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DYNAMICAL SYSTEM APPROACH TO SOLVING LINEAR PROGRAMMING PROBLEMS AND APPLICATIONS IN ECONOMICS MODELLING

ПІДХІД ДО РОЗВ'ЯЗАННЯ ЗАДАЧ ЛІНІЙНОГО ПРОГРАМУВАННЯ З ТОЧКИ ЗОРУ ДИНАМІЧНИХ СИСТЕМ ТА ЗАСТОСУВАННЯ В МОДЕЛЮВАННІ ЕКОНОМІЧНИХ ПРОЦЕСІВ

I. P. Tverdokhlib

I. Franko National University, L'viv, Ukraine

O. A. Prykarpats'ky

University of Koln, Germany

A class of dynamycal systems on symplectic manifolds solving linear programming problems is described. The structure of orbit space is analyzed in the framework of the Marsden – Weinstein reduction scheme. Some examples having applications in modern macro-economical modelling are treated in detail.

Описано клас динамічних систем на симплектичному многовиді, що розв'язують задачі лінійного програмування. Проведено аналіз простору орбіт з точки зору принципу редукції Марсдена і Вайнштейна. Детально розглянуто приклади застосувань у моделюванні сучасних макроекономічних процесів.

1. Introduction. The methods of the theory of dynamical systems were effectively applied for the first time in [1], where a new polynomial-time algorithm for linear programming was devised and approved further by numerical experiments. Amongst very important problems of linear programming we chose for studying here the one described below.

Let $a_i \in \mathbb{R}^n, i = \overline{1,m}$, be linearly independent constant vectors. Consider functions $f_i \in \in D(\mathbb{R}^{2n}), f_i(x,c) := \langle c, a_i\rangle, i = \overline{1,m}$, where $u := (x,c) \in \mathbb{R}^n \times \mathbb{R}^n$ is a given vector and $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^n$. It is assumed further that $\{f_i, f_j\} = 0$ for all $i, j = \overline{1,m}$, where $\{\cdot,\cdot\}$ is a Poisson structure on $\mathbb{R}^{2n}$, to be specified in detail in what follows, related naturally with some sympletic structure $\omega^{(2)} \in \Lambda^2(\mathbb{R}^{2n})$ on $\mathbb{R}^{2n}$. Denote also by $Y_i : \mathbb{R}^{2n} \to T(\mathbb{R}^{2n}), i = \overline{1,m}$, the corresponding Hamiltonian vector fields equal to $\{f_i, u\}, i = \overline{1,m}$, commuting on $\mathbb{R}^{2n}$ and generating orbit spaces related with polyhedra arising here naturally via a construction specified below.

Consider a polyhedron $P \subset \mathbb{R}^n$ which is described as

$$\langle a_i, x\rangle = b_i, \qquad x_j \geq 0, \tag{1}$$

for $i = \overline{1,m}, j = \overline{1,n}$. Take now $x \in \mathbb{R}^n$ and denote the set $\{j \in [1,n] : x_j = 0\}$ by $J(x)$. A point $x \in P$ is called an extreme point of P if the following implication holds:

$$\{y \in \mathbb{R}^n : \langle a_i, y\rangle = 0,\ i = \overline{1,m}\} \wedge \{y_j = 0 : j \in J(x)\} \Longrightarrow y = 0. \tag{2}$$

The set of extreme points will be denoted by $E(P)$. We shall also call a point $x \in E(P)$ simple if the vectors

$$a_j, j = \overline{1,m}, \quad \text{together with basis vectors} \quad e_j,\ j \in J(x), \tag{3}$$

form a basis in $\mathbb{R}^n$. A polyhedron P defined by (2) and (3) is said to be simple if each of its extreme points is simple. The following lemmas [2] are dealing with the critical set $E(p)$.

