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A VAN DER POL OSCILLATOR SUBJECTED TO COMPLICATED EXCITATIONS CHAOTIC PHENOMENON*

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This paper deals with the Van der Pol oscillator subjected to complicated excitations. Section 1 presents the Van der Pol oscillator with a variable friction force. Section 2 is concerned with a Van der Pol oscillator under a simultaneous influence of forced and nonlinear parametric excitations. Section 3 is devoted to the regular oscillation and chaotic phenomenon in a forced, strongly nonlinear Van der Pol's oscillator. The stationary oscillations, their stability and transitional regimes and also chaotic phenomenon are of special interest. The asymptotic method of nonlinear mechanics and the method of harmonic balance, in combination with a computer, are used.

AMS Subject Classification: 34E10

1. Van der Pol Oscillator with Variable Friction Force

Let us consider a generalized Van der Pol equation:

$$\ddot{x} + \omega^2 x = \varepsilon f = \varepsilon \left\{ \omega \Delta x - \omega \gamma x^3 + h \left[1 - k(x + q \cos \nu t)^2 \right] \dot{x} \right\}, \tag{1}$$

where dot denotes the derivative with respect to time t, $h > 0, k > 0, \omega, \nu, \gamma, q$ are constants, and ε is a small parameter, q is a parameter characterizing the intensity of the variable part of the nonlinear friction. When $q = \Delta = \gamma = 0$ we have the well-known Van der Pol equation, for which there exists only one stable stationary oscillation

$$x = 2 \cos(\omega t + \alpha).$$

To investigate the oscillations described by equation (1) we first transform it into the standard form by means of the formulae

$$\begin{aligned} x &= a \cos \psi, \\ \dot{x} &= -a\omega \sin \psi, \\ \psi &= \omega t + \theta, \end{aligned} \tag{2}$$

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and then use the averaging method of nonlinear oscillations [1]. In the first approximation we have the following equations for a and θ:

$$\begin{aligned} \frac{da}{dt} &= \varepsilon a f_1(a,\theta), \\ \frac{ad\theta}{dt} &= \varepsilon a f_2(a,\theta), \end{aligned} \tag{3}$$

here

$$f_1 = h\left\{\frac{1}{2} - k\langle(a\cos\psi + q\cos\nu t)^2\sin^2\psi\rangle\right\},$$

$$f_2 = -\frac{1}{2}\left\{\Delta - \frac{3}{4}\gamma a^2 + 2hk\langle(a\cos\psi + q\cos\nu t)^2\sin\psi\cos\psi\rangle\right\},$$

and $\langle f\rangle$ denotes the averaged value of a function f.

Equations (3) have a zero solution $a = a_{**} = 0$ and the other nontrivial stationary solutions $a = a_* \neq 0, \theta = \theta_*$ are determined by the system of equations

$$f_1(a_*,\theta_*) = 0, \quad f_2(a_*,\theta_*) = 0.$$

The zero solution is stable if $(\varepsilon > 0)$

$$f_1(0,\theta_{**}) < 0,$$

θ_{**} is arbitrary.

The conditions of stability for nontrivial solutions are

$$\begin{aligned} S &= a_*\left(\frac{\partial f_1}{\partial a}\right)_* + \left(\frac{\partial f_2}{\partial \theta}\right)_* < 0, \\ R &= \left(\frac{\partial f_1}{\partial a}\right)_*\left(\frac{\partial f_2}{\partial \theta}\right)_* - \left(\frac{\partial f_1}{\partial \theta}\right)_*\left(\frac{\partial f_2}{\partial a}\right)_* > 0. \end{aligned} \tag{4}$$

We shall now consider various resonance cases that correspond to concrete values of the frequencies ω and ν.

The Subharmonic Resonance Case $\nu \approx 3\omega$. In this case the averaged equations (3) take the form

$$\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon a}{4}h\left[2 - k\left(\frac{a^2}{2} + q^2\right) + kaq\cos 3\theta\right], \\ a\frac{d\theta}{dt} &= -\frac{\varepsilon a}{2}\left(\Delta - \frac{3}{4}\gamma a^2 + \frac{hk}{2}qa\sin 3\theta\right). \end{aligned} \tag{5}$$

Stationary amplitudes are determined by

$$W = h^2\left[2 - k\left(\frac{a^2}{2} + q^2\right)\right]^2 + 4\left(\Delta - \frac{3}{4}\gamma a^2\right)^2 - k^2h^2q^2a^2 = 0.$$

The stability conditions (4) for stationary solutions $(da/dt = d\theta/dt = 0)$ of equations (5) are

$$S = h\left[1 - \frac{k}{2}(a^2 + q^2)\right] < 0 \quad \text{or} \quad a^2 > \frac{2}{k} - q^2$$

$$R = \frac{3a}{16} \quad \frac{\partial W}{\partial a^2} > 0.$$

In the plane (a^2, Δ) the resonance curve $W = 0$ is an ellipse. Figure 1 shows the resonance curve for the case $q = 1.1$, $\gamma = 0.1$, $h = 0.1$, $k = 2$ and Fig. 2 presents the case $h = k = 1$, $q = 1.1$, $\gamma = 0.6$.

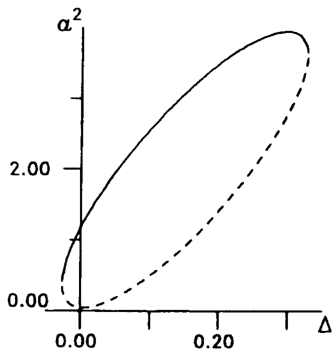

Fig. 1. The resonance curve for $q = 1.1, \gamma = 0.1,\ h = 0.1,\ k = 2.$

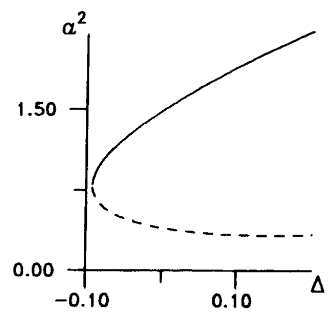

Fig. 2. The resonance curve for $h = k = 1, q = 1.1,\ \gamma = 0.6.$

Stationary Oscillations in the Case $\boldsymbol{\nu \not\approx \omega, 3\omega}$. For these cases the averaged equations become quite simple,

$$\frac{da}{dt} = \frac{\varepsilon a}{2}h\left[1 - k\left(\frac{a^2}{4} + \frac{q^2}{2}\right)\right],$$

$$a\frac{d\theta}{dt} = -\frac{\varepsilon a}{2}\left(\Delta - \frac{3}{4}\gamma a^2\right).$$

It is easy to show that the zero solution $a = 0$ is stable if $q^2 \geq 2/k$, and unstable if $-\sqrt{2/k} < q < \sqrt{2/k}$.

The nontrivial stationary solution,

$$a_*^2 = 4\left(\frac{1}{k} - \frac{q^2}{2}\right),$$

is stable for the values $q^2 < 2/k$. Thus, for $q^2 < 2/k$ the system oscillates with the stationary amplitude a_*:

$$a_*^2 = \frac{4}{k} - 2q^2.$$

For $q^2 > 2/k$ the damping of oscillations takes place (Fig. 3).

