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ASYMPTOTIC NONLINEAR MULTIMODAL MODELING OF LIQUID SLOSHING IN AN UPRIGHT CIRCULAR CYLINDRICAL TANK. PART 1: MODAL EQUATIONS

АСИМПТОТИЧНЕ НЕЛІНІЙНЕ МУЛЬТИМОДАЛЬНЕ МОДЕЛЮВАННЯ ХЛЮПАННЯ РІДИНИ У ВЕРТИКАЛЬНОМУ КРУГОВОМУ ЦИЛІНДРИЧНОМУ РЕЗЕРВУАРІ. Ч. 1. МОДАЛЬНІ РІВНЯННЯ

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Combining the variational method by Lukovsky – Miles and the Narimanov – Moiseev asymptotics, a nonlinear modal system describing the resonant liquid sloshing in an upright circular cylindrical tank is derived. Sloshing occurs due to a small-amplitude periodic or almost-periodic excitation with the forcing frequency close to the lowest natural sloshing frequency. In contrast to the existing nonlinear modal systems based on the Narimanov – Moiseev asymptotic intermodal relationships, the derived modal equations: (i) contain all the necessary (infinitely many) generalized coordinates of the second and the third orders, (ii) include exclusively nonzero hydrodynamic coefficients for which (iii) rather simple computational formulas are found. As a consequence, the modal equations can be used in analytical studies of nonlinear sloshing phenomena that will be demonstrated in the forthcoming Part 2.

Комбінуючи варіаційний метод Луковського – Майлса та асимптотику Наріманова – Моісеєва, побудовано нелінійну модальну систему, що описує резонансні коливання рідини у вертикальному круговому циліндричному резервуарі. Коливання відбуваються завдяки періодичному чи майже періодичному збуренню з частотою, близькою до першої власної частоти. На відміну від існуючих нелінійних модальних систем, які базуються на асимптотичних співвідношеннях Наріманова – Моісеєва, побудовані модальні рівняння: (i) включають всі необхідні (нескінченну кількість) узагальнені координати другого та третього порядку, (ii) утримують виключно ненульові гідродинамічні коефіцієнти, для яких (iii) знайдено достатньо прості обчислювальні формули. Як наслідок, модальні рівняння можна використати в аналітичних дослідженнях нелінійних явищ, що буде продемонстровано в наступній частині 2.

1. Introduction. Accounting for liquid sloshing loads is of importance for designing the engineering constructions carrying a liquid cargo. Safety, reliability, stability, and control analysis of the liquid containing structures have been extensively studied in the context of aircraft and spacecraft applications, for cargo tanks of automotive vehicles, offshore platforms, and seismic analysis of the elevated water tanks. The studies require comprehensive quantitative and qualitative examining the coupled fluid-structure dynamics, its modeling and simulation on the real-time scale. Liquid sloshing response becomes most severe in resonance conditions when the carrying structure oscillates with a frequency close to the lowest natural sloshing frequency. Those resonant free-surface motions are strongly nonlinear and must be described by solving an evolution free-boundary problem in which both instant shapes of the free surface $\Sigma(t)$ and the velocity field in the liquid domain $Q(t)$ are the unknowns.

Under certain circumstances, one can distinguish *three* different approaches to solving the *nonlinear liquid sloshing* problem. The *first approach* is the Computational Fluid Dynamics (CFD). A broad variety of numerical methods exists which could be divided into two subclasses comprising potential flow, the Navier – Stokes method, and, sometimes, their hybrids typically based on the domain decomposition method [8, 39]. The CFD methods are universal, accurate and efficient, especially on the short-time scale when focus is on transient waves. Their drawback is that they are, generally speaking, computational time demanding, especially for three-dimensional problems. Furthermore, their applicability can be rather limited when the tasks consist in simulation and classification of the so-called steady-state wave regimes occurring on the long-time scale and, therefore, requiring long-time simulations with different initial scenarios.

The *second approach* is purely analytical. It is developed for studying the steady-state (periodic) solution expected for prescribed small-amplitude harmonic tank excitations. The analytical approach employs asymptotic methods which have been created by great mathematicians of the XIX century in the theory of nonlinear ocean waves [3]. An extension of these methods to nonlinear resonant sloshing in closed basins is often referred to the pioneering paper by Moiseev [32]. More mathematical details on constructing the steady-state asymptotic solution by solving a series of recurrence boundary value problems and deriving the so-called secularity condition (the necessary solvability condition) coupling the forcing frequency and the dominant response amplitude can be found in [5, 36, 37, 11, 12, 8]. The asymptotic steady-state solution technique changes with the mean liquid depth. For finite liquid depths, the Taylor expansion of the nonlinear free-surface conditions with respect to the mean (unperturbed) free surface leads to cubic algebraic secularity equations and yields the so-called third-order Moiseev asymptotics causing the dominant response amplitude be of the order $O(\epsilon^{1/3})$ where ϵ is the nondimensional forcing amplitude. The asymptotic solution methods are generally not applicable to transient waves, and for modeling the fluid-structure interaction. Furthermore, the asymptotic steady-state solution is only valid in a matching forcing-frequency range and for a relatively small forcing amplitude. Forcing frequencies away from this range and increasing the forcing amplitude can lead to the so-called internal (secondary, combinatory) resonance and, thereby, cause a failure of the Moiseev intermodal asymptotic relationships (Moiseev's asymptotics).