Lemma 1. *Given $x \in P$, where P is a simple compact polyhedron. Then there always exists $p \in E(p)$ such that $J(x) \subset J(p)$.*

Lemma 2. *Let $x, y \in M = \Phi^{-1}(P)$, where $R^n \ni x \xrightarrow{\Phi} \{1/2x_j^2 : j = \overline{1,n}\} \in R^n$, be such that $J(x) \subset J(y)$ and $\Phi(y) \in E(P)$. Then the vectors $D(x)a_i, i = \overline{1,m}$, and $e_j, j \in J(y)$, form a basis in R^n, where $D(x) = \operatorname{diag}\{x_j : j = \overline{1,n}\}$.*

Now the linear programming problem in the polyhedron P can be formulated as follows: find $\overline{x} \in M$ such that

$$\max_{x\in M} g_c(x) = g_c(\overline{x}) \tag{4}$$

under the conditions

$$\langle a_i, \Phi(x)\rangle = b_i, \quad i = \overline{1,m}, \tag{5}$$

where $g_c := \langle c, \Phi(x)\rangle, \quad x \in \mathbb{R}^n$. For its analysis, we shall apply methods of the Marsden – Weinstein reduction theory [3].

2. Reduction algorithm and linear programming. Subject to the Poisson bracket $\{\cdot,\cdot\}$ on $\mathbb{R}^{2n}$, conditions (1) can be considered evidently as constraints of first class [3, 4]. The latter, in particular, means that there exists a reduced Poisson bracket $\{\cdot,\cdot\}^r$ on the orbit space $\bar{M}_c := := \mathbb{R}^{2n}/T^m$, T^m is a Lie group isomorphic to an Abelian Lie group generated by the mutually commuting complete vector fields $Y_i : \mathbb{R}^{2n} \to T(\mathbb{R}^{2n})$, $i = \overline{1,m}$, defined before. With respect to this Poisson bracket $\{\cdot,\cdot\}^r$ on $\bar{M}_c$, one can consider the vector field $\bar{K}_c : \bar{M}_c \to T(\bar{M}_c)$ generated by the Hamiltonian function $H_c := \langle c, c\rangle/2$ reduced on $\bar{M}_c$ and study all of its asymptotically stable limit points. The theorems following below make it possible to associate these limit points with points $\overline{x} \in M$ solving the linear programming problem posed above. Consider the distribution $D_c : \mathbb{R}^{2n} \to T(\mathbb{R}^{2n})$ spanned by the Hamiltonian vector fields

$$K_j(u) = \{H_j, u\},$$

where $u \in \mathbb{R}^{2n}$ and by definition,

$$H_j = \langle a_j, \Phi(x)\rangle,\ j = \overline{1,m},$$

which is evidently integrable on $\mathbb{R}^{2n}$. Its maximal integral submanifolds are naturally imbedded into $\mathbb{R}^{2n}$ and are usually called symplectic leaves of the bracket $\{\cdot,\cdot\}$ with respect to which we can build the reduced manifold $\overline{M}_c = \mathbb{R}^{2n}/D_c$.

Construct now the second distribution $D_b : \mathbb{R}^n \to T(\mathbb{R}^n)$ spanned by gradient vector fields $\nabla g_j : \mathbb{R}^n \to T(\mathbb{R}^n),\ g_j(x) := \langle a_j, \Phi(x)\rangle,\ j = \overline{1,m}$. Then evidently the reduced space $\overline{M}_b := \mathbb{R}^n/D_b$ is diffeomorphic to $M = \Phi^{-1}(P)$ and one can define the reduction mapping

$$r_b : \mathbb{R}^n \to \overline{M}_b.$$

Choose now the distribution D_c and let us denote by E_c the foliation induced by D_c on $\mathbb{R}^n$. The leaves of E_c are the intersections of $\mathbb{R}^{2n}$ with leaves of D_c. One can prove that E_c is sufficiently regular for the quotient space $\overline{M}_c = \mathbb{R}^{2n}/E_c$ to be a manifold, which is satisfied for the case of a compact simple polyhedron $P \in \mathbb{R}^n$ assumed before. Thereby one can define the canonical projection $r_c : \mathbb{R}^{2n} \to \overline{M}_c$ and apply the standard Marsden – Weinstein reduction algorithm to find the reduced vector fields $\bar{K}_c : \bar{M}_c \to T(\bar{M}_c)$ and $\bar{K}_b : \bar{M}_b \to T(\bar{M}_b)$ on the reduced manifolds $\overline{M}_c$ and $\overline{M}_b$ corresponding, respectively, to the vector fields $K_b := \nabla_{g_c} : \mathbb{R}^n \to \to T(\mathbb{R}^n)$ and $K_c := \{H_c, \cdot\} : \mathbb{R}^{2n} \to T(\mathbb{R}^{2n})$.