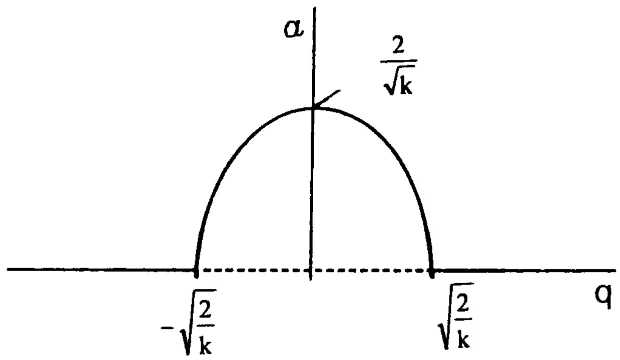

Fig. 3. The dependence of the amplitude a on the parameter q.

Oscillations in the Case $\nu \approx \omega$. Now, the averaged equations (3) are of the form

$$\dot{a} = -\frac{\varepsilon a}{2} f_0, \quad a\dot{\theta} = -\frac{\varepsilon a}{2} g_0, \tag{6}$$

where

$$f_0 = h\left(\frac{1}{4}ka^2 + \frac{1}{2}kq^2 - 1\right) + \frac{1}{2}hkqa\cos\theta - \frac{1}{4}hkq^2\cos 2\theta,$$

$$g_0 = \Delta - \frac{3}{4}\gamma a^2 + \frac{1}{2}hkqa\sin\theta + \frac{1}{4}hkq^2\sin 2\theta.$$

The stationary values a_0 of a and θ_0 of θ are determined from the equations

$$f_0 = 0, \quad g_0 = 0. \tag{7}$$

We eliminate 2θ from (7) by using the combination

$$\begin{aligned} f &= (2a + q\cos\theta) f_0 - q\sin\theta g_0 = A\sin\theta + B\cos\theta - E = 0, \\ g &= -q\sin\theta f_0 + (2a - q\cos\theta) g_0 = G\sin\theta + H\cos\theta - K = 0. \end{aligned} \tag{8}$$

The resonance curves consist of two parts: the regular part C_1 and the critical part C_2. The regular part C_1 lies in the equivalence region, where equations (7) and (8) are equivalent, that is, where the determinant of the transformation (8) is different from zero,

$$\begin{vmatrix} 2a + q\cos\theta & -q\sin\theta \\ -q\sin\theta & 2a - q\cos\theta \end{vmatrix} = 4a^2 - q^2 \neq 0.$$

The critical part C_2 lies in the nonequivalence region, where $D = 0$ and satisfies the compatibility conditions $D_1 = EH - KB = 0$, $D_2 = AK - BE = 0$ and also the trigonometric conditions $A^2 + B^2 \geq E^2$, $G^2 + H^2 \geq K^2$.

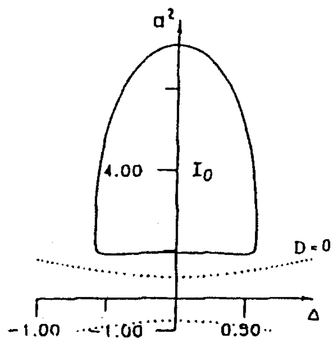

Fig. 4. The resonance curve for $q = 0.8$.

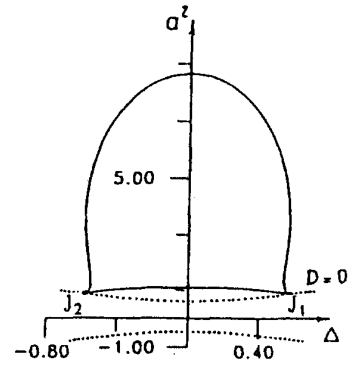

Fig. 5. The resonance curve for $q = 0.94$.

The regular part of resonance curves is obtained by eliminating the phase θ from equations (8):

$$\begin{aligned} W(\Delta, a^2) &= 16a^2h^2q^2\left(\Delta - \frac{3}{4}\gamma a^2\right)^2(3T^* + X^2) \\ &\quad + 4a^2q^2\left\{\left(\Delta - \frac{3}{4}\gamma a^2\right)^2 + h^2(T^* + X)(3T^* - X)\right\}^2 \\ &\quad - q^4\left\{\left(\Delta - \frac{3}{4}\gamma a^2\right)^2 - h^2(5T^* + X)(3T^* - X)\right\}^2 = 0, \end{aligned} \tag{9}$$

where

$$T^* = \frac{k}{16}(4a^2 - q^2), \quad X = \frac{9}{16}\left(kq^2 - \frac{16}{9}\right).$$

The resonance curves are represented in Figs 4–14 for the case $\gamma = 0$.

For given values of h and k we increase q to observe a change in the forms of the resonance curve. Taking, for example, $h = k = 1$, we have the level of the self-excited oscillation of the original system: $a_0^2 = 4$. In the plane $R(\Delta, a^2)$, a representative point of this oscillating regime is the point $I_0\left(\Delta = (3/4)\gamma a^2, a^2 = a_0^2 = 4/k = 4\right)$.

If $0 < kq^2 < 8/9$, the critical points still do not appear. The resonance curve is a simple oval encircling the point I_0. In Fig. 4 the resonance curve corresponds to $q = 0.8$.

If $kq^2 = 8/9$, the resonance curve is still a simple oval, but two critical points J_1, J_2 appear. They are returning points as shown in Fig. 5 for $q = 0.94 \simeq \sqrt{8/9}$.

If $\dfrac{8}{9} < kq^2 < 4/3$, two critical points J_1, J_2 move down and become two nodes. The resonance curve has two cycles which are tied at J_1, J_2, as shown in Fig. 6 for $q = 1.1$.

If $kq^2 = 4/3$, critical point I_2 appears at the origin. It is a degenerated node, at which two branches of the resonance curve are tangential to each other and to the abscissa axis Δ. The resonance curve in Fig. 7 corresponds to the value $q = 1.155 \simeq \sqrt{4/3}$.

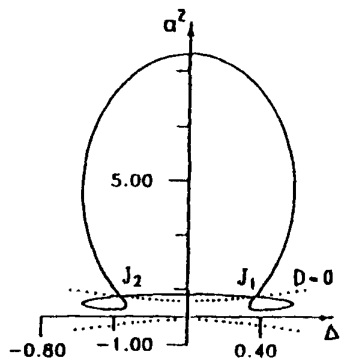

Fig. 6. The resonance curve for $q = 1.1$.

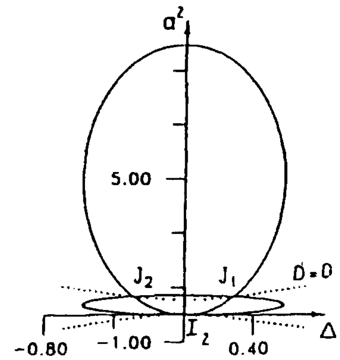

Fig. 7. The resonance curve for $q = 1.155$.