The *third approach* is associated with nonlinear multimodal methods whose application assumes an ideal liquid with irrotational flow and no overturning and breaking waves allowed. In this paper, we follow this approach to derive an infinite-dimensional system of nonlinear asymptotic-type modal equations for sloshing in an upright circular cylindrical tank by combining the variational multimodal method by Lukovsky – Miles [19, 20, 29] and the Narimanov – Moiseev intermodal asymptotic relationships [34, 35, 32] which can follow from the second approach or, simply, be postulated. Distinguished details for such a combined variational-and-asymptotic version of multimodal methods, its difference from others are outlined in reviews [27, 11]. Readers interested in employing other versions of multimodal methods for liquid sloshing in an upright cylindrical tank are referred to [34, 4, 35, 20, 10] (Narimanov's modal-type perturbation method), [33, 38, 15] (fully-nonlinear [non-asymptotic] multimodal [Perko-type] method), [16, 17, 18] (combining the Lagrange variational principle and perturbation method), [20, 7, 8] (combining the Bateman – Luke variational principle and perturbation method), and references therein.

Sloshing of an ideal liquid with irrotational flow introduces a nonlinear free boundary problem with the two unknowns that are the instant free-surface shapes and the velocity potential.

According to the multimodal method *concept* for liquid sloshing in tanks with upright walls, the instant free-surface shapes should be defined by the Fourier-type solution with unknown time-dependent coefficients $q_i(t)$ (furthermore, *generalized coordinates*) in the front of $f_i(y,z) = \varphi_i(0,y,z)$, i.e.,

$$x = f(y,z,t) = \sum_i q_i(t) f_i(y,z), \tag{1}$$

where $\varphi_i(x,y,z)$ are the so-called natural sloshing modes. Analogous Fourier-type solution involving $\varphi_i(x,y,z)$ is used for the velocity potential. Even though there exist different versions of nonlinear multimodal methods, all of them are developed to transform the original problem to an infinite-dimensional system of nonlinear ordinary differential (modal) equations coupling the generalized coordinates q_i. However, since derivation of nonlinear multimodal equations is a difficult mathematical task, each a version proposes a proper analytical way pursuing the modal equations of desirable structure.

Except for the Perko-type methods, the derivations require a postulation of asymptotic relationships between the generalized coordinates q_i assuming a small set of dominant generalized coordinates. Neglecting the nonlinear terms in q_i which have the order higher than the forcing input signal associated with the highest-order term, $O(\epsilon)$, leads to the so-called *asymptotic nonlinear modal equations*. Using the asymptotic modal equations helps to avoid physically-unrealistic higher harmonics which give negligible contribution to liquid response, but may cause the stiffness of the differential (modal) equations as it was observed for the Perko-type simulations [15].

The Narimanov – Moiseev asymptotics [34, 35, 32] is the most-often accepted asymptotic relationships used for deriving the asymptotic nonlinear modal systems. They follow from Moiseev's asymptotic solution (the second approach) or can be postulated as it has been done by Narimanov in his classical works [34, 35]. Adopting the asymptotic relationships reduces the problem to calculation of the *non-zero* hydrodynamic coefficients at the polynomial-type quantities in the asymptotic modal equations. Usually, the number of the nonzero coefficients is quite limited. Bearing in mind *analytical studies* based on the asymptotic nonlinear modal equations, i.e., considering the nonlinear sloshing as an object of either *applied mathematics or theoretical mechanics*, strongly requires to exclude the zeros from the modal equations as well as to provide simple and compact formulas for the nonzero hydrodynamic coefficients. Of course, using the asymptotic nonlinear multimodal equations as a computational tool [18, 7], i.e., considering the multimodal method as a CFD approach, does not need the analytical extraction of the zeros.

For rectangular cross-section, the Narimanov – Moiseev asymptotic relationships lead, due to trigonometric relations between the natural sloshing modes f_i, to a nine-dimensional nonlinear asymptotic modal system. This system is explicitly derived and analytically studied in [6, 8]. Other cylindrical tank shapes yield, generally speaking, infinite-dimensional asymptotic multimodal systems. The latter is also true for the case of circular cross-section. The literature presents various analytically-given finite-dimensional asymptotic modal systems [20 – 22] but these systems couple only a few of second and third-order generalized coordinates. To the best authors' knowledge, the present paper firstly derives *in an analytical form* the infinite-dimensional Narimanov – Moiseev' asymptotic modal system for the circle-based tank providing that the modal equations (i) contain all the necessary generalized coordinates of the

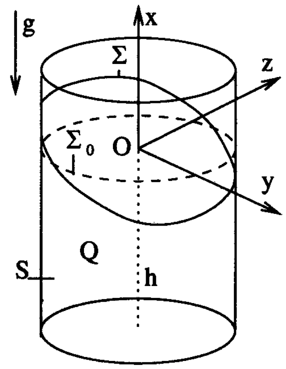

Fig. 1. Sketch of an upright circular cylindrical tank and the adopted nomenclature. The geometrical and physical characteristics are scaled in our analysis by the dimensional tank radius R_0 so that, e.g., h in the figure is the ratio between the mean liquid depth and R_0.

second and third order following from the Narimanov – Moiseev asymptotic intermodal relationships, (ii) include exclusively nonzero hydrodynamic coefficients for which (iii) rather simple computational formulas are found. In the forthcoming Part 2, we will present analytical studies of the nonlinear resonant sloshing based on the derived modal system and compare the analytical results with experiments.

2. Statement of the problem. ***2.1. Free boundary problem.*** An upright circular cylindrical tank is considered which is partly filled with an inviscid incompressible liquid with irrotational flow. Fig. 1 introduces the basic notations. No overturning waves are assumed. The time-dependent liquid domain $Q(t)$ is bounded by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$. The mean liquid depth is equal to h. Henceforth in all the mathematical expressions, we suggest that the liquid characteristics, including h and the gravity acceleration g, are scaled by the tank radius R_0 so that, everywhere in our forthcoming analysis the theoretical radius of the tank is nondimensional and equal to 1.

