UDC 517 . 9

ON N-th ORDER NONLINEAR ORDINARY RANDOM DIFFERENTIAL EQUATIONS

ПРО ЗВИЧАЙНІ НЕЛІНІЙНІ ВИПАДКОВІ ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ ПОРЯДКУ N

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In this paper, an existence result for a nonlinear n-th order ordinary random differential equation is proved under Carathéodory condition. Two existence results for extremal random solutions are also proved for Carathéodory as well as discontinuous cases of the nonlinearity involved in the equations. Our investigations are placed in the Banach space of continuous real-valued functions on closed and bounded intervals of real line together with the application of random version of Leray – Schauder principle.

Доведено результат про існування розв'язку нелінійного звичайного випадкового диференціального рівняння за виконання умови Каратеодорі. Наведено два результати про існування екстремальних випадкових розв'язків: у випадку виконання умови Каратеодорі та у випадку, коли нелінійність не є неперервною. Дослідження проведено в банаховому просторі неперервних дійснозначних функцій на замкнених і обмежених інтервалах дійсної осі з застосуванням випадкової версії принципу Лере – Шаудера.

1. Introduction. Let $\mathbb{R}$ denote the real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Let $C^1(J, \mathbb{R})$ denote the class of real-valued functions defined and continuously differentiable on J. Given a measurable space $(\Omega, \mathcal{A})$ and for a given measurable function $x : \Omega \to C^{(n-1)}(J, \mathbb{R})$, consider the initial value problem of n-th order ordinary random differential equations (in short RDE),

$$\begin{aligned} x^{(n)}(t,\omega) &= f(t, x(t,\omega), \omega) \quad \text{a.e.} \quad t \in J, \\ x^{(i)}(0,\omega) &= q_i(\omega), \quad i \in \{0, \ldots, n-1\}, \end{aligned} \tag{1.1}$$

for all $\omega \in \Omega$, where $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$, $q_i : \Omega \to \mathbb{R}$.

By a *random solution* of the RDE (1.1) we mean a measurable function $x : \Omega \to AC^{(n-1)}(J, \mathbb{R})$ that satisfies the equations in (1.1), where $AC^{(n-1)}(J, \mathbb{R})$ is the space of real-valued functions whose $(n-1)$-th exists and is absolutely continuously differentiable on J.

The RDE (1.1) is not new to the theory random differential equations. When the random parameter ω is absent, the RDE (1.1) reduces to the classical RDE of n-th order ordinary differential equations (in short ODE),

$$\begin{aligned} x^{(n)}(t) &= f(t, x(t)) \quad \text{a.e.} \quad t \in J, \\ x^{(i)}(0) &= q_i \in \mathbb{R}, \quad i \in \{0, \ldots, n-1\}, \end{aligned} \tag{1.2}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$.

The classical ODE (1.2) has been studied in the literature by several authors for different aspects of the solutions. See for example, Heikkilä and Lakshmikantham [1] and the references therein. In this paper, we discuss the RDE (1.1) for existence as well as for existence of the extremal solutions under suitable conditions of the nonlinearity f which thereby generalize several existence results of the RDE (1.2) proved in the above mentioned papers. Our analysis rely on the random versions of nonlinear alternative of Leray – Schauder type (see Dhage [2, 3]) and an algebraic random fixed point theorem of Dhage [2].

The rest of the paper is organized as follows. In Section 2 we give some preliminaries and definitions needed in the sequel. The main existence result is given in Section 3, while the results on extremal solutions is given in Section 4. Finally, in Section 5, an example is presented to illustrate the abstract results proved in Section 3.

2. Auxiliary results. Let E denote a Banach space with the norm $\|\cdot\|$ and let $Q : E \to E$. Then Q is called **compact** if $Q(E)$ is a relatively compact subset of E. Q is called **totally bounded** if $Q(B)$ is totally bounded subset of E for any bounded subset B of E. Q is called **completely continuous** if is continuous and totally bounded on E. Note that every compact operator is totally bounded, but the converse may not be true. However, both the notions coincide on bounded sets in the Banach space E.

We further assume that the Banach space E is separable, i.e., E has a countable dense subset and let β_E be the σ-algebra of Borel subsets of E. We say a mapping $x : \Omega \to E$ is measurable if for any $B \in \beta_E$,

$$x^{-1}(B) = \{\omega \in \Omega \mid x(\omega) \in B\} \in \mathcal{A}.$$

Similarly, a mapping $x : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has

$$x^{-1}(B) = \{(\omega, x) \in \Omega \times E \mid x(\omega, x) \in B\} \in \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E those defined in Ω and E respectively. The details of the different types of measurablity concepts of the functions appears in Himmelberg [4]. Note that a continuous map f from a Banach space E into itself is measurable, but the converse may not be true.