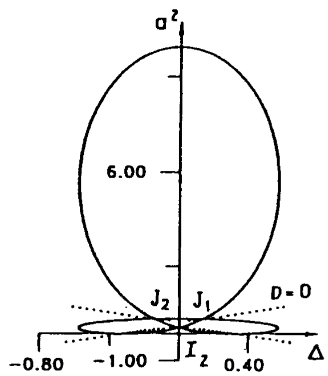

Fig. 8. The resonance curve for $q = 1.25$.

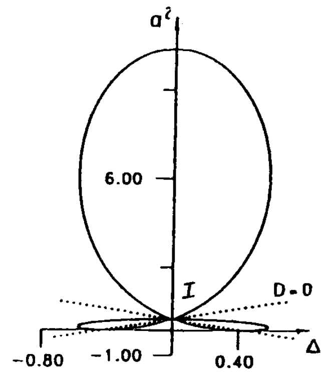

Fig. 9. The resonance curve for $q = 1.33$.

If $4/3 < kq^2 < 16/9$, the critical point I_2 moves up while the two other critical points continue to move down. All these three points are nodes and the resonance curve has the form shown in Fig. 8 for $q = 1.25$.

If $kq^2 = 16/9$, three critical nodes I_2, J_1, J_2 coincide at a special critical point I of the nonequivalence line. The resonance curve has the form shown in Fig. 9 for $q = 1.33 \simeq 4/3$.

If $16/9 < kq^2 < 8/3$, three nodes appear again, but I_2 lies higher and J_1, J_2 lie lower. The resonance curve has the form shown in Fig. 10 for $q = 1.45$.

If $kq^2 = 8/3$, two nodes J_1 and J_2 lie on the abscissa axis and become two returning points as shown in Fig. 11 for $q = 1.63 \approx \sqrt{8/3}$.

If $8/3 < kq^2 < 4$, the resonance curve has only one node I_2 and takes the form in Fig. 12 for $q = 1.75$.

If $kq^2 = 4$, we have the resonance curve in the form of the number eight which is tangential to the abscissa axis as shown in Fig. 13 for $q = 2$.

If $kq^2 > 4$, the resonance curve still has the form of the number eight but lies above the abscissa axis as shown in Fig. 14 for $q = 2.8$.

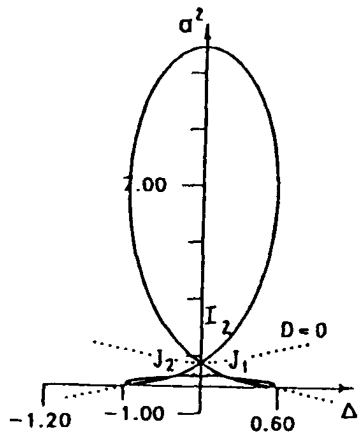

Fig. 10. The resonance curve for $q = 1.45$.

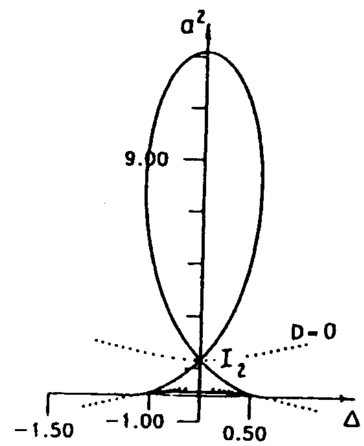

Fig. 11. The resonance curve for $q = 1.63$.

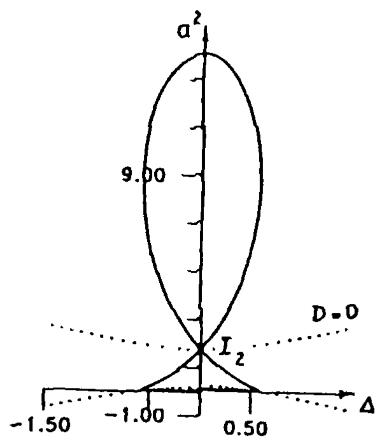

Fig. 12. The resonance curve for $q = 1.75$.

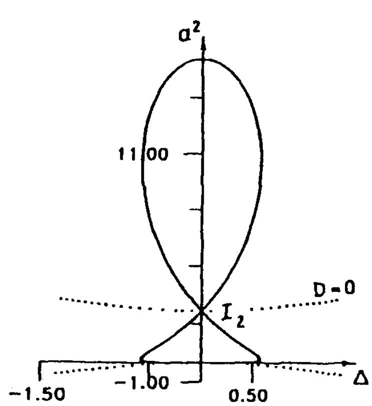

Fig. 13. The resonance curve for $q = 2$.

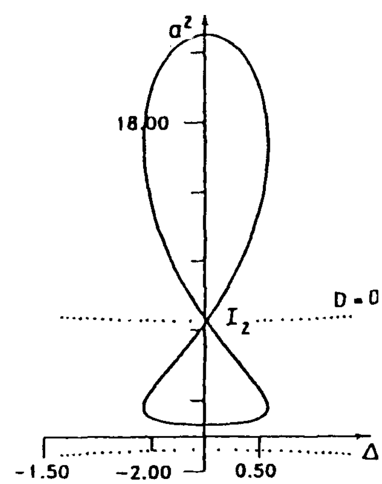

Fig. 14. The resonance curve for $q = 2.8$.

Stability of Stationary Oscillations. The stability conditions obtained from the original equations have a well-known form. However, for the system under consideration, the transformation of stability conditions is more complicated.

The first stability condition is

$$S_1 = a\frac{\partial f_0}{\partial a} + \frac{\partial g_0}{\partial \theta} = a\left\{\frac{1}{2}(hka + hkq\cos\theta)\right\} + \frac{1}{2}\left\{hkqa\cos\theta + hkq^2\cos 2\theta\right\} > 0.$$

Based on the original equations (6), we can elimitate $\cos 2\theta$,

$$S_1 = h\{(ka^2 + kq^2 - 2) + 2kqa\cos\theta\} > 0. \tag{10}$$

For the regular oscillation in the equivalence region, from the associated equations we get $\cos\theta = D_2/D$. Substituting this value into the last equation we have

$$S_1 = \frac{h}{D}\{(ka^2 + kq^2 - 2)D + 2kqaD_2\} = \frac{hq^2}{D}\bar{S}_1 > 0, \tag{11}$$

where

$$\bar{S}_1 = (ka^2 + kq^2 - 2)\left\{\Delta^2 - h^2\left(\frac{5}{4}ka^2 + \frac{1}{4}kq^2 - 1\right)\left(34ka^2 - \frac{3}{4}kq^2 + 1\right)\right\}$$

$$+ 4ka^2\left\{\Delta^2 + h^2\left(\frac{3}{4}ka^2 - \frac{3}{4}kq^2 + 1\right)\left(\frac{1}{4}ka^2 + \frac{1}{2}kq^2 - 1\right)\right\}.$$

Hence, in the plane $R(\Delta, a^2)$ and in the equivalence region, based on the sign of the functions D and $\bar{S}_1$, we can identify the branches which satisfy and do not satisfy the first stability condition for the regular part C_1 of the resonance curves.

For the critical points in the equivalence region, we use form (10) if the phase is known. However, since the critical phase is the limit of the phase on the regular branch leading to the critical point, we can use form (11). More simply, we can deduce the first stability condition at the critical point from the first stability condition on the regular branch which is considered as containing the critical point.