The liquid motions are considered in the tank-fixed coordinate system $Oxyz$ whose origin is situated in the center of the mean free-surface Σ_0. The Ox-axis is superposed with the tank symmetry axis. For brevity, we concentrate on the case when the tank moves translatory with the velocity $\mathbf{v}_0(t)$ relative to an absolute Earth-fixed coordinate system $Ox'y'z'$. Small-magnitude angular forcing terms can also be accounted for by assuming that these terms are of the highest order in the Narimanov – Moiseev asymptotic ordering. The latter procedure is extensively discussed in [8].

The absolute velocity potential $\Phi(x, y, z, t)$, and the free surface $\Sigma(t)$ are the two unknowns which should be found from the following nonlinear free-boundary problem

$$\nabla^2\Phi = 0, \quad \mathbf{r} \in Q(t), \tag{2}$$

$$\frac{\partial\Phi}{\partial\nu} = \mathbf{v}_0 \cdot \nu + \frac{f_t}{\sqrt{1 + |\nabla f|^2}}, \quad \mathbf{r} \in \Sigma, \tag{3}$$

$$\frac{\partial\Phi}{\partial\nu} = \mathbf{v}_0 \cdot \nu, \quad \mathbf{r} \in S(t), \tag{4}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\,|\nabla\Phi|^2 - \nabla\Phi\cdot\mathbf{v}_0 + U = 0, \quad \mathbf{r}\in\Sigma(t). \tag{5}$$

Here ν is the outer normal vector, $U = (\mathbf{g}\cdot\mathbf{r})$ is the gravity potential with $\mathbf{r} = (x, y, z)$, $\mathbf{g} = (-g, 0, 0)$ is the gravity acceleration vector, and $x = f(y, z, t)$ is the free-surface equation.

For the free-boundary problem (2), typical initial conditions (at $t = 0$) define the initial liquid shape and velocity field and take the form

$$f(y, z, 0) = \xi_0(y, z), \quad \Phi(x, y, z, 0) = \Phi_0(x, y, z), \quad \mathbf{r}\in Q(0). \tag{6}$$

2.2. Variational formulation. In 1976, Miles [29] and Lukovsky [19] independently proposed to use the Bateman – Luke variational principle for derivation of nonlinear modal systems. History of the Bateman – Luke principle starts from 1908, when R. Hargneaves [13] has noted that the pressure integral can play the role of the Lagrangian in variational formulations of diverse hydrodynamic problems. The canonical formulation of this principle for an incompressible ideal liquid is given by Bateman [1]. Furthermore, this formulation was generalized by Luke [28] for ocean waves and by Lukovsky [20] for liquid sloshing in a tank performing arbitrary spatial motions. The Bateman – Luke principle for a compressible fluid can be found in [24, 25, 26, 2].

According to Lukovsky [20], the Bateman – Luke principle for (2) – (5) can be formulated as follows: *The free-boundary problem (2) – (5) is associated with the necessary extrema of the action*

$$W = \int\limits_{t_1}^{t_2} L\,dt, \tag{7}$$

where the Lagrangian L is defined by the pressure integral

$$L = \int\limits_{Q(t)} (p - p_o)\,dQ = -\rho \int\limits_{Q(t)} \left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}\,|\nabla\Phi|^2 - \nabla\Phi\cdot\mathbf{v}_0 + U\right] dQ \tag{8}$$

and trial functions satisfy the conditions

$$\delta\Phi(x, y, z, t_1) = \delta\Phi(x, y, z, t_2) = \delta f(y, z, t_1) = \delta f(y, z, t_2) = 0. \tag{9}$$

3. Nonlinear multimodal modeling. The nonlinear multimodal modeling is based on the Fourier-type solution (1) in which $q_i(t)$ are treated as *generalized coordinates* of the considered hydromechanical system. Here $f_i(x, y)$ is a complete orthogonal system of functions satisfying the volume conservation condition $\int_{\Sigma_0} f_i(x, y)\,dx\,dy = 0$. In addition, one should introduce the Fourier-type representation of the velocity potential

$$\Phi(x, y, z, t) = \mathbf{v}_0\cdot\mathbf{r} + \sum_n Q_n(t)\,\varphi_n(x, y, z), \tag{10}$$

where the complete set of harmonic functions $\varphi_n(x, y, z)$ satisfies both the Laplace equation in the whole tank domain and the zero Neumann boundary conditions on the wetted body surface.

Normally, φ_n and $f_n(x,y) = \varphi_n(x,y,0)$ are the eigenfunctions (natural sloshing modes) of the spectral boundary problem

$$\nabla^2\varphi_n = 0, \quad \vec{r} \in Q_0; \quad \frac{\partial \varphi_n}{\partial \nu} = \kappa_n \varphi_n, \quad \vec{r} \in \Sigma_0; \quad \frac{\partial \varphi_n}{\partial \nu} = 0, \quad \vec{r} \in S_0, \tag{11}$$

where Q_0 is the mean liquid domain and S_0 is the mean wetted tank surface. The natural sloshing frequencies are defined by the eigenvalues κ_n via $\sigma_n = \sqrt{g\kappa_n}$.

The aim of the multimodal modeling is to derive a system of ordinary differential equations (modal equations) with respect to generalized coordinates $q_i(t)$. There are different analytical schemes (multimodal methods) how to do that; these are shortly outlined in Introduction. According to [19, 20, 29], the derivation can employ the Bateman – Luke principle instead of the free-boundary problem (2).