Let $Q : \Omega \times E \to E$ be a mapping. Then Q is called a random operator if $Q(\omega, x)$ is measurable in ω for all $x \in E$ and it expressed as $Q(\omega)x = Q(\omega, x)$. In this case we also say that $Q(\omega)$ is a random operator on E. A random operator $Q(\omega)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $Q(\omega, x)$ is continuous (resp. compact, totally bounded and completely continuous) in x for all $\omega \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [5]. The study of random operator equations and their solutions have been discussed in Bharucha-Reid [6] and Hans [7] which is further applied to different types of random equations such as random differential and random integral equations etc. See Itoh [5], Bharucha-Reid [8] and the references therein. In this paper, we employ the following random nonlinear alternative in proving the main result of this paper.

Theorem 2.1 (Dhage [2, 3]). *Let E be a separable Banach space E and let $Q : \Omega \times E \to E$ be a completely continuous random operator. Then, either*

(i) the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to E$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$, or

(ii) the set $\mathcal{E} = \{x : \Omega \to E \quad \text{is measurable} \quad | \lambda(\omega)Q(\omega)x = x\}$ is unbounded for some measurable $\lambda : \Omega \to \mathbb{R}$ with $0 < \lambda(\omega) < 1$ on Ω.

An immediate corollary to above theorem in applicable form is the following.

Corollary 2.1. *Let E be a separable Banach space E and let $Q : \Omega \times E \to E$ be a completely continuous random operator. Then, either*

(i) the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to E$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$, or

(ii) the set $\mathcal{E} = \{x : \Omega \to E \quad \text{is measurable} \quad | \lambda Q(\omega)x = x\}$ is unbounded for all $\omega \in \Omega$ satisfying $0 < \lambda < 1$.

The following theorem is often used in the study of nonlinear discontinuous random differential equations. We also need this result in the subsequent part of this paper.

Theorem 2.2 (Carathéodory). *Let $Q : \Omega \times E \to E$ be a mapping such that $Q(\cdot, x)$ is measurable for all $x \in E$ and $Q(\omega, \cdot)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \mapsto Q(\omega, x)$ is jointly measurable.*

The following lemma is useful in the study of n-th order initial value problems of ordinary random differential equations via fixed point techniques.

Lemma 2.1. *For any function $h : J \to \in L^1(J, \mathbb{R})$, a function $x : J \to C^{(n-1)}(J, \mathbb{R})$ is a solution to the differential equation*

$$\begin{gathered} x^{(n)}(t) = h(t) \quad \text{a.e.} \quad t \in J, \\ x^{(i)}(0) = q_i \in \mathbb{R}, \end{gathered} \tag{2.1}$$

if and only if it is a solution of the integral equation

$$x(t) = \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, h(s)\, ds. \tag{2.2}$$

3. Existence result. We seek the random solutions of RDE (1.1) in the Banach space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We equip the space $C(J, \mathbb{R})$ with the supremum norm $\| \cdot \|$ defined by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

It is known that the Banach space $C(J, \mathbb{R})$ is separable. By $L^1(J, \mathbb{R})$ we denote the space of Lebesgue measurable real-valued functions defined on J. By $\| \cdot \|_{L^1}$ we denote the usual norm in $L^1(J,\mathbb{R})$ defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\,dt.$$

Definition 3.1. *A function $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random Carathéodory if*
(i) the map $(t,\omega) \to f(t,x,\omega)$ is jointly measurable for all $x \in \mathbb{R}$, and
(ii) the map $x \to f(t,x,\omega)$ is continuous for all $t \in J$ and $\omega \in \Omega$.

Definition 3.2. *A Carathéodory function $f : J\times\mathbb{R}\times\Omega \to \mathbb{R}$ is called random L^1-Carathéodory if for each real number $r > 0$ there is a measurable and bounded function $h_r : \Omega \to L^1(J,\mathbb{R})$ such that*

$$|f(t,x,\omega)| \le h_r(t,\omega) \quad \text{a.e.} \quad t \in J$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$ with $|x| \le r$. Similarly, Carathéodory function f is called random $L^1_{\mathbb{R}}$-Carathéodory if there is a measurable and bounded function $h : \Omega \to L^1(J,\mathbb{R})$ such that

$$|f(t,x,\omega)| \le h(t,\omega) \quad \text{a.e.} \quad t \in J$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$.

We consider the following set of hypotheses in what follows:

(H_0) The functions $q_i : \Omega \to \mathbb{R}$ are measurable and bounded with $Q_i = \operatorname{ess\,sup}_{\omega\in\Omega} |q_i(\omega)|$ for $i = 0,1,\ldots,n-1$.