For the regular oscillation in the nonequivalence region, we can use initial forms (10), (11). For the critical oscillation in the nonequivalence region, i.e., at the special critical point, we can use either initial form (10) or form (11).

For the critical point $I\left(\Delta = 0,\ ka^2 = 4/9\right)$ corresponding to $kq^2 = 16/9$ in the equivalence region, we have three phases $\theta = 0,\ \theta = \pm 2\pi/3$. Substituting these values into (10) we obtain

for $\theta = 0,\ S_1 = h\left\{4/9 + 16/9 - 2 + 2a\sqrt{k}q\sqrt{k}\cos\theta\right\} = 2h > 0,$

for $\theta = \pm 2\pi/3,\ S_1 = -2h/3 < 0.$

Thus, only the oscillation with the phase $\theta = 0$ satisfies the first stability condition and here the critical point I is considered as belonging to the horizontal resonance branch.

We have the following second stability condition:

$$S_2 = \frac{\partial f_0}{\partial a}\frac{\partial g_0}{\partial \theta} - \frac{\partial f_0}{\partial \theta}\frac{\partial g_0}{\partial a} = \left(\frac{hka}{2} + \frac{hk}{2}q\cos\theta\right)\left(\frac{hk}{2}qa\cos\theta + \frac{hk}{2}q^2\cos 2\theta\right)$$

$$-\left(-\frac{hka}{2}qa\sin\theta + \frac{hk}{2}q^2\sin 2\theta\right)\frac{hk}{2}q\sin\theta > 0.$$

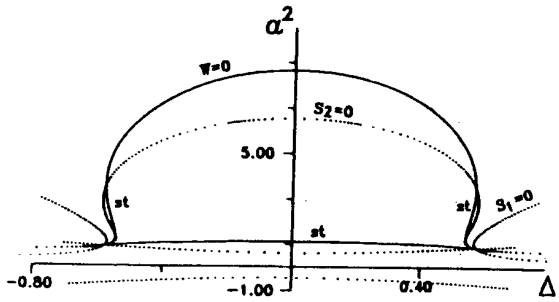

Fig. 15. Stability (st) branches of resonance curves in the case $q = 0.94$.

Using original equations (6) we can eliminate $\sin 2\theta, \cos 2\theta$ to get

$$S_2 = \frac{hk}{4}\left\{4ah\left(\frac{k}{4}a^2 + \frac{k}{2}q^2 - 1\right) + 3hkq^2a + 4\Delta q\sin\theta\right.$$

$$\left. + \left[3hkqa^2 + 4hq\left(\frac{k}{4}a^2 + \frac{k}{2}q^2 - 1\right)\right]\cos\theta\right\} > 0. \tag{12}$$

By substituting $\sin\theta = D_1/D, \cos\theta = D_2/D$, we obtain the second stability condition for the regular oscillation in the equivalence region. For other oscillations, we can use either form (12) or its corresponding limit form. For example, for special critical points corresponding to $\theta = 0$, the second stability condition (12) gives

$$S_2 = \frac{4}{3}h^2\sqrt{k} > 0.$$

However, it is more convenient to use the abbreviated form [2] of the second stability condition for the regular oscillation in the equivalence region:

$$S_2 = \frac{a}{TD}\frac{\partial W}{\partial a^2} > 0,$$

or

$$S_2 = \frac{ka}{16D}\frac{\partial W_0}{\partial a^2} > 0.$$

In Figs 15, 16, 17, the branches marked with "st" correspond to stable oscillations.

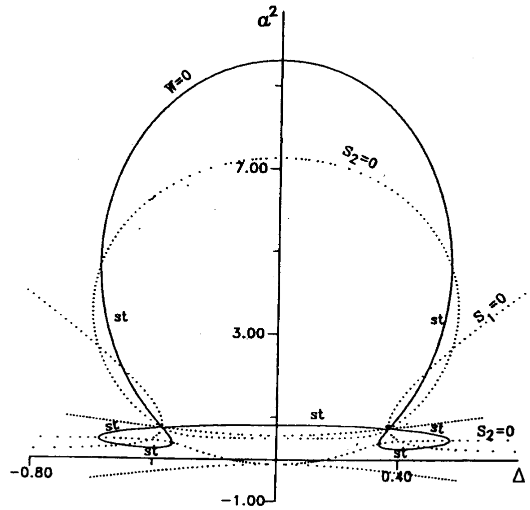

Fig. 16. Stability (st) branches of resonance curves in the case $q = 1.1$.

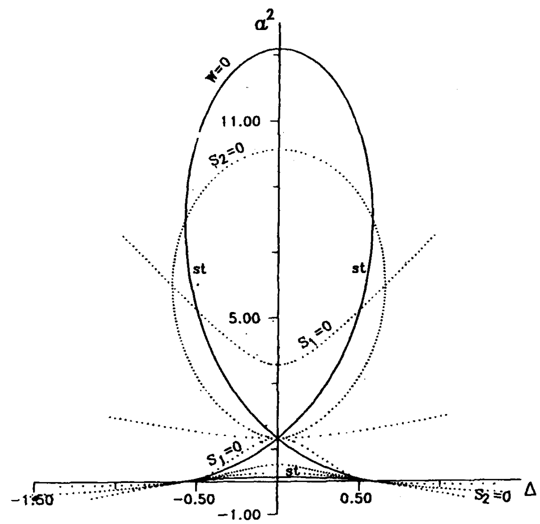

Fig. 17. Stability (st) branches of resonance curves in the case $q = 1.63$.

2. Van der Pol Oscillator under External and Quadratic Parametric Excitations

Let us consider a quasilinear system described by the differential equation

$$\ddot{x} + \omega^2 x = \varepsilon\{\Delta x - \gamma x^3 + h(1 - kx^2)\dot{x} + 2px^2 \cos \omega t + e \cos(\omega t + \sigma)\}. \tag{13}$$

The solution of equation (13) will be found in form (2). Applying the asymptotic method [1] we obtain the following averaged differential equations:

$$\dot{a} = \frac{-\varepsilon}{2\omega} f = \frac{-\varepsilon}{2\omega} \left\{ h\omega \left(\frac{k}{4} a^2 - 1 \right) a + \frac{1}{2} p a^2 \sin\theta + e \sin(\theta - \sigma) \right\},$$

$$a\dot{\theta} = \frac{-\varepsilon}{2\omega} g = \frac{-\varepsilon}{2\omega} \left\{ \Delta a - \frac{3}{4}\gamma a^3 + \frac{3}{2} p a^2 \cos\theta + e\cos(\theta - \sigma) \right\}.$$

The constant amplitude and dephase of stationary oscillations satisfy the equations:

$$f = A\sin\theta + B\cos\theta - E = 0,$$

$$g = G\sin\theta + H\cos\theta - K = 0,$$

where

$$A = \frac{1}{2} p a^2 + e\cos\sigma, \quad B = -e\sin\sigma, \quad E = -h\omega\left(\frac{k}{4}a^2 - 1\right),$$

$$G = e\sin\sigma, \quad H = \frac{3}{2} p a^2 + e\cos\sigma, \quad K = -\Delta a + \frac{3}{4}\gamma a^3.$$