3.1. Lukovsky – Miles' variational method. Lukovsky [20] showed how to use the Bateman – Luke principle for deriving the nonlinear modal equations coupling $q_i(t)$ and $Q_n(t)$. The result for translatory tank excitations is the following infinite-dimensional system of nonlinear ordinary differential equations:

$$\sum_i \frac{\partial A_n}{\partial q_i}\dot{q}_i - \sum_k A_{nk}Q_k = 0, \quad n = 1, 2, \ldots, \tag{12}$$

$$\sum_n \frac{\partial A_n}{\partial q_i}\dot{Q}_n + \frac{1}{2}\sum_{n,k}\frac{\partial A_{nk}}{\partial q_i}Q_n Q_k + \sum_{j=1}^{3}(\dot{v}_{Oj} - g_j)\frac{\partial l_j}{\partial q_i} = 0, \quad i = 1, 2, \ldots, \tag{13}$$

where

$$\frac{\partial l_1}{\partial q_i} = \int\limits_{\Sigma_0} f_i^2\, dS\, q_i, \quad \frac{\partial l_2}{\partial q_i} = \int\limits_{\Sigma_0} y f_i\, dS, \quad \frac{\partial l_3}{\partial q_i} = \int\limits_{\Sigma_0} z f_i\, dS, \tag{14}$$

$\mathbf{g} = (g_1, g_2, g_3) = (-g, 0, 0)$, and

$$A_n = \int\limits_{Q(t)} \varphi_n\, dQ, \quad A_{nk} = \int\limits_{Q(t)} \nabla\varphi_n \cdot \nabla\varphi_k\, dQ. \tag{15}$$

The nonlinear modal equations (12), (13) are a full analogy of the original free-boundary problem. Direct simulations by the modal equations (12), (13) imply the so-called Perko's numerical method (see Introduction). Lukovsky & Timokha [27] pointed out that these simulations can be stiff for resonant sloshing and, therefore, a certain numerical time-integration becomes numerically unstable. This physically-unrealistic stiffness is caused by amplification of higher harmonics which, in the reality, are highly damped due to different dissipative mechanisms. An alternative is to introduce asymptotic relationship between generalized coordinates and, thereby, exclude ("filter") the unrealistically high harmonics.

3.2. Narimanov – Moiseev' asymptotic intermodal relationships for circular-base tank. For the circular-base tank, the modal solution (1) can be rewritten in the cylindrical coordinate system as follows:

$$x = f(\xi, \eta, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(r_{m,n}(t)\sin(m\eta) + p_{m,n}(t)\cos(m\eta))f_{mn}(\xi), \tag{16}$$

where $f_{mn} = \dfrac{J_m(k_{m,n}\xi)}{J_m(k_{m,n})}$ ($J_m(\cdot)$ is the Bessel function) and $J'_m(k_{m,n}) = 0$. The zeros of the last equation defines the eigenvalues $\kappa_{m,n}$ and the natural sloshing frequencies $\sigma_{m,n}$ by the formulas

$$\kappa_{m,n} = k_{m,n}\tanh(k_{m,n}h) \quad \text{and} \quad \sigma^2_{m,n} = g\kappa_{m,n}. \tag{17}$$

The generalized coordinates q_i as well as $r_{m,n}$ and $p_{m,n}$ are nondimensional (scaled by the tank radius) and one can introduce asymptotic relations between them. When the forcing frequency σ is close to the lowest natural frequency $\sigma_{1,1}$ associated with the two generalized coordinates $p_{1,1}(t)$ and $r_{1,1}(t)$, the Narimanov – Moiseev asymptotics [32, 20, 30, 31, 27] requires the asymptotic relation

$$p_{1,1} \sim r_{1,1} = O(\epsilon^{1/3}), \tag{18}$$

where $\epsilon \ll 1$ implies the nondimensional forcing magnitude.

Postulating (18) and using the trigonometric algebra with respect to the angular coordinate η, one can establish the second- and third-order generalized coordinates

$$\begin{gathered} p_{0,n} \sim p_{2,n} \sim r_{2,n} = O(\epsilon^{2/3}); \quad p_{3,n} \sim r_{3,n} = O(\epsilon), \quad n = 1, 2, \ldots, \\ p_{1,m} \sim r_{1,m} = O(\epsilon), \quad m = 2, 3, \ldots. \end{gathered} \tag{19}$$

Remaining generalized coordinates are of the order $o(\epsilon)$ and can be neglected in our nonlinear asymptotic multimodal analysis.

4. Nonlinear asymptotic multimodal equations. The most general analytical scheme for combining the Lukovsky – Miles variational method and the Narimanov – Moiseev asymptotics is described in [27]. Accounting for (18) – (19), the scheme suggests the following steps:

1. Using the Taylor expansion, one should find polynomial expressions (in terms of nondimensional generalized coordinates q_i) for $\partial A_n/\partial q_k$ and A_{nk} keeping up to the $O(\epsilon^{2/3})$-order and $\partial A_{nk}/\partial q_i$ keeping the $O(\epsilon^{1/3})$-terms.

2. We should find the asymptotic solution $Q_i = F(q_k, \dot{q}_k)$ from modal equations (12) by substituting previously-found asymptotic expressions for $\partial A_n/\partial q_k$ and A_{nk}. This solution should neglect the $o(\epsilon)$-terms.

3. We should substitute expressions $Q_i = F(q_k, \dot{q}_k)$ from the previous step into modal equations (13) and keep up to the $O(\epsilon)$-terms. This will give the desirable asymptotic modal equations.

The scheme was *fully* realized only for upright cylindrical tanks of rectangular shape. By generalizing [20], the paper [23] showed that the scheme can also be applied for a circular cylindrical tank. It is implemented in the present paper to obtain the required *analytically*-given asymptotic nonlinear modal equations.

