

Let us first prove the uniqueness. Assume that x_1, x_2 are two T-periodic solutions of Eq. (3). The objective is to prove that the difference $d(t) = x_1(t) - x_2(t)$ is a solution of a second order linear equation (15) in the conditions of Proposition 1, then the unique periodic solution of (15) would be the trivial one, so $d(t) \equiv 0$.

By subtracting the equations,

$$\ddot{d} + f(x_1)\dot{x}_1 - f(x_2)\dot{x}_2 + h(x_1) - h(x_2) = 0. \tag{16}$$

By using the Mean Value Theorem

$$f(x_1)\dot{x}_1 - f(x_2)\dot{x}_2 = [f(x_1) - f(x_2)]\dot{x}_1 + f(x_2)[\dot{x}_1 - \dot{x}_2] = \dot{f}(\xi(t))d(t)\dot{x}_1(t) + f(x_2(t))\dot{d}(t)$$

with $\xi(t)$ a value between $x_1(t)$ and $x_2(t)$, hence verifying $m_1 < \xi(t) < m_2$ for all t. Similarly, $h(x_1) - h(x_2) = \dot{h}(\nu(t))d(t)$ with $m_1 < \nu(t) < m_2$ for all t. Inserting this information into (16), one finds that $d(t)$ is a solution of a second order linear differential equation like (15) with

$$p(t) = f(x_2(t)) = \frac{1}{\tau}\left(1 + e^{2x_2}\right), \quad q(t) = \dot{f}(\xi(t))\dot{x}_1(t) + \dot{h}(\nu(t)) = \frac{2}{\tau}e^{2\xi}\dot{x}_1 + \frac{\tilde{g}_{th}}{\tau}e^{2\nu}.$$

Let us prove that such coefficients verify the conditions of Proposition 1. First, note that condition (1) is trivially satisfied because $p(t)$ is positive. On the other hand, by using Lemma 1 and the monotonicity of the exponential,

$$-\frac{1}{\tau}\left(-2m_3e^{2m_2} + \tilde{g}_{th}e^{2m_1}\right) < q(t)\frac{1}{\tau}\left(2m_3 + \tilde{g}_{th}\right)e^{2m_2}, \quad p(t) < \frac{1}{\tau}\left(1 + e^{2m_2}\right).$$

In consequence,

$$\int_0^T q(t)dt + 2\|p\|_\infty < \frac{T}{\tau}\left(2m_3 + \tilde{g}_{th}\right)e^{2m_2} + \frac{2}{\tau}\left(1 + e^{2m_2}\right).$$

Therefore, (2) holds if

$$\frac{T}{\tau}\left(2m_3 + \tilde{g}_{th}\right)e^{2m_2} + \frac{2}{\tau}\left(1 + e^{2m_2}\right) \le \frac{4}{T}.$$

After simple computations, one realizes that this is just condition (i) of Theorem 2. Similarly,

$$q(t) > \frac{1}{\tau}\left(-2m_3e^{2m_2} + \tilde{g}_{th}e^{2m_1}\right), \quad p(t)^2 < \frac{1}{\tau^2}\left(1 + e^{2m_2}\right)^2.$$

Hence, condition (3) holds if $-8m_3e^{2m_2} + 4\tilde{g}_{th}e^{2m_1} \ge \frac{1}{\tau}\left(1 + e^{2m_2}\right)^2$, and this is equivalent to condition (ii) of Theorem 2. Therefore, the proof of uniqueness is concluded.

The proof of asymptotic stability is similar. Let $x(t)$ be the unique T-periodic solution of (3). The linearized equation along $x(t)$ is $\ddot{y} + p(t)\dot{y} + q(t)y = 0$, where

$$p(t) = \frac{1}{\tau}\left(1 + e^{2x(t)}\right), \quad q(t) = \frac{2}{\tau}e^{2x(t)}\dot{x}(t) + \frac{\tilde{g}_{th}}{\tau}e^{2x(t)}.$$

By performing exactly the same bounds as in the proof of uniqueness, p, q are in the conditions of Proposition 1. In consequence, the linearized equation along $x(t)$ is asymptotically stable and the proof is finished.