Eliminating the phase θ gives the relationship for the frequency amplitude,

$$W(\Delta, a^2) = D_1^2 + D_2^2 - D^2 = a^2 \left\{ \left(\frac{3}{2} p a^2 + e\cos\sigma\right) h\omega \left(\frac{k}{4}a^2 - 1\right) + \left(\Delta - \frac{3}{4}\gamma a^2\right) e\sin\sigma \right\}^2$$

$$+ a^2 \left\{ \left(\frac{1}{2} p a^2 + e\cos\sigma\right)\left(\Delta - \frac{3}{4}\gamma a^2\right) - h\omega\left(\frac{k}{4}a^2 - 1\right) e\sin\sigma \right\}^2$$

$$- \left\{ \left(\frac{1}{2} p a^2 + e\cos\sigma\right)\left(\frac{3}{2} p a^2 + e\cos\sigma\right) + e^2\sin^2\sigma \right\}^2 = 0, \tag{14}$$

where

$$D = \left(\frac{1}{2} p a^2 + e\cos\sigma\right)\left(\frac{3}{2} p a^2 + e\cos\sigma\right) + e^2\sin^2\sigma,$$

$$D_1 = -a \left\{ \left(\frac{3}{2} p a^2 + e\cos\sigma\right) h\omega \left(\frac{k}{4}a^2 - 1\right) + \left(\Delta - \frac{3}{4}\gamma a^2\right) e\sin\sigma \right\},$$

$$D_2 = -a \left\{ \left(\frac{1}{2} p a^2 + e\cos\sigma\right)\left(\Delta - \frac{3}{4}\gamma a^2\right) - h\omega\left(\frac{k}{4}a^2 - 1\right) e\sin\sigma \right\}.$$

The resonance curve consists of two parts: the regular part C_1 and the critical one C_2.

The regular part C_1 satisfies (14) and lies in the regular region R_1,

$$D(\omega, a) \neq 0.$$

The critical part C_2 lies in the critical region R_2,

$$D(\omega, a) = 0 \quad \text{or} \quad \frac{3}{2}p^2a^4 + 2pa^2e\cos\sigma + e^2 = 0,$$

and satisfies the following:

the compatibility conditions,

$$D_1 = 0 \quad \text{or} \quad \left(\frac{3}{2}pa^2 + e\cos\sigma\right)h\omega\left(\frac{k}{4}a^2 - 1\right) + \left(\Delta - \frac{3}{4}\gamma a^2\right)e\sin\sigma = 0,$$

$$D_2 = 0 \quad \text{or} \quad \left(\frac{1}{2}pa^2 + e\cos\sigma\right)\left(\Delta - \frac{3}{4}\gamma a^2\right) - h\omega\left(\frac{k}{4}a^2 - 1\right)e\sin\sigma = 0;$$

the trigonometrical conditions,

$$A^2 + B^2 \geq E^2 \quad \text{or} \quad \left(\frac{1}{2}pa^2 + e\cos\sigma\right)^2 + e^2\sin^2\sigma \geq h^2\omega^2\left(\frac{k}{4}a^2 - 1\right)^2 a^2,$$

$$G^2 + H^2 \geq K^2 \quad \text{or} \quad e^2\sin^2\sigma + \left(\frac{3}{2}pa^2 + e\cos\sigma\right)^2 \geq \left(\Delta - \frac{3}{4}\gamma a^2\right)^2 a^2.$$

The critical region R_2 exists (in the upper half-plane $R(\Delta, a^2 > 0)$ if

$$\frac{5\pi}{6} \leq \sigma \leq \frac{7\pi}{6},$$

and consists of two straight lines D' and D'', respectively, of the ordinates:

$$D', D'' : a^2 = a_{1,2}^2 = \frac{4e}{3p}\left\{-\cos\sigma \mp \sqrt{\cos^2\sigma - \frac{3}{4}}\right\},$$

D' and D'' coincide if $\sigma = 5\pi/6$ or $\sigma = 7\pi/6$.

There exists another interesting property: the resonance curve always contains two points — denoted by J_0^-, J_0^+ — of abscissa not depending on h and situated on the line $a^2 = a_0^2 = 4/k$ (the amplitude of the purely self-excited oscillation).

Indeed, substituting $a^2 = a_0^2$ into (14) we obtain a quadratic equation for unknown Δ,

$$W(\Delta, a_0^2) = a_0^2\left\{e^2\sin^2\sigma + \left(\frac{1}{2}pa_0^2 + e\cos\sigma\right)^2\right\}\left(\Delta - \frac{3}{4}\gamma a_0^2\right)^2$$

$$- \left\{\left(\frac{1}{2}pa_0^2 + e\cos\sigma\right)\left(\frac{3}{2}pa_0^2 + e\cos\sigma\right) + e^2\sin^2\sigma\right\}^2 = 0, \qquad (15)$$

from which it follows that

$$\Delta = \frac{3}{4}\gamma a_0^2 \pm \bar{\Delta} = \frac{3}{4}\gamma a_0^2 \pm \frac{\left|\left(\frac{1}{2}pa_0^2 + e\cos\sigma\right)\left(\frac{3}{2}pa_0^2 + e\cos\sigma\right) + e^2\sin^2\sigma\right|}{a_0\sqrt{e^2\sin^2\sigma + \left(\frac{1}{2}pa_0^2 + e\sin\sigma\right)^2}},$$

$3\gamma a_0^2/4 \pm \bar{\Delta}$ are thus the abscissa of J_0^- and J_0^+ respectively. By J_0 we denote the segment

$$J_0^- J_0^+ \equiv J_0 : \frac{3}{4}\gamma a^2 - \bar{\Delta} \leq \Delta \leq \frac{3}{4}\gamma a^2 + \bar{\Delta}, \quad a^2 = a_0^2.$$

If $a_0^2 \neq a_1^2, a_2^2$, the two ends J_0^-, J_0^+ are located in the regular region R_1, thus they belong to the regular part C_1 of the resonance curve. If $a_0^2 = a_1^2$ or $a_0^2 = a_2^2$, the two points J_0^- and J_0^+ coincide, and the segment J_0 is reduced to a critical point (of C_2).