Implementing the analytical scheme with $3N$ ($N \to \infty$) generalized coordinates of the second order and $4N$ generalized coordinates of the third order leads to the nonlinear asymptotic modal equations which include the following two differential equations for the lowest-order generalized coordinates $p_{1,1}$ and $r_{1,1}$:

$$\mu_{1,1}\left[\ddot{p}_{1,1}+\sigma_{1,1}^{2}p_{1,1}\right]+p_{1,1}\sum_{n=1}^{N}d_{0,n}^{(2)}\ddot{p}_{0,n}+\sum_{n=1}^{N}d_{0,n}^{(3)}\left(\ddot{p}_{1,1}p_{0,n}+\dot{p}_{1,1}\dot{p}_{0,n}\right)+$$
$$+d_{1}\left(p_{1,1}^{2}\ddot{p}_{1,1}+p_{1,1}\dot{p}_{1,1}^{2}+r_{1,1}p_{1,1}\ddot{r}_{1,1}+p_{1,1}\dot{r}_{1,1}^{2}\right)+$$
$$+d_{2}\left(r_{1,1}^{2}\ddot{p}_{1,1}+2r_{1,1}\dot{r}_{1,1}\dot{p}_{1,1}-r_{1,1}p_{1,1}\ddot{r}_{1,1}-2p_{1,1}\dot{r}_{1,1}^{2}\right)+$$
$$+\sum_{n=1}^{N}d_{2,n}^{(3)}\left(\ddot{p}_{1,1}p_{2,n}+\ddot{r}_{1,1}r_{2,n}+\dot{p}_{1,1}\dot{p}_{2,n}+\dot{r}_{1,1}\dot{r}_{2,n}\right)+$$
$$+\sum_{n=1}^{N}d_{2,n}^{(2)}\left(p_{1,1}\ddot{p}_{2,n}+r_{1,1}\ddot{r}_{2,n}\right)=-\frac{\mu_{1,1}\kappa_{1,1}}{k_{1,1}^{2}-1}\dot{v}_{01}, \tag{20a}$$

$$\mu_{1,1}\left[\ddot{r}_{1,1}+\sigma_{1,1}^{2}r_{1,1}\right]+r_{1,1}\sum_{n=1}^{N}d_{0,n}^{(2)}\ddot{p}_{0,n}+\sum_{n=1}^{N}d_{0,n}^{(3)}\left(\ddot{r}_{1,1}p_{0,n}+\dot{r}_{1,1}\dot{p}_{0,n}\right)+$$
$$+d_{1}\left(r_{1,1}^{2}\ddot{r}_{1,1}+r_{1,1}\dot{r}_{1,1}^{2}+r_{1,1}p_{1,1}\ddot{p}_{1,1}+r_{1,1}\dot{p}_{1,1}^{2}\right)+$$
$$+d_{2}\left(p_{1,1}^{2}\ddot{r}_{1,1}+2p_{1,1}\dot{r}_{1,1}\dot{p}_{1,1}r_{1,1}p_{1,1}\ddot{p}_{1,1}-2r_{1,1}\dot{p}_{1,1}^{2}\right)+$$
$$+\sum_{n=1}^{N}d_{2,n}^{(3)}\left(\ddot{p}_{1,1}r_{2,n}-\ddot{r}_{1,1}p_{2,n}+\dot{p}_{1,1}\dot{r}_{2,n}-\dot{r}_{1,1}\dot{p}_{2,n}\right)+$$
$$+\sum_{n=1}^{N}d_{2,n}^{(2)}\left(p_{1,1}\ddot{r}_{2,n}-r_{1,1}\ddot{p}_{2,n}\right)=-\frac{\mu_{1,1}\kappa_{1,1}}{k_{1,1}^{2}-1}\dot{v}_{02}. \tag{20b}$$

These equations include both the lowest- and second-order generalized coordinates, but the third-order generalized coordinates are absent here. Notations for $k_{m,n}$ (the roots of the equation $J_m'(k_{m,n})=0$), $\kappa_{m,n}$ (see eq. (17)), $\sigma_{m,n}$ (natural sloshing frequency) as well as the translatory velocity components $\dot{v}_{01}(t)$ and $\dot{v}_{02}(t)$ were explained before. The nondimensional hydrodynamic coefficients at the nonlinear terms are defined by the end of this section.

The differential equations for finding the second-order generalized coordinates $p_{0,n}$, $p_{2,n}$ and $r_{2,n}$ take the form

$$2\mu_{0,n}\left[\ddot{p}_{0,n}+\sigma_{0,n}^{2}p_{0,n}\right]+d_{0,n}^{(1)}\left(\dot{p}_{1,1}^{2}+\dot{r}_{1,1}^{2}\right)+d_{0,n}^{(2)}\left(\ddot{p}_{1,1}p_{1,1}+\ddot{r}_{1,1}r_{1,1}\right)=0, \tag{21a}$$

$$\mu_{2,n}\left[\ddot{p}_{2,n}+\sigma_{2,n}^{2}p_{2,n}\right]+d_{2,n}^{(1)}\left(\dot{p}_{1,1}^{2}-\dot{r}_{1,1}^{2}\right)+d_{2,n}^{(2)}\left(\ddot{p}_{1,1}p_{1,1}-\ddot{r}_{1,1}r_{1,1}\right)=0, \tag{21b}$$

$$\mu_{2,n}\left[\ddot{r}_{2,n}+\sigma_{2,n}^{2}r_{2,n}\right]+2d_{2,n}^{(1)}\dot{r}_{1,1}\dot{p}_{1,1}+d_{2,n}^{(2)}\left(\ddot{p}_{1,1}r_{1,1}+\ddot{r}_{1,1}p_{1,1}\right)=0. \tag{21c}$$