Let now $p \in E(p)$ be an extreme point of P. Since P is assumed to be simple, there always exist vectors $h_j(p) \in \mathbb{R}^n,\ j \in J(p)$, such that $\langle a_i, h_j(p)\rangle = 0, i = \overline{1,m}$, and $\langle e_k, h_j(p)\rangle = = \delta_{jk},\ k, j \in J(p)$. Then one says that a vector $c \in \mathbb{R}^n$ is in general position (relative to P) if $\langle c, h_j(p)\rangle \neq 0$ for all $p \in E(p),\ j \in J(p)$.

Theorem 1. *Let a vector $c \in \mathbb{R}^n$ be in general position. Then the set of stationary points of the reduced vector fields $\bar{K}_c : \bar{M}_c \to T(\bar{M}_c)$ and $\bar{K}_b : \bar{M}_b \to T(\bar{M}_b)$ coincides with the set $E(p)$.*

A proof of this statement is easy and can be obtained from [2].

Theorem 2. *Consider the following corresponding commutative diagrams*

$$\begin{array}{ccc} T^*(\mathbb{R}^{2n}) & \xleftarrow{r_c^*} & T^*(\overline{M}_c) \\ \downarrow & & \downarrow \\ \mathbb{R}^{2n} & \xrightarrow{r_c} & \overline{M}_c \end{array},$$

$$\begin{array}{ccc} T(\mathbb{R}^{n}) & \xrightarrow{r_{b*}} & T(\bar{M}_b) \\ \downarrow & & \downarrow \\ \mathbb{R}^{n} & \xrightarrow{r_b} & \bar{M}_b \end{array}$$

and the reduced symplectic structure $\overline{\omega}_c^{(2)} \in \Lambda^2(\overline{M}_c)$ on $\overline{M}_c$ and the reduced vector fields $\overline{K}_b : \overline{M}_b \to T(\overline{M}_b)$ and $\bar{K}_c : \bar{M}_c \to T(\bar{M}_c)$ related to them via the relations:

$$r_{b_*} K_b = \bar{K}_b \cdot r_b, \quad r_{c_*} K_c = \bar{K}_c \cdot r_c, \quad \omega^{(2)} = r_c^* \bar{\omega}_c^{(2)},$$

where $K_b := \nabla g_c : \mathbb{R}^n \to T(\mathbb{R}^n)$, defining them uniquely, giving rise to the analytical criteria for finding particular points $\overline{x} \in \mathbb{R}^n$ solving the linear programming problem (4), (5) via the limiting procedure,

$$\lim_{t\to\infty} r_b(x(t;x_0)) = \bar{x}_b, \quad \lim_{t\to\infty} r_c(x(t;x_0)) = \bar{u}_c, \tag{6}$$

for all $x_0 \in R^n$, together with evident conditions, $\bar{K}_b(\overline{x}_b) = 0, \quad \bar{K}_c(\bar{u}_c) = 0$.

Here, by definition, $r_b(\overline{x}) = \overline{x}_b \in \bar{M}_b$, $r_c(\overline{x}, c) = \bar{u}_c \in \bar{M}_c$, *and the reduced vector field* $\bar{K}_c : \bar{M}_c \to T(\bar{M}_c)$ *is defined simultaneously also by the relations*

$$i_{\overline{K}_c}\overline{\omega}^{(2)} = -d\overline{H}_c, \qquad i_{K_c}\omega^{(2)} = -dH_c, \tag{7}$$

where $\overline{H}_c := H_c|_{\bar{M}_c}$. *Moreover,* $\bar{M}_c \simeq M \times \mathbb{R}^n$ *and the function* $\bar{g}_c := g_c|_{\bar{M}_b} : \bar{M}_b \to \mathbb{R}$ *serves as a Liapunov function for the reduced vector field* $\bar{K}_b : \bar{M}_b \to T(\bar{M}_b)$.