In the particular case $\sigma = \pi, a_0^2 = a_2^2 = 2e/p = 4/k$ if we substitute k with its value $2\pi/e$, the frequency amplitude relationship (14) can be written as

$$W(\Delta, a^2) = \frac{1}{e^2}\left(\frac{1}{2}pa^2 - e\right)^2 \times \left\{a^2\left(\frac{3}{2}pa^2 - e\right)^2 h^2\omega^2 + a^2e^2\left(\Delta - \frac{3}{4}\gamma a^2\right)^2 - e^2\left(\frac{3}{2}pa^2 - e\right)^2\right\} = 0. \quad (16)$$

Since $\left(1/2pa^2 - e\right)^2$ is a factor of W, the line $a^2 = a_0^2$ forms a branch of the curve $W = 0$. As will be seen below, on this line there exists a critical segment $J_1 J_2(|\Delta - (3/4)\gamma a^2| \leq \sqrt{2ep}; a^2 = a_0^2)$ which can be regarded as the aforesaid segment J_0 (substituting $\sigma = \pi$ into (15), in the limit $a^2 \to a_0^2 = 2e/p$, we obtain $\bar{\Delta} = \sqrt{2ep}$). Other branches of the resonance curve are given by

$$a^2\left(\frac{3}{2}pa^2 - e\right)^2 h^2\omega^2 + a^2e^2\left(\Delta - \frac{3}{4}\gamma a^2\right)^2 - e^2\left(\frac{3}{2}pa^2 - e\right)^2 = 0.$$

Since h is small enough, these branches intersect the segment $J_1 J_2$ at two points J', J'' of the abscissa determined by the equation

$$\left(\Delta - \frac{3}{4}\gamma a^2\right)^2 + 4h^2\left(\Delta - \frac{3}{4}\gamma a^2\right) + 4h^2 - 2ep = 0 \quad or$$

$$\Delta = \frac{3}{4}\gamma a^2 - 2h^2 \mp \sqrt{4h^4 - 4h^2 + 2ep}.$$

On the basis of formula (14) the resonance curves are plotted in Figs 18–27 for $\gamma = 0.15$ and in Fig. 28 for $\gamma = -0.15$. With the positive (negative) value of γ the resonance curve leans toward the right (left).

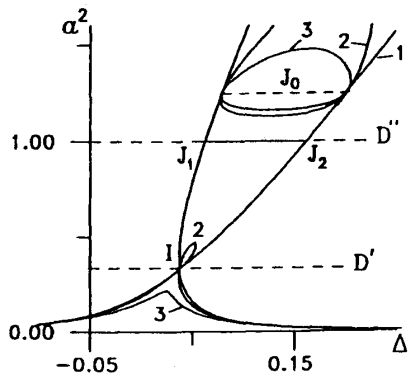

Fig. 18. The resonance curves for $\gamma = 0.15$, $\sigma = \pi$, $k = 3.2$, and different values of $h(0, 0.03, 0.05)$.

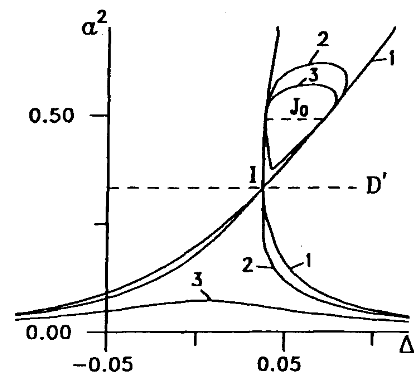

Fig. 19. The resonance curves for $\gamma = 0.15$, $\sigma = \pi$, $k = 8$, and different values of $h(0, 0.05, 0.1)$.

The case $\sigma = \pi$. If $k = 3.2$, we have $a_0^2 = 1.25 > 1 = a_2^2$ and the segment J_0 lies above the second critical line D'' . In Fig. 18 the curve *1* including its nodal point I together with the segment J_1J_2 form the resonance curve corresponding to $h = 0$, the resonance curves *2, 3* correspond to $h = 0, 0.03, 0.05$, respectively. We see that under the action of the self-excitation, the critical segment disappears and the resonance curve is divided into two branches, separated and located, respectively, above and under the critical line D''. Passing through two points J_0^-, J_0^+ and encircling the segment J_0, the upper branch, as h increases, is contracted and "tends" to J_0. For small h, the lower branch has a loop, tied at I. As h increases, this loop, always tied at I, becomes narrower. For $h = \bar{h} \approx 0.039$, the loop disappears, the nodal point I is degenerated into a returning point. Further increasing h, the lower branch leaves I (which now becomes an isolated point, not belonging to the resonance curve), moves down and approaches the abscissa axis Δ. Thus, when h is large enough, the amplitude level of the combined oscillation is the same as that of the purely self - excited one (the effect of the external and parametric excitations only results in the phase determination and the expansion of the resonance zone).

For $k = 8$, we have $a_0^2 = 1/2$, the segment J_0 is situated between the two critical lines D', D''. When $h = 0$, the form of the resonance curve is like that of its corresponding curve in the above presented case. When $h > 0$, the resonance curve is also divided into two branches, separated and respectively located above and under the critical line D''. The upper branch moves up as h increases. Various forms of the lower branch as shown in Fig. 19 (*1, 2, 3*) correspond to $h = 0, 0.05, 0.1$, respectively. We see that for small h, there is also a loop; for large h, the lower branch divides itself into two sub-branches, the upper branch is a single closed curve encircling and tending to the segment J_0.

For $k = 12$, we have $a_0^2 = 1/2 = a_1^2$; the segment J_0 is reduced to nodal point I. In Fig. 20 the curves *1, 2, 3* are lower branches of the resonance curve corresponding to $h = 0, 0.005, 0.1$. We see that, for a large enough h, the upper sub-branch takes the form of the number eight, tied at I.

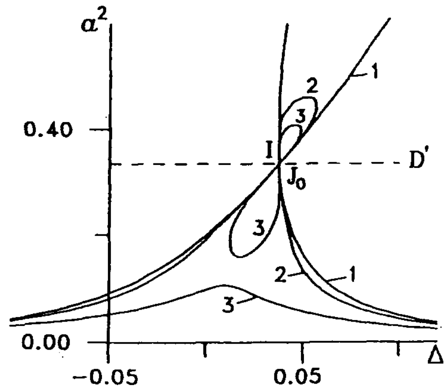

Fig. 20. The resonance curves for $\gamma = 0.15$, $\sigma = \pi$, $k = 12$, and different values of $h(0, 0.005, 0.1)$.

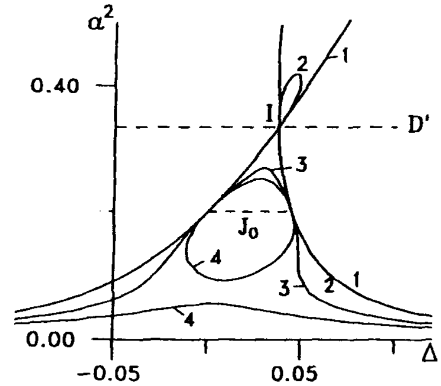

Fig. 21. The resonance curves for $\gamma = 0.15$, $\sigma = \pi$, $k = 20$, and different values of $h(0, 0.02, 0.1, 0.138)$.

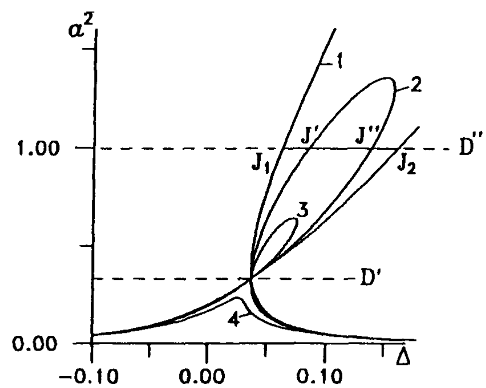

Fig. 22. The resonance curve for $\gamma = 0.15$, $\sigma = \pi, k = 4$, and different values of $h(0, 0.02, 0.03, 0.05)$.