(H_1) The function f is random Carathéodory on $J \times \mathbb{R} \times \Omega$.

(H_2) There exists a measurable and bounded function $\gamma : \Omega \to L^2(J,\mathbb{R})$ and a continuous and nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ such that

$$|f(t,x,\omega)| \le \gamma(t,\omega)\psi(|x|) \quad \text{a.e.} \quad t \in J$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$.

We note the hypotheses (H_0) through (H_1) are natural and have been used widely in the literature on nonlinear differential equations. Our main existence result is the following.

Theorem 3.1. *Assume that the hypotheses* (H_0) – (H_2) *hold. Suppose that*

$$\int_C^\infty \frac{dr}{\psi(r)} > \frac{T^{n-1}}{(n-1)!}\,\|\gamma(\omega)\|_{L^1} \tag{3.1}$$

for all $\omega \in \Omega$, where $C = \sum_{i=0}^{n-1} \dfrac{Q_i\,T^i}{i!}$. Then, the RDE (1.1) has a random solution defined on J.

Proof. Set $E = C(J,\mathbb{R})$ and define a mapping $Q : \Omega \times E \to E$ by

$$Q(\omega)x(t) = \sum_{i=0}^{n-1} \frac{q_i(\omega)\,t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\,f(s,x(s,\omega),\omega)\,ds \tag{3.2}$$

for all $t \in J$ and $\omega \in \Omega$.

Now the map $t \mapsto \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!}$ is continuous for all $\omega \in \Omega$. Again, as indefinite integral is continuous on J, $Q(\omega)$ defines a mapping $Q : \Omega \times E \to E$. We show that Q satisfies all the conditions of Corollary 2.1 on E.

First, we show that Q is a random operator on $\Omega \times E$. Since $f(t, x, \omega)$ is random Carathéodory, the map $\omega \mapsto f(t, x, \omega)$ is measurable in view of Theorem 2.2. Similarly, the product $\frac{(t-s)^{n-1}}{(n-1)!} \times$ $\times f(s, x(s, \omega), \omega)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$\omega \mapsto \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, x(s, \omega), \omega)\, ds = Q(\omega)x(t)$$

is measurable. As a result, Q is a random operator on $\Omega \times E$ into E.

Let B be a bounded subset of E. Then, there is a real number $r > 0$ such that $\|x\| \leq r$ for all $x \in B$. Next, we show that the random operator $Q(\omega)$ is continuous on B. Let $\{x_n\}$ be a sequence of points in B converging to the point x in B. Then it is enough to prove that $\lim_{n\to\infty} Q(\omega)x_n(t) = Q(\omega)x(t)$ for all $t \in J$ and $\omega \in \Omega$. By dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} Q(\omega)x_n(t) &= \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \lim_{n\to\infty} \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, x_n(s, \omega), \omega)\, ds = \\
&= \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!} \left[\lim_{n\to\infty} f(s, x_n(s, \omega), \omega)\right] ds = \\
&= \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, x(s, \omega), \omega)\, ds = Q(\omega)x(t)
\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a continuous random operator on E.

Now, we show that $Q(\omega)$ is a totally bounded random operator on E. We prove that $Q(\omega)(B)$ is totally bounded subset of E for each bounded subset B of E. To finish, it is enough to prove that $Q(\omega)(B)$ is a uniformly bounded and equicontinuous set in E for each $\omega \in \Omega$. Since the map $\omega \mapsto \gamma(t, \omega)$ is bounded and $\gamma \in L^2(J, \mathbb{R}) \subset L^1(J, \mathbb{R})$, by hypothesis ($\mathrm{H}_2$), there is a constant c such that $\|\gamma(\omega)\|_{L^1} \leq c$ for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed. Then for any $x : \Omega \to B$, one has

$$
\begin{aligned}
|Q(\omega)x(t)| &\le \sum_{i=0}^{n-1} \frac{|q_i(\omega)|\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, |f(s, x(s,\omega), \omega)|\, ds \le \\
&\le \sum_{i=0}^{n-1} \frac{|q_i(\omega)|\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, \gamma(s,\omega)\psi(|x(s,\omega)|)\, ds \le \\
&\le \sum_{i=0}^{n-1} \frac{Q_i\, T^i}{i!} + \int\limits_0^T \frac{T^{n-1}}{(n-1)!}\, \gamma(s,\omega)\psi(\|x(\omega)\|)\, ds \le \\
&\le \sum_{i=0}^{n-1} \frac{Q_i\, T^i}{i!} + \int\limits_0^T \frac{T^{n-1}}{(n-1)!}\, \gamma(s,\omega)\psi(r)\, ds \le \\
&\le \sum_{i=0}^{n-1} \frac{Q_i\, T^i}{i!} + \frac{T^{n-1}}{(n-1)!}\, \|\gamma(\omega)\|_{L^1} \psi(r) \le K_1
\end{aligned}
$$

for all $t \in J$, where $K_1 = \sum_{i=0}^{n-1} \dfrac{Q_i\, T^i}{i!} + c\, \dfrac{T^{n-1}}{(n-1)!}\, \psi(r)$. This shows that $Q(\omega)(B)$ is a uniformly bounded subset of E for each $\omega \in \Omega$.