Here $n=1,\ldots,N$, i.e., there is $3N$ ordinary differential equations for these generalized coordinates. Note that equations (21) contain $p_{1,1}$ and $r_{1,1}$ defined by (21) and, therefore, one can say that the first and second-order generalized coordinates are nonlinearly coupled by our modal equations. However, the third-order generalized coordinates $p_{3,n}$ and $r_{3,n}$ are not presented in (21). Equations for these generalized coordinates take the form

$$\mu_{3,n}\left[\ddot r_{3,n}+\sigma^2_{3,n}r_{3,n}\right]+d_3\left(r_{1,1}\dot p^2_{1,1}+2p_{1,1}\dot p_{1,1}\dot r_{1,1}-r_{1,1}\dot r^2_{1,1}\right)+$$
$$+d_4\left(p^2_{1,1}\ddot r_{1,1}+2r_{1,1}p_{1,1}\ddot p_{1,1}-r^2_{1,1}\ddot r_{1,1}\right)+\sum_{n=1}^{N}d^{(1)}_{3,n}\left(\dot p_{1,1}\dot r_{2,n}+\dot r_{1,1}\dot p_{2,n}\right)+$$
$$+\sum_{n=1}^{N}d^{(2)}_{3,n}\left(p_{1,1}\ddot r_{2,n}+r_{1,1}\ddot p_{2,n}\right)+\sum_{n=1}^{N}d^{(3)}_{3,n}\left(\ddot p_{1,1}r_{2,n}+\ddot r_{1,1}p_{2,n}\right)=0, \quad (22a)$$

$$\mu_{3,n}\left[\ddot p_{3,n}+\sigma^2_{3,n}p_{3,n}\right]+d_3\left(p_{1,1}\dot p^2_{1,1}-2r_{1,1}\dot p_{1,1}\dot r_{1,1}-p_{1,1}\dot r^2_{1,1}\right)+$$
$$+d_4\left(p^2_{1,1}\ddot p_{1,1}-2p_{1,1}r_{1,1}\ddot r_{1,1}-r^2_{1,1}\ddot p_{1,1}\right)+\sum_{n=1}^{N}d^{(1)}_{3,n}\left(\dot p_{1,1}\dot p_{2,n}-\dot r_{1,1}\dot r_{2,n}\right)+$$
$$+\sum_{n=1}^{N}d^{(2)}_{3,n}\left(p_{1,1}\ddot p_{2,n}-r_{1,1}\ddot r_{2,n}\right)+\sum_{n=1}^{N}d^{(3)}_{3,n}\left(\ddot p_{1,1}p_{2,n}-\ddot r_{1,1}r_{2,n}\right)=0, \quad (22b)$$

$$\mu_{1,n}\left[\ddot r_{1,n}+\sigma^2_{1,n}r_{1,n}\right]+d_5\left(\ddot r_{1,1}r^2_{1,1}+r_{1,1}p_{1,1}\ddot p_{1,1}\right)+$$
$$+d_6\left(r_{1,1}\dot r^2_{1,1}+r_{1,1}\dot p^2_{1,1}\right)+d_7\left(\ddot r_{1,1}p^2_{1,1}-r_{1,1}p_{1,1}\ddot p_{1,1}\right)+$$
$$+d_8\left(\dot r_{1,1}\dot p_{1,1}p_{1,1}-r_{1,1}\dot p^2_{1,1}\right)+\sum_{n=1}^{N}d^{(1)}_{4,n}\left(\dot p_{1,1}\dot r_{2,n}-\dot r_{1,1}\dot p_{2,n}\right)+$$
$$+\sum_{n=1}^{N}d^{(2)}_{4,n}\left(p_{1,1}\ddot r_{2,n}-r_{1,1}\ddot p_{2,n}\right)+\sum_{n=1}^{N}d^{(3)}_{4,n}\left(\ddot p_{1,1}r_{2,n}-\ddot r_{1,1}p_{2,n}\right)+$$
$$+\dot r_{1,1}\sum_{n=1}^{N}d^{(1)}_{5,n}\dot p_{0,n}+r_{1,1}\sum_{n=1}^{N}d^{(2)}_{5,n}\ddot p_{0,n}+\ddot r_{1,1}\sum_{n=1}^{N}d^{(3)}_{5,n}p_{0,n}=-\frac{\mu_{1,n}\kappa_{1,n}}{k^2_{1,n}-1}\dot v_{02}, \quad (22c)$$

$$
\begin{aligned}
\mu_{1,n}\left[\ddot p_{1,n}+\sigma_{1,n}^2 p_{1,n}\right] &+ d_5\left(\ddot p_{1,1}p_{1,1}^2+r_{1,1}p_{1,1}\ddot r_{1,1}\right)+\\
&+d_6\left(p_{1,1}\dot p_{1,1}^2+p_{1,1}\dot r_{1,1}^2\right)+d_7\left(\ddot p_{1,1}r_{1,1}^2-r_{1,1}p_{1,1}\ddot r_{1,1}\right)+\\
&+d_8\left(\dot r_{1,1}\dot p_{1,1}p_{1,1}-p_{1,1}\dot r_{1,1}^2\right)+\sum_{n=1}^{N}d_{4,n}^{(1)}\left(\dot p_{1,1}\dot p_{2,n}+\dot r_{1,1}\dot r_{2,n}\right)+\\
&+\sum_{n=1}^{N}d_{4,n}^{(2)}\left(r_{1,1}\ddot r_{2,n}+p_{1,1}\ddot p_{2,n}\right)+\sum_{n=1}^{N}d_{4,n}^{(3)}\left(\ddot p_{1,1}p_{2,n}+\ddot r_{1,1}r_{2,n}\right)+\\
&+\dot p_{1,1}\sum_{n=1}^{N}d_{5,n}^{(1)}\dot p_{0,n}+p_{1,1}\sum_{n=1}^{N}d_{5,n}^{(2)}\ddot p_{0,n}+\ddot p_{1,1}\sum_{n=1}^{N}d_{5,n}^{(3)}p_{0,n}=-\frac{\mu_{1,n}\kappa_{1,n}}{k_{1,n}^2-1}\dot v_{01},
\end{aligned}
\tag{22d}
$$

where $n=1,\ldots,N$. Equations (22) are linear in $p_{3,n}$ and $r_{3,n}$ and contain nonlinear quantities in terms of the first- and second-order generalized coordinates.