For $k = 20$, we have $a_0^2 = 0,2 < a_1^2 = 1/3$, the segment J_0 is located under the first critical line $D^{'}$. In Fig. 21 (*1, 2, 3, 4*) are lower branches of the resonance curve corresponding to $h = 0, 0.02, 0.1, 0.138$, respectively.

Finally, let us examine a particular case $k = 4$. We have $a_0^2 = a_2^2 = 1$, the line $a^2 = a_2^2$ coincides with the critical line $D^{''}$ and the segment J_0 coincides with the critical segment J_1J_2. In Fig. 22, the resonance curves *1, 2, 3, 4* correspond to $h = 0, 0.02, 0.03, 0.05$, respectively. The critical segment J_1J_2 is a part of all the resonance curves. For small h, the segment J_1J_2 intersects other branches of the resonance curve at two points $J^{'}, J^{''}$ of abscissae determined by (2.14). For large h, there remains only J_1J_2 (the amplitude is absolutely constant for the whole resonance zone).

For the case $\boldsymbol{\pi \neq \sigma \in [5\pi/6, 7\pi/6]}$, let us choose $\sigma = 17\pi/8, p = 0.05, e = 0,025$. We have $a_1^2 \approx 0.34, a_2^2 \approx 0.97$ and two critical points I_1 and I_2 on D' and D'' appear.

In Fig. 23 , for $k = 3.2$, the resonance curves *1, 2, 3, 4* correspond to $h = 0, 0.006, 0.02, 0.05$, respectively; J_0 is above $D^{''}, I_1$ and I_2 move to the right.

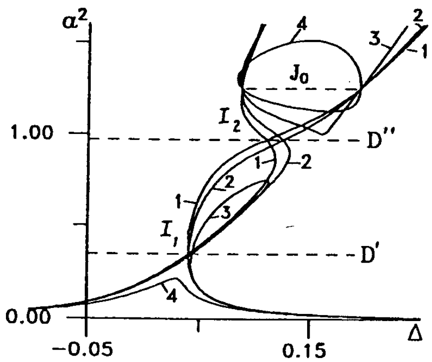

Fig. 23. The resonance curves for $\gamma = 0.15$, $\sigma = 17\pi/8$, $k = 3.2$, and different values of $h(0, 0.006, 0.02, 0.05)$.

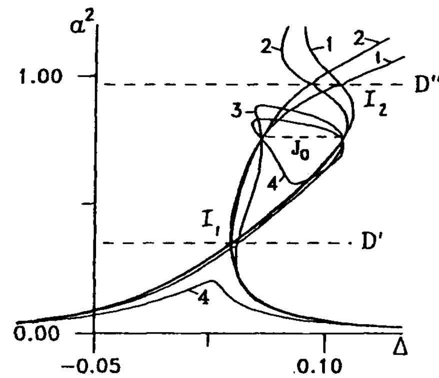

Fig. 24. The resonance curves for $\gamma = 0.15$, $\sigma = 17\pi/8$, $k = 5.3$, and different values of $h(0, 0.006, 0.02, 0.05)$.

In Fig. 24, for $k = 5.3$, the resonance curves *1, 2, 3, 4* correspond to $h = 0, 0.006, 0.03, 0.06$, respectively; J_0 is situated between $D^{'}$ and $D^{''}$, I_1 moves to the right while I_2 moves to the left.

In Fig. 25 for $k = 20$, the resonance curves *1, 2, 3, 4* correspond to $h = 0, 0.01, 0.08, 0.14$, respectively; J_0 is under $D^{'}$, I_1 and I_2 move to the left.

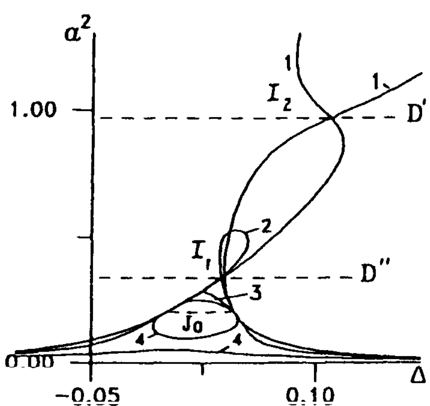

Fig. 25. The resonance curves for $\gamma = 0.15$, $\sigma = 17\pi/8$, $k = 20$, and different values of $h(0, 0.01, 0.08, 0.14)$.

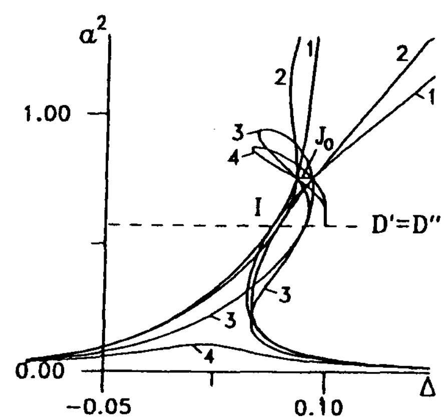

Fig. 26. The resonance curves for $\gamma = 0.15$, $\sigma = 5\pi/6$, $k = 5.3$, and different values of $h(0, 0.01, 0.05, 0.08)$.

In the case $\sigma = 5\pi/6$, we have a "double" critical line $D^{'} \equiv D^{''}$ and two points I_1 and I_2 coincide at I.

In Fig. 26, for $k = 5.3$, the resonance curves *1, 2, 3, 4* correspond to $h = 0, 0.01, 0.05, 0.08$, respectively; J_0 is above the double critical line, the critical point I moves to the right.

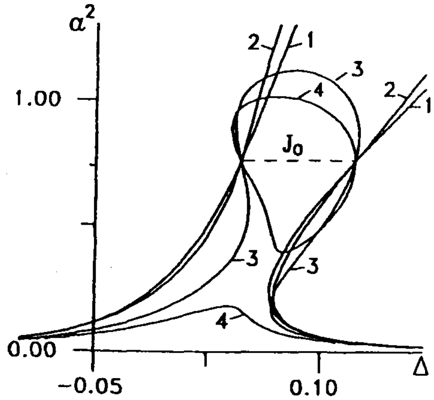

Fig. 27. The resonance curves for $\gamma = 0.15$, $\sigma = 2\pi/3, k = 5.3$, and different values of $h(0, 0.01, 0.05, 0.07)$.

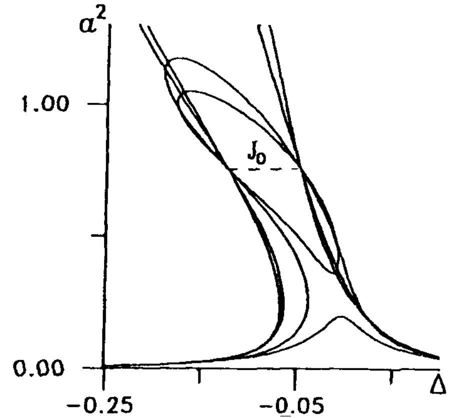

Fig. 28. The resonance curves for $\gamma = -0.15$, $\sigma = 2\pi/3, k = 5.3$, and different values of $h(0, 0.01, 0.05, 0.07)$.