Next, we show that $Q(\omega)(B)$ is an equicontinuous set in E. Let $x \in B$ be arbitrary. Then, for any $t_1, t_2 \in J$, one has

$$
\begin{aligned}
|Q(\omega)x(t_1) - Q(\omega)x(t_2)| &\le \left| \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t_1^i}{i!} - \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t_2^i}{i!} \right| + \\
&+ \left| \int\limits_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!}\, f(s, x(s,\omega), \omega)\, ds - \int\limits_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!}\, f(s, x(s,\omega), \omega)\, ds \right| \le \\
&\le \sum_{i=0}^{n-1} \frac{q_i(\omega)}{i!}\, |t_1^i - t_2^i| + \left| \int\limits_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!}\, f(s, x(s,\omega), \omega)\, ds - \int\limits_0^{t_1} \frac{(t_2-s)^{n-1}}{(n-1)!}\, f(s, x(s,\omega), \omega)\, ds \right| + \\
&+ \left| \int\limits_0^{t_1} \frac{(t_2-s)^{n-1}}{(n-1)!}\, f(s, x(s,\omega), \omega)\, ds - \int\limits_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!}\, f(s, x(s,\omega), \omega)\, ds \right| \le \\
&\le \sum_{i=0}^{n-1} \frac{q_i(\omega)}{i!}\, |t_1^i - t_2^i| + \int\limits_0^{t_1} \left| \frac{(t_1-s)^{n-1}}{(n-1)!} - \frac{(t_2-s)^{n-1}}{(n-1)!} \right| |f(s, x(s,\omega), \omega)|\, ds +
\end{aligned}
$$

$$+\left|\int\limits_{t_2}^{t_1}\frac{(t_2-s)^{n-1}}{(n-1)!}\,f(s,x(s,\omega),\omega)\,ds\right|\le$$

$$\le\sum_{i=0}^{n-1}\frac{Q_i}{i!}\,|t_1^i-t_2^i|+\int\limits_0^{t_1}\left|\frac{(t_1-s)^{n-1}}{(n-1)!}-\frac{(t_2-s)^{n-1}}{(n-1)!}\right|\gamma(s,\omega)\psi(|x(s,\omega)|)\,ds+$$

$$+\left|\int\limits_{t_2}^{t_1}\frac{(t_2-s)^{n-1}}{(n-1)!}\,\gamma(s,\omega)\psi(|x(s,\omega)|)\,ds\right|\le$$

$$\le\sum_{i=0}^{n-1}\frac{Q_i}{i!}\,|t_1^i-t_2^i|+\int\limits_0^{T}\left|\frac{(t_1-s)^{n-1}}{(n-1)!}-\frac{(t_2-s)^{n-1}}{(n-1)!}\right|\gamma(s,\omega)\psi(r)\,ds+$$

$$+\left|\int\limits_{t_2}^{t_1}\frac{T^{n-1}}{(n-1)!}\,\gamma(s,\omega)\psi(r)\,ds\right|\le$$

$$\le\sum_{i=0}^{n-1}\frac{Q_i}{i!}\,|t_1^i-t_2^i|+\left(\int\limits_0^{T}\left|\frac{(t_1-s)^{n-1}}{(n-1)!}-\frac{(t_2-s)^{n-1}}{(n-1)!}\right|^2ds\right)^{1/2}\left(\int\limits_0^{T}\gamma^2(s,\omega)\,ds\right)^{1/2}\psi(r)+$$

$$+\left|\int\limits_{t_2}^{t_1}\frac{T^{n-1}}{(n-1)!}\,\gamma(s,\omega)\psi(r)\,ds\right|\le$$

$$\le\sum_{i=0}^{n-1}\frac{Q_i}{i!}\,|t_1^i-t_2^i|+\left(\int\limits_0^{T}\left|\frac{(t_1-s)^{n-1}}{(n-1)!}-\frac{(t_2-s)^{n-1}}{(n-1)!}\right|^2ds\right)^{1/2}\times$$

$$\times\left(\int\limits_0^{T}\gamma^2(s,\omega)\,ds\right)^{1/2}\psi(r)+|p(t_1,\omega)-p(t_2,\omega)| \tag{3.3}$$

for all $\omega\in\Omega$, where $p(t,\omega)=\dfrac{T^{n-1}}{(n-1)!}\displaystyle\int_0^t\gamma(s,\omega)\psi(r)\,ds$.