Thus, we obtained the effective analytical algorithm (6), (7) for solving linear programming problems in a simple compact polyhedron P like (4), (5) based on its symmetry properties [5] related with polyhedron constrains (1).

3. An application of the method of dynamical systems to modelling regional demand. Below we shall discuss an example of an economics-mathematical model to which the method of dynamical systems can be successfully applied. In [6] there was suggested the AIDS-model of demand and described a technique of its application that based on dynamical statistical data. Since one of the problems in modelling transition economical processes is a lack of true statistical data for a long enough period of time, in [7] there was well grounded the following priority demand model which is an extension of the known [6] AIDS-model: to find such a symmetric matrix $X : E^n \to E^n$, which would give

$$\min_{\{\vec{\alpha};\eta\}} (\mathrm{Sp}X + \langle \overrightarrow{b}, X\overrightarrow{b} \rangle) \tag{8}$$

under the condition

$$\overrightarrow{w} = \overrightarrow{\alpha} + X\overrightarrow{a} + c\vec{\beta} \tag{9}$$

and the constrains

$$\langle \overrightarrow{a}, \overrightarrow{\alpha} \rangle + \langle \overrightarrow{a}, X\overrightarrow{a} \rangle / 2 = k, \quad \overrightarrow{e} = (1, 1, \ldots, 1)^T \in E^n,$$
$$\langle \overrightarrow{\alpha}, \overrightarrow{e} \rangle = 1, \quad \langle \overrightarrow{\beta}, \overrightarrow{e} \rangle = 0, \quad X\overrightarrow{e} = 0, \tag{10}$$

where $\overrightarrow{w}$, $\overrightarrow{b}$, $\overrightarrow{\beta}$ and $\overrightarrow{a} \in E^n$ are some specified vectors; $\overrightarrow{\alpha} \in E^n$ in an unknown vector, $c, k \in \mathbb{R}^1$ are some constants; E^n is the n-dimensional Euclidean vector space with the usual scalar product $\langle \cdot, \cdot \rangle$; $\mathrm{Sp}X$ is the trace of a matrix $X : E^n \to E^n$ and $\eta \in E^n$ are intrinsic hidden stabilization parameters.

Attempts to solve problem (8) – (10) making use of the usual simplex-method did not lead to a success. That is why there was devised a special technique for mathematical analysis of this problem basing on the Lagrangian multipliers method. With its help one could determine a solution to problem (8) – (10) at a given vector $\overrightarrow{\beta} \in E^n$. But a vector $\overrightarrow{\beta} \in E^n$, as a rule, is unknown in general, that is why in [7] it was suggested to seek it by means of solving an additional linear programming problem. It allows to pose the additional optimization problem (8) – (10) subject to vectors $\overrightarrow{\beta} \in E^n$.

Let us consider a model which includes conditions (9), (10) and the functional

$$\min_{\{\vec{\alpha};\vec{\beta};\eta\}} (\mathrm{Sp}X + \langle \vec{b}, X\vec{b}\rangle). \tag{11}$$

Having used the Hilbert–Schmidt representation for a symmetric matrix $X : E^n \to E^n$, one gets the following expression:

$$X = \sum_{j=1}^{n-1} \lambda_j \vec{\eta}_j \otimes \vec{\eta}_j, \tag{12}$$

where $\lambda_j \in \mathbb{R}^1,\ j = \overline{1, n-1}$, are eigenvalues of the matrix $X; \vec{\eta}_j \in E^n,\ j = \overline{1, n-1}$, are the corresponding eigenvectors, which are mutually orthonormal and orthogonal with the vector $\vec{e} \in E^n, \otimes$ is the usual tensor product. Having substituted (12) info (9), (10), one can determine from that