The most *important result* of the present paper is that *nonzero* hydrodynamic coefficients in (20) – (22) can be effectively calculated by the following quite simple formulas:

$$
d_{0,n}^{(1)}=d_{0,n}^{(2)}-\frac{d_{0,n}^{(3)}}{2},\quad d_{0,n}^{(2)}=\frac{\pi}{2}\left[2-\frac{k_{0,n}^2}{\kappa_{0,n}\kappa_{1,1}}\right]j_{(0,n)(1,1)^2},
$$

$$
d_{0,n}^{(3)}=\pi\left[j_{(0,n)(1,1)^2}-\frac{1}{\kappa_{1,1}^2}\left(j_{(0,n)}^{(1,1)^2}+i_{(0,n)(1,1)^2}\right)\right],\quad d_{2,n}^{(1)}=d_{2,n}^{(2)}-\frac{d_{2,n}^{(3)}}{2},
$$

$$
d_{2,n}^{(2)}=\frac{\pi}{2}\left[j_{(2,n)(1,1)^2}-\frac{1}{\kappa_{2,n}\kappa_{1,1}}\left(j_{(1,1)}^{(2,n)(1,1)}+2i_{(2,n)(1,1)^2}\right)\right],
$$

$$
d_{2,n}^{(3)}=\frac{\pi}{2}\left[j_{(2,n)(1,1)^2}-\frac{1}{\kappa_{1,1}^2}\left(j_{(2,n)}^{(1,1)^2}-i_{(2,n)(1,1)^2}\right)\right],
$$

$$
d_1=\frac{\pi}{2\kappa_{1,1}}\left[\frac{k_{0,1}^4\left(j_{(0,1)(1,1)^2}\right)^2}{4\kappa_{01}\kappa_{1,1}}\frac{1}{j_{(0,1)^2}}+i_{(1,1)^4}-j_{(1,1)^2}^{(1,1)^2}\right]+d_2,
$$

$$
d_2=\frac{\pi}{4\kappa_{1,1}}\left[\frac{\left(j_{(1,1)}^{(1,1)(2,1)}+2i_{(2,1)(1,1)^2}\right)^2}{\kappa_{1,1}\kappa_{2,1}}\frac{1}{j_{(2,1)^2}}-3i_{(1,1)^4}-j_{(1,1)^2}^{(1,1)^2}\right],
$$

$$
\begin{aligned}
d_3=\frac{\pi}{4\kappa_{1,1}}&\frac{\left(j_{(1,1)}^{(1,1)(2,1)}+2i_{(2,1)(1,1)^2}\right)}{\kappa_{1,1}\kappa_{2,1}}\frac{\left(2i_{(1,1)(2,1)(3,1)}-j_{(3,1)}^{(1,1)(2,1)}\right)}{j_{(2,1)^2}}+\\
&+\frac{\pi}{4\kappa_{1,1}}\left[j_{(1,1)(3,1)}^{(1,1)^2}-i_{(1,1)^3(3,1)}\right]+2d_4,
\end{aligned}
$$

$$d_4 = \frac{\pi}{4\kappa_{1,1}} \frac{\left(j^{(1,1)(2,1)}_{(1,1)} + 2i_{(2,1)(1,1)^2}\right)}{\kappa_{2,1}\kappa_{3,1}} \frac{\left(6i_{(1,1)(2,1)(3,1)} + j^{(2,1)(3,1)}_{(1,1)}\right)}{j_{(2,1)^2}} -$$

$$- \frac{\pi}{4\kappa_{1,1}} \left[\frac{(\kappa_{1,1} + \kappa_{3,1})}{2\kappa_{3,1}} \left(3i_{(1,1)^3(3,1)} + j^{(1,1)(3,1)}_{(1,1)^2}\right)\right],$$

$$d^{(1)}_{3,n} = d^{(2)}_{3,n} + d^{(3)}_{3,n} - \frac{\pi}{2}\left[j_{(1,1)(2,n)(3,1)} - \frac{j^{(1,1)(2,n)}_{(3,1)} - 2i_{(1,1)(2,n)(3,1)}}{\kappa_{1,1}\kappa_{2,n}}\right],$$

$$d^{(2)}_{3,n} = \frac{\pi}{2}\left[j_{(1,1)(2,n)(3,1)} - \frac{j^{(2,n)(3,1)}_{(1,1)} + 6i_{(1,1)(2,n)(3,1)}}{\kappa_{2,n}\kappa_{3,1}}\right],$$

$$d^{(3)}_{3,n} = \frac{\pi}{2}\left[j_{(1,1)(2,n)(3,1)} - \frac{j^{(1,1)(3,1)}_{(2,n)} + 3i_{(1,1)(2,n)(3,1)}}{\kappa_{1,1}\kappa_{3,1}}\right],$$