Hence, for all $t_1,t_2\in J$,

$$|Q(\omega)x(t_1)-Q(\omega)x(t_2)|\to 0\quad\text{as}\quad t_1\to t_2,$$

uniformly for all $x\in B$ and $\omega\in\Omega$. Therefore, $Q(\omega)(B)$ is an equicontinuous set in E. As $Q(\omega)(B)$ is uniformly bounded and equicontinuous, it is compact by Arzelá – Ascolli theorem for each $\omega\in\Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on B.

Finally, we prove that the set $\mathcal{E}$ given in conclusion (ii) of Corollary 2.1 does not hold. Let $u\in\mathcal{E}$ be arbitrary and let $\omega\in\Omega$ be fixed. Then $u(t,\omega)=\lambda Q(\omega)u(t)$ for all $t\in J$ and $\omega\in\Omega$, where $0 < \lambda < 1$. Then, one has

$$|u(t,\omega)| \le \lambda |Q(\omega)u(t)| \le \sum_{i=0}^{n-1} \frac{|q_i(\omega)|\,t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\,|f(s,u(s,\omega),\omega)|\,ds \le$$

$$\le \sum_{i=0}^{n-1} \frac{Q_i\,T^i}{i!} + \int\limits_0^t \frac{T^{n-1}}{(n-1)!}\,\gamma(s,\omega)\psi(|u(s,\omega)|)\,ds \le$$

$$\le C + \frac{T^{n-1}}{(n-1)!} \int\limits_0^t \gamma(s,\omega)\psi(|u(s,\omega)|)\,ds \tag{3.4}$$

for all $t \in J$ and $\omega \in \Omega$, where $C = \sum_{i=0}^{n-1} \dfrac{Q_i\,T^i}{i!}$.

Let $m(t,\omega) = \sup_{s\in[0,t]} |u(s,\omega)|$. Then, there is a $t^* \in [0,t]$ such that $m(t,\omega) = |u(t^*,\omega)|$. Now, the function $t \mapsto m(t,\omega)$ is monotone increasing for each $\omega \in \Omega$, and so, it is integrable on J. Again, since ψ is continuous and nondecreasing, the composite function $t \mapsto \psi(m(t,\omega))$ is integrable on J. Hence, from inequalities (3.3) and (3.4), it follows that

$$m(t,\omega) = |u(t^*,\omega)| \le \sum_{i=0}^{n-1} \frac{Q_i\,T^i}{i!} + \frac{T^{n-1}}{(n-1)!} \int\limits_0^t \gamma(s,\omega)\psi(|u(s,\omega)|)\,ds \le$$

$$\le C + \frac{T^{n-1}}{(n-1)!} \int\limits_0^t \gamma(s,\omega)\psi(m(s,\omega))\,ds.$$

Put

$$w(t,\omega) = C + \frac{T^{n-1}}{(n-1)!} \int\limits_0^t \gamma(s,\omega)\psi(m(s,\omega))\,ds$$

for $t \in J$. Now differentiating this w.r.t. t, we obtain

$$w'(t,\omega) = \frac{T^{n-1}}{(n-1)!}\,\gamma(t,\omega)\psi(m(t,\omega)),$$

$$w(0,\omega) = C$$

for all $t \in J$. From the above inequality, we obtain

$$w'(t,\omega) \le \frac{T^{n-1}}{(n-1)!}\,\gamma(t,\omega)\psi(w(t,\omega)),$$

$$w(0,\omega) = C$$

or

$$\frac{w'(t,\omega)}{\psi(w(t,\omega))} \le \frac{T^{n-1}}{(n-1)!}\gamma(t,\omega),$$

$$w(0,\omega) = C.$$

Integrating w.r.t. t from 0 to t,

$$\int_0^t \frac{w'(s,\omega)}{\psi(w(s,\omega))} \le \frac{T^{n-1}}{(n-1)!}\int_0^t \gamma(s,\omega)\,ds.$$

By change of variable,

$$\int_C^{w(s,\omega)} \frac{dr}{\psi(r)} \le \frac{T^{n-1}}{(n-1)!}\|\gamma(\omega)\|_{L^1} < \int_C^{\infty}\frac{dr}{\psi(r)}.$$

Now by an application of mean value theorem for integral calculus, there exists a constant $M > 0$ such that

$$u(t,\omega) \le m(t,\omega) \le w(t,\omega) \le M$$

for all $t \in J$ and $\omega \in \Omega$. Hence the conclusion (ii) of Corollary 2.1 does not hold. As a result, the conclusion (i) holds and the operator equation $Q(\omega)x = x$ has a random solution. This further implies that the random differential equation (1.1) has a random solution defined on $\Omega \times J$.