$$\lambda_j = \frac{\langle \vec{w}, \vec{\eta}_j\rangle - \langle \vec{\alpha}, \vec{\eta}_j\rangle - c\langle \vec{\beta}, \vec{\eta}_j\rangle}{\langle \vec{\eta}_j, \vec{a}\rangle} \tag{13}$$

for $j = \overline{1, n-1}$. Having taken then into account (12) and (13) the problem above reduces to the equivalent one: to find a system of vectors $\vec{\eta}_j \in E^n,\ j = \overline{1, n-1}$, such that

$$\min_{\{\vec{\alpha};\vec{\beta};\eta\}} F(\overline{\alpha}; \vec{\beta}; \eta) = \min_{\{\vec{\alpha};\vec{\beta};\eta\}} \sum_{j=1}^{n-1} \left\{ \frac{\langle \vec{w} - \vec{\alpha} - c\vec{\beta}, \vec{\eta}_j\rangle}{\langle \vec{a}, \vec{\eta}_j\rangle} \times \right.$$

$$\left. \times \left[1 + \langle \vec{b}, \vec{\eta}_j\rangle^2\right] \right\} \tag{14}$$

under the conditions

$$\begin{gathered} \langle \vec{a}, \vec{\alpha}\rangle - c\langle \vec{a}, \vec{\beta}\rangle = \overline{k}, \qquad \overline{k} := 2k - \langle \vec{a}, \vec{w}\rangle, \\ \langle \vec{\eta}_j, \vec{\eta}_k\rangle = \delta_{jk}\ (j, k = \overline{1, n-1}), \quad \langle \vec{e}, \vec{\alpha}\rangle = 1, \\ \langle \vec{e}, \vec{\beta}\rangle = 0, \qquad \langle \vec{\eta}_j, \vec{e}\rangle = 0\ (j = \overline{1, n-1}). \end{gathered} \tag{15}$$

Having gone over from the model (11) and (8), (9) to the problem (14), (15), we have realized some justification of hidden parameters $\vec{\eta}_j \in E^n, j = \overline{1, n-1}$, specifying the wanted matrix $X : E^n \to E^n$. But the analysis of the optimization problem (14), (15) by means of the Lagrangian multiplier method does not lead to a success.

Thereby we are forced to apply, for solving this problem, the dynamical system approach described before. For it to be implemented, it is necessary to construct the corresponding phase space Q and the constraint functions $f_i : Q \to \mathbb{R}^1, i = \overline{1, m}$. As a result of some analysis of this problem, we found that

$$Q = \left\{(\vec{\alpha}, \vec{\beta}) \in \mathbb{R}^{2n}\right\} \times \left\{(\vec{a}, \vec{b}) \in \mathbb{R}^{2n}\right\} \times T^*(\mathbf{SO}(n-1)) \tag{16}$$

under the constraints

$$f_1(x;a,b) := \langle \overrightarrow{a}, \overrightarrow{\alpha} \rangle - c\langle \overrightarrow{a}, \overrightarrow{\beta} \rangle, \; f_2(x;a,b) := \langle \overrightarrow{e}, \overrightarrow{\alpha} \rangle,$$

$$f_3(x;a,b) = \langle \overrightarrow{e}, \overrightarrow{\beta} \rangle,$$

where the elements $(a,b) \in \mathbb{R}^{2n}$, together with the space $T_g^*(\mathbf{SO}(n-1)),\; g \in \mathbf{SO}(n-1)$, have to be considered as complementary functions of the phase space subject to the Poisson bracket $\{\cdot,\cdot\}$ defined on the whole symplectic space Q.

Concerning the phase space (16) we have also the following representation: $\vec{\eta}_j = \sum_{j=1}^{n-1} g_{ij}\vec{e}_i$, where $j = \overline{1, n-1}$ and the matrix $g \in \mathbf{SO}(n-1)$ in accordance with conditions (15), which should be substituted into expression (14). Now one can make use of the dynamical system method to solve the nonlinear programming problem (14) following the scheme described in Chapter 2.

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