Proof of Corollary 1. It is clear that conditions (i), (ii) are verified for τ large enough. For convenience, let us write explicitly a concrete value of τ_0. Multiplying (i) by τ, we obtain

$$\tau^2 - \tau\left[\frac{T^2}{4}\tilde{g}_{th}e^{2m_2} + \frac{T}{2}(e^{2m_2}+1)\right] - \frac{1}{4}AT^3e^{2m_2} \geq 0.$$

The left-hand side of this inequality is a second-order polynomial, so this is equivalent to assume that τ is above the positive root of such a polynomial, that is,

$$\tau \geq R_1 := \frac{T^2}{8}\tilde{g}_{th}e^{2m_2} + \frac{T}{4}(e^{2m_2}+1) + \frac{1}{2}\left(\left[\frac{T^2}{4}\tilde{g}_{th}e^{2m_2} + \frac{T}{2}(e^{2m_2}+1)\right]^2 + AT^3\,e^{2m_2}\right)^{\frac{1}{2}}.$$

On the other hand, (ii) holds if $\tau \geq R_2 := \frac{e^{-2m_1}}{\tilde{g}_{th}}\left[\frac{1}{4}\left(e^{2m_2}+1\right)^2 + ATe^{2m_2}\right]$. The proof is finished by taking

$$\tau_0 = \max\{R_1, R_2\}. \tag{17}$$

4. Concluding remarks. In this paper, we have proved (Theorem 1) a necessary and sufficient condition for existence of T-periodic solution of system (1), which model the synchronization of two optically coupled lasers pumped by an alternating current. Explicit bounds for the solution are given (Lemma 1).

Besides, a sufficient condition in terms of the involved parameters is given in order to get uniqueness and asymptotic stability of such a solution (Theorem 2). In Corollary 1, the stability condition is interpreted in the following way: the effective relaxation time of the active medium τ should be greater than a given computable quantity τ_0. From the physical point of view, this condition makes sense because τ is much more higher than the unit time ($\tau >> 1$, see [3]).

Surely, the sufficient condition for stability is far from being optimal. To improve it, there are two possibilities: (1) to apply other stability criteria for the linear second order equation, (2) to improve the bounds obtained in Lemma 1. The first way opens as many variants as stability criteria available in the literature (see for instance [9–11] and their references). We have chosen Erbe's result for the sake of simplicity.

As for the option of improving the bounds in Lemma 1, one can use a recursive procedure as follows. Once we know that every solution verifies $m_1 < x(t) < m_2$, (9) can be improved to

$$\frac{1+e^{2m_1}}{\tau}\|\dot{x}\|_2^2 < \frac{1}{\tau}\int\limits_0^T (1+e^{2x})\dot{x}^2\,dt = \frac{1}{2\tau}\int\limits_0^T g_0(t)\dot{x}dt \leq \frac{1}{2\tau}\|g_0(t)\|_2\,\|\dot{x}\|_2.$$

Hence, (10) is improved to

$$\|\dot{x}\|_2 < \frac{1}{2(1+e^{2m_1})}\,\|g_0(t)\|_2 = \frac{A}{2(1+e^{2m_1})}\sqrt{\frac{3T}{2}},$$

and repeating the arguments, one gets that $m_1^1 < x(t) < m_2^1$, with

$$m_1^1 := \frac{1}{2}\ln\left(\frac{A}{\tilde{g}_{th}} - 1\right) - \frac{\sqrt{3}AT}{2\sqrt{2}(1+e^{2m_1})}, \quad m_2^2 := \frac{1}{2}\ln\left(\frac{A}{\tilde{g}_{th}} - 1\right) + \frac{\sqrt{3}AT}{2\sqrt{2}(1+e^{2m_1})}.$$

This trick can be repeated recursively giving rise to monotone and convergent sequences m_1^n, m_2^n such that $m_1^n < x(t) < m_2^n$ for every $n \in \mathbb{N}$.

Finally, we observe that different bounds for $x(t)$ can be derived by applying (6) and (8) into the expression $x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(s)\,ds$, thus obtaining

$$\frac{1}{2}\ln\left(\frac{A}{\tilde{g}_{th}} - 1\right) - \frac{AT^2}{2\tau} < x(t) < \frac{1}{2}\ln\left(\frac{A}{\tilde{g}_{th}} - 1\right) + \frac{AT^2}{2\tau}.$$

Such bound are sharper than m_1, m_2 in the case when τ is a high value.

In a private communication, professors D. M. Lila and A. A. Martynyuk pointed out that on this model the physical range for the dimensionless field amplitude E is $[0, 1]$. In this sense, the problem would amount to find a non-positive periodic solution $x(t) \leq 0$ of Eq. (3). In view of (8), the necessary condition can be fixed more accurately as $\tilde{g}_{th} < A \leq 2\tilde{g}_{th}$. The question if this condition is also sufficient for existence of a non-positive periodic solution of Eq. (3) is an interesting open problem. By using Lemma 2, the more conservative estimate $\tilde{g}_{th} < A \leq \tilde{g}_{th}(1 + e^{-AT\sqrt{3/2}})$ can be given as a sufficient condition. The author warmly thanks professors D. M. Lila and A. A. Martynyuk for this remark.

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