Theorem 3.1 is proved.

4. Extremal random solutions. It is sometimes desirable that one should be interested in knowing the realistic behavior of the random solutions for a given dynamical system in question. Therefore, in the following section we prove the existence of extremal positive random solutions for the RDE (1.1) defined on $\Omega \times J$. We need the following definition in what follows.

A closed set K of the Banach space E is called a cone with vertex θ if

(i) $K + K \subseteq K$,

(ii) $\lambda K \subset K$ for all $\lambda \in \mathbb{R}_+$, and

(iii) $\{-K\} \cap K = \{\theta\}$,

where θ is the zero element of E. We introduce an order relation $\le$ in E with the help of cone K in E as follows. Let $x, y \in E$, then we define

$$x \le y \iff y - x \in K. \tag{4.1}$$

A cone K in the Banach space E is called normal, if the the norm $\|\cdot\|$ is semimonotone on K, i.e., if $x, y \in K$, then $\|x+y\| \le \|x\| + \|y\|$. Again a cone K is called regular, if every nondecreasing order bounded sequence in E converges in norm. Similarly, a cone K is called fully regular, if every nondecreasing norm-bounded sequence converges in E. The details of different types of cones and their properties appear in Deimling [9], Heikkilä and Lakshmikantham [1] and Zeidler [10].

We introduce an order relation $\leq$ in $C(J,\mathbb{R})$ with the help of a cone K in it defined by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geq 0 \text{ for all } t \in J\}.$$

Thus, we have

$$x \leq y \Longrightarrow x(t) \leq y(t) \quad \text{for all} \quad t \in J.$$

It is known that the cone K is normal in $C(J,\mathbb{R})$. For any function $a, b : \Omega \to C(J,\mathbb{R})$ we define a random interval $[a,b]$ in $C(J,\mathbb{R})$ by

$$[a,b] = \{x \in C(J,\mathbb{R}) \mid a(\omega) \leq x \leq b(\omega)\, \forall\, t \in J\} = \bigcap_{\omega\in\Omega} [a(\omega), b(\omega)].$$

Definition 4.1. *A operator $Q : \Omega \times E \to E$ is called nondecreasing if $Q(\omega)x \leq Q(\omega)y$ for all $\omega \in \Omega$ and for all $x, y \in E$ for which $x \leq y$.*

We use the following random fixed point theorem of Dhage [2, 11]in what follows.

Theorem 4.1 (Dhage [11]). *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a,b]$ be a random order interval in the separable Banach space E. Let $Q : \Omega \times [a,b] \to [a,b]$ be a completely continuous and nondecreasing random operator. Then Q has a least fixed point x_* and a greatest random fixed point y^* in $[a,b]$. Moreover, the sequences $\{Q(\omega)x_n\}$ with $x_0 = a$ and $\{Q(\omega)y_n\}$ with $y_0 = b$ converge to x_* and y^* respectively.*

We need the following definitions in the sequel.

Definition 4.2. *A measurable function $a : \Omega \to C^{(n-1)}(J,\mathbb{R})$ is called a lower random solution for the RDE (1.1) if*

$$a^{(n)}(t,\omega) \leq f(t, a(t,\omega), \omega) \quad \textit{a.e.} \quad t \in J,$$

$$a^{(i)}(0,\omega) \leq q_i(\omega), \quad i \in \{0, \ldots, n-1\},$$

for all $t \in J$ and $\omega \in \Omega$. Similarly, a measurable function $b : \Omega \to C^{(n-1)}(J,\mathbb{R})$ is called an upper random solution for the RDE (1.1) if

$$b^{(n)}(t,\omega) \geq f(t, b(t,\omega), \omega) \quad \textit{a.e.} \quad t \in J,$$

$$b^{(i)}(0,\omega) \geq q_i(\omega), \quad i \in \{0, \ldots, n-1\},$$

for all $t \in J$ and $\omega \in \Omega$.

Note that a random solution for the RDE (1.1) is lower as well as upper random solution for the RDE (1.1) defined on J.

Definition 4.3. *A random solution r_M for the RDE (1.1) is called maximal if for all random solutions of the RDE (1.1), one has $x(t,\omega) \leq r_M(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$. Similarly, a minimal random solution to the RDE (1.1) on J is defined.*

Definition 4.4. *A function $f : J \times \mathbb{R} \times \Omega$ is called random **Chandrabhan** if*
(i) the map $(t,\omega) \mapsto f(t,x,\omega)$ is jointly measurable,

(ii) the map $x \mapsto f(t,x,\omega)$ is continuous and nondecreasing for all $t \in J$ and $\omega \in \Omega$.