$$d^{(1)}_{4,n} = d^{(2)}_{4,n} - d^{(3)}_{3,n} - \frac{\pi}{2}\left[j_{(1,1)(2,n)(1,2)} - \frac{j^{(1,1)(2,n)}_{(1,2)} + 2i_{(1,1)(2,n)(1,2)}}{\kappa_{1,1}\kappa_{2,n}}\right],$$

$$d^{(2)}_{4,n} = \frac{\pi}{2}\left[j_{(1,1)(2,n)(1,2)} - \frac{j^{(2,n)(1,1)}_{(1,1)} + 2i_{(1,1)(2,n)(1,2)}}{\kappa_{2,n}\kappa_{1,2}}\right],$$

$$d^{(3)}_{4,n} = \frac{\pi}{2}\left[j_{(1,1)(2,n)(1,2)} - \frac{j^{(1,1)(1,2)}_{(2,n)} - i_{(1,1)(2,n)(1,2)}}{\kappa_{1,1}\kappa_{1,2}}\right],$$

$$d^{(1)}_{5,n} = d^{(2)}_{5,n} + d^{(3)}_{5,n} - \pi\left[j_{(0,n)(1,1)(1,2)} - \frac{j^{(0,n)(1,1)}_{(1,2)}}{\kappa_{1,1}\kappa_{0,n}}\right],$$

$$d^{(2)}_{5,n} = \pi\left[j_{(0,n)(1,1)(1,2)} - \frac{j^{(0,n)(1,2)}_{(1,1)}}{\kappa_{0,n}\kappa_{1,2}}\right],$$

$$d^{(3)}_{5,n} = \pi\left[j_{(0,n)(1,1)(1,2)} - \frac{j^{(1,1)(1,2)}_{(0,n)} + i_{(0,n)(1,1)(1,2)}}{\kappa_{1,1}\kappa_{1,2}}\right]$$

in which, by definition,

$$j^{(c,d)}_{(a,b)} = \int \xi \left(\prod f_{a,b}(k_{a,b}\xi)\right)\left(\prod \frac{d}{d\xi} f_{c,d}(k_{c,d}\xi)\right) d\xi,$$

$$i^{(c,d)}_{(a,b)} = \int \frac{1}{\xi} \left(\prod f_{a,b}(k_{a,b}\xi)\right)\left(\prod \frac{d}{d\xi} f_{c,d}(k_{c,d}\xi)\right) d\xi$$

and there are special indexing rules for i and j exemplified by the formula

$$j_{(0,2)(2,2)(1,1)}^{(1,2)(0,1)(1,2)} = j_{(0,2)(1,1)(2,2)}^{(0,1)(1,2)^2} =$$

$$= \int_0^1 \xi(f_{0,2}(k_{0,2}\xi)f_{1,1}(k_{1,1}\xi)f_{2,2}(k_{2,2}\xi)) \left(\frac{d}{d\xi} f_{0,1}(k_{0,1}\xi) \left(\frac{d}{d\xi} f_{1,2}(k_{1,2}\xi)\right)^2\right) d\xi.$$

Eqs. (22c) and (22d) contain coefficients d_5, d_6, d_7, and d_8 which are computed by the formulas:

$$d_5 = -\frac{0,51201}{h_{1,1}} - \frac{0,16879}{h_{1,2}} + \frac{0,50224}{h_{1,2}h_{1,1}h_{0,1}} + \frac{0,17969}{h_{1,2}h_{2,1}h_{1,1}},$$

$$d_6 = -\frac{1,34899}{h_{1,1}} - \frac{0,3376}{h_{1,2}} + \frac{1,00448}{h_{1,2}h_{1,1}h_{0,1}} + \frac{0,37908}{h_{1,1}^2 h_{0,1}} + \frac{0,35938}{h_{1,2}h_{2,1}h_{1,1}} + \frac{0,23782}{h_{1,1}^2 h_{2,1}},$$

$$d_7 = -\frac{0,11748}{h_{1,1}} - \frac{0,00307}{h_{1,2}} + \frac{0,17969}{h_{1,2}h_{2,1}h_{1,1}},$$

$$d_8 = -\frac{0,68799}{h_{1,1}} - \frac{0,17186}{h_{1,2}} + \frac{0,37908}{h_{1,2}^2 h_{0,1}} + \frac{0,35938}{h_{1,2}h_{2,1}h_{1,1}} + \frac{0,50224}{h_{1,2}h_{1,1}h_{0,1}},$$

where $h_{m,n} = \tanh(k_{m,n}h)$ depends on the nondimensional depth.

It may be important for applications that the modal equations (20) – (22) can be rewritten in the following matrix form:

$$Q(\vec{q})\ddot{\vec{q}} + C\vec{q} + \vec{\Psi}(\vec{q};\dot{\vec{q}}) = V, \tag{23}$$

where $\vec{q} = (q_{1,1}; q_{1,2}; \ldots; q_{1,n}; q_{2,1}; q_{2,2}; \ldots; q_{2,n}; \ldots; q_{7,1}; q_{7,2}; \ldots; q_{7,n})^T$.

5. Conclusions. Bearing in mind analytical studies of nonlinear resonant sloshing in an upright circular-base tank, the present paper analytically derives a system of nonlinear ordinary differential equations (modal system) facilitating an approximate modeling of sloshing phenomena. The derivation uses the Narimanov – Moiseev intermodal asymptotic relationships which cause for this tank shape an infinite number of the generalized coordinates coupled by the system. In contrast to the existing *analytically-given* modal equations, the derived system (i) contains all the necessary generalized coordinates, (ii) includes exclusively nonzero hydrodynamic coefficients for which (iii) rather simple computational formulas are found. A use of the modal equations in *analytical studies* of the nonlinear resonant sloshing will be demonstrated in the forthcoming Part 2.

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