Definition 4.5. *A function $f(t,x,\omega)$ is called random L^1-**Chandrabhan** if for each real number $r > 0$ there exists a measurable function $h_r : \Omega \to L^1(J,\mathbb{R})$ such that for all $\omega \in \Omega$*

$$|f(t,x,\omega)| \le h_r(t,\omega) \quad a.e. \quad t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

We consider the following set of assumptions:

(H_3) The function f is random Chandrabhan on $J \times \mathbb{R} \times \Omega$.

(H_4) The RDE (1.1) has a lower random solution a and upper random solution b with $a \le b$ on J.

(H_5) The function $h : J \times \Omega \to \mathbb{R}_+$ defined by

$$h(t,\omega) = |f(t,a(t,\omega),\omega)| + |f(t,b(t,\omega),\omega)|$$

is Lebesgue integrable in t for all $\omega \in \Omega$.

Remark 4.1. If the hypotheses (H_3) and (H_5) hold, then for each $\omega \in \Omega$,

$$|f(t,x(t,\omega),\omega)| \le h(t,\omega)$$

for all $t \in J$ and $x \in [a,b]$ and the map $\omega \to h(t,\omega)$ is measurable on Ω.

Remark 4.2. Hypothesis (H_3) is natural and used in several research papers on random differential and integral equations (see Dhage [2, 11] and the references given therein). Hypothesis (H_4) holds, in particular, if there exist measurable functions $u, v : \Omega \to C^{(n-1)}(J,\mathbb{R})$ such that for each $\omega \in \Omega$,

$$u(t,\omega) \le f(t,x,\omega) \le v(t,\omega)$$

for all $t \in J$ and $x \in \mathbb{R}$. In this case, the lower and upper random solutions to the RDE (1.1) are given by

$$a(t,\omega) = \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, u(s,\omega)\, ds$$

and

$$b(t,\omega) = \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, v(s,\omega)\, ds$$

respectively. The details about the lower and upper random solutions for different types of random differential equations may be found in Ladde and Lakshmikantham [12]. Finally, hypothesis (H_5) remains valid if the function f is L^1-Carathéodory on $J \times \mathbb{R} \times \Omega$.

Theorem 4.2. *Assume that the hypotheses (H_1), (H_3) – (H_5) hold. Then the RDE (1.1) has a minimal random solution $x_*(\omega)$ and a maximal random solution $y^*(\omega)$ defined on J. Moreover,*

$$x_*(t,\omega) = \lim_{n\to\infty} x_n(t,\omega) \quad and \quad y^*(t,\omega) = \lim_{n\to\infty} y_n(t,\omega)$$

for all $t \in J$ and $\omega \in \Omega$, where the random sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are given by

$$x_{n+1}(t,\omega) = \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, x_n(s,\omega),\omega)\, ds, \quad n \geq 0 \quad \textit{with} \quad x_0 = a,$$

and

$$y_{n+1}(t,\omega) = \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, y_n(s,\omega),\omega)\, ds, \quad n \geq 0 \quad \textit{with} \quad y_0 = b$$

for all $t \in J$ and $\omega \in \Omega$.

Proof. Set $E = C(J, \mathbb{R})$ and define an operator $Q : \Omega \times [a,b] \to E$ by (3.2). We show that Q satisfies all the conditions of Theorem 4.1 on $[a,b]$.

It can be shown as in the proof of Theorem 3.1 that Q is a random operator on $\Omega \times [a,b]$. We show that it is L^1-Chandrabhan. First we show that $Q(\omega)$ is nondecreasing on $[a,b]$. Let $x, y : \Omega \to [a,b]$ be arbitrary such that $x \leq y$ on Ω. Then,

$$Q(\omega)x(t) \leq \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, x(s,\omega),\omega)\, ds \leq$$

$$\leq \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, y(s,\omega),\omega)\, ds = Q(\omega)y(t)$$

for all $t \in J$ and $\omega \in \Omega$. As a result, $Q(\omega)x \leq Q(\omega)y$ for all $\omega \in \Omega$ and that Q is a nondecreasing random operator on $[a,b]$.

Secondly, by hypothesis (H_4),

$$a(t,\omega) \leq Q(\omega)a(t) = \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, a(s,\omega),\omega)\, ds \leq$$

$$\leq \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, x(s,\omega),\omega)\, ds = Q(\omega)x(t) \leq Q(\omega)b(t) =$$

$$= \sum_{i=0}^{n-1} \frac{q_i(\omega)\, t^i}{i!} + \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, f(s, b(s,\omega),\omega)\, ds \leq b(t,\omega)$$

for all $t \in J$ and $\omega \in \Omega$. As a result Q defines a random operator $Q : \Omega \times [a,b] \to [a,b]$.

Next, since (H_5) holds, the hypothesis (H_2) is satisfied with $\gamma(t,\omega) = h(t,\omega)$ for all $(t,\omega) \in \in J \times \Omega$ and $\psi(r) = 1$ for all real numbers $r \geq 0$. Now it can be shown as in the proof of Theorem 3.1 that the random operator $Q(\omega)$ is completely continuous on $[a,b]$ into itself.

Thus, the random operator $Q(\omega)$ satisfies all the conditions of Theorem 4.1 and so the random operator equation $Q(\omega)x = x(\omega)$ has a least and a greatest random solution in $[a, b]$. Consequently, the RDE (1.1) has a minimal and a maximal random solution defined on J.

Theorem 4.2 is proved.

Remark 4.3. The conclusion of the Theorem 4.2 also remains true if we replace the hypotheses (H_3) and (H_5) with the following one.

(H_6) The function f is random L^1-Chandrabhan on $J \times \mathbb{R} \times \Omega$.

To see this, let hypothesis (H_6) hold. Since the cone K in $C(J, \mathbb{R})$ is normal, the random order interval $[a, b]$ is norm-bounded. Hence there is a real number $r > 0$ such that $\|x\| \leq r$ for all $x \in [a, b]$. Now f is L^1-Chandrabhan, so there is a measurable function $h_r : \Omega \to C(J, \mathbb{R})$ such that

$$|f(t, x, \omega)| \leq h_r(t, \omega) \quad \text{a.e.} \quad t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$ and for all $\omega \in \Omega$. Hence, hypotheses (H_3) and (H_5) hold with $h(t, \omega) = h_r(t, \omega)$ for all $t \in J$ and $\omega \in \Omega$.

5. Example. Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$ and let $J = [0, 1]$ be a closed and bounded interval in $\mathbb{R}$. Given a measurable function $x : \Omega \to C^1(J, \mathbb{R})$, consider the following RDE:

$$\begin{aligned} x^{(n)}(t, \omega) &= \frac{t\,\omega^2\, x^2(t, \omega)}{(1 + \omega^2)[1 + x^2(t, \omega)]} \quad \text{a.e.} \quad t \in J, \\ x^{(i)}(0, \omega) &= \sin^{(i)}(\omega), \quad i \in \{0, \ldots, n - 1\}, \end{aligned} \tag{5.1}$$

for all $\omega \in \Omega$.

Here,

$$f(t, x, \omega) = \frac{t\,\omega^2\, x^2}{(1 + \omega^2)[1 + x^2]}$$

for all $(t, x, \omega) \in J \times \mathbb{R} \times \Omega$, and

$$q_i(0, \omega) = \sin^{(i)}(\omega), \quad i = 0, 1, \ldots, n - 1,$$

for all $\omega \in \Omega$.

Clearly, the map $(t, \omega) \mapsto f(t, x, \omega)$ is jointly continuous for all $x \in \mathbb{R}$ and hence jointly measurable for all $x \in \mathbb{R}$. Also the map $x \mapsto f(t, x, \omega)$ is continuous for all $t \in J$ and $\omega \in \Omega$. So the function f is Carathéodory on $J \times \mathbb{R} \times \Omega$. Moreover,

$$\left| \frac{t\,\omega^2\, x^2}{(1 + \omega^2)[1 + x^2]} \right| \leq t = \gamma(t, \omega)\psi(|x|),$$

where $\gamma(t, \omega) = t$ for all $t \in [0, 1]$ and $\psi(r) = 1$ for all real numbers $r \geq 0$. Clearly, γ defines a measurable and bounded function $\gamma : \Omega \to L^2(J, \mathbb{R})$. Similarly, ψ defines a continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\int_C^\infty \frac{dr}{\psi(r)} = \int_C^\infty dr = \infty$$

for all $C \geq 0$. Again, the functions $q_i : \Omega \to \mathbb{R}$, $i = 0, 1, \ldots, n-1$, are measurable and bounded with $\operatorname{ess\,sup}_{\omega\in\Omega} |q_i(\omega)| \leq 1$ for all $i = 0, 1, \ldots, n-1$. Now,

$$\|\gamma(\omega)\|_{L^1} = \int_0^1 \gamma(t,\omega)\,dt = \frac{1}{2} < \int_C^\infty \frac{dr}{\psi(r)},$$

where $C = \sum_{i=0}^{n-1} \dfrac{1}{i!}$. Therefore, the condition (3.1) is satisfied. Hence, by Theorem 3.1, the RDE (5.1) has a random solution defined on $[0, 1]$.

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Received 09.03.09,
after revision — 06.09.09