For the case $\boldsymbol{\sigma \notin [5\pi/6,\ 7\pi/6]}$ the critical region, and consequently, the critical part C_2 does not exist. For $\sigma = 2\pi/3, p = 0.05, e = 0.025, k = 5.3$ in Fig. 27 ($\gamma = 0.15$) and Fig. 28 ($\gamma = -0.15$) the resonance curves *1, 2, 3, 4* correspond to $h = 0, 0.01, 0.05, 0.07$. When h is small, the resonance curve consists of two branches separated by the back-bone $\Delta = 3\gamma a^2/4$. These two branches approach each other as h increases. At a certain value $\bar{h}$, they will be joined at a singular point (belonging to C_1, not to C_2). As h exceeds $\bar{h}$, the above mentioned ordinary singular joint disappears at once, the resonance curve once again divides itself into two branches: the upper, encircling J_0, becomes increasing by narrower; the lower, moving down, runs near the Δ-axis.

Stability conditions. The two stability conditions are of well-known forms. The first stability condition can be transformed into

$$S_1 = a\frac{\partial f}{\partial a} + \frac{\partial g}{\partial \theta} = h\omega a(ka^2 - 2) > 0, \quad i.e., \quad a^2 > \frac{2}{k} = \frac{a_0^2}{2}.$$

The second stability condition can be written as

$$\begin{aligned} S_2 &= \frac{\partial f}{\partial a}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial a} \\ &= \left\{h\omega\left(\frac{3}{4}ka^2 - 1\right) + pa\sin\theta\right\}\left\{-pa^2\sin\theta + h\omega a\left(\frac{k}{4}a^2 - 1\right)\right\} \\ &\quad + \left\{pa^2\cos\theta + \left(\Delta - \frac{3}{4}\gamma a^2\right)a\right\}\left\{\Delta - \frac{3}{4}\gamma a^2 + 3pa\cos\theta\right\} > 0. \end{aligned}$$

This form can be used directly for studying the stability of critical stationary oscillations. For example, for the segment J_1J_2, among the critical stationary oscillations represented by J_1J_2 and having $\sin\theta < 0$, only those corresponding to subsegments J_1J' and $J''J_2$ are stable. For large h, J' and J'' disappear, the whole segment J_1J_2 with $\sin\theta < 0$ is stable.

For regular stationary oscillations, it is better to use the abbreviated form of the second stability condition [2],

$$S_2 = \frac{a}{D}\frac{\partial W}{\partial a^2} > 0.$$

Stable and unstable portions of the resonance curve can thus be obtained by examining the sign distribution of two functions $D(\Delta, a^2)$ and $W(\Delta, a^2)$ on the plane (Δ, a^2) [1].

We have identified various forms of the resonance curve of a system of the Van der Pol type, simultaneously subjected to an external and quadratic parametric excitations. Critical representative points corresponding to critical stationary oscillations have been shown to be useful to this end.

3. Regular Oscillation and Chaotic Phenomenon in a Forced Strongly Nonlinear Van der Pol's Oscillator

Let us consider a system described by the following differential equation:

$$\ddot{x} - \delta(1 - \beta x^2)\dot{x} + x + \gamma x^3 = Q\nu^2 \cos \nu t, \tag{17}$$

where parameters $\delta > 0, \beta > 0$ and γ, Q are constants and are not necessarily small quantities.

The self-excited oscillation in the system under consideration, when the external force is absent $Q = 0$, can be found by putting in (17)

$$x = a_0 \cos \omega_0 t, \quad \dot{x} = -a_0\omega_0 \sin \omega_0 t, \quad \ddot{x} = -a_0\omega_0^2 \cos \omega_0 t,$$

and equating the coefficients of the harmonics $\sin \omega_0 t$ and $\cos \omega_0 t$ separately to zero,

$$a_0^2 = \frac{4}{\beta}, \quad \omega_0^2 = 1 + \frac{3}{4}\gamma a_0^2 = 1 + \frac{3\gamma}{\beta}.$$

We see that the frequency ω_0 depends on the amplitude a_0 of self-oscillation. This results from the presence of the nonlinear restoring terms.

In the present article, the case we will be dealing with is the oscillation in a vicinity of the main resonance, so that ν is close to ω_0. The solution of (17) characterizing the forced oscillation can be approximated by the form

$$\begin{aligned} x &= a \cos \psi, \quad \dot{x} = -a\nu \sin \psi, \\ \ddot{x} &= -a\nu^2 \cos \psi, \quad \psi = \nu t - \varphi, \end{aligned} \tag{18}$$

where a is the amplitude of the forced oscillation and φ is the angle of the phase displacement relative to the forced excitation.

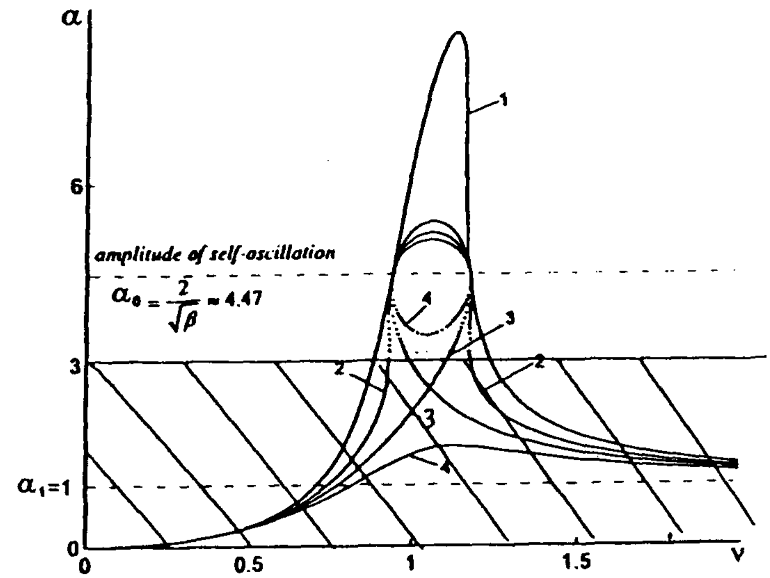

Fig. 29. The resonance curves for the case $Q = 1, \gamma = 0.005, \beta = 0.2$, and for various values of δ.

To find the equations for determining the amplitude a and phase φ, we use the method of harmonic balance which consists of substituting the assumed solution (18) into the equation of motion (17) and comparing the terms of the harmonic components $\cos\psi$ and $\sin\psi$. This gives a system of nonlinear algebraic equations with respect to a and φ,

$$\begin{aligned} a(1-\nu^2) + \frac{3}{4}\gamma a^3 &= Q\nu^2\cos\varphi, \\ \delta\left(1 - \frac{1}{4}\beta a^2\right) a\nu &= -Q\nu^2\sin\varphi. \end{aligned} \tag{19}$$

Eliminating the phase φ from these equations we obtain the following formula for the amplitude a:

$$W(a^2,\nu) = a^2\left[\left(1-\nu^2+\frac{3}{4}\gamma a^2)^2 + \delta^2\nu^2(1-\frac{1}{4}\beta a^2\right)^2\right] - Q^2\nu^4 = 0, \tag{20}$$

and the phase φ:

$$\operatorname{tg}\varphi = \frac{-\delta\nu\left(1-\frac{1}{4}\beta a^2\right)}{1-\nu^2+\frac{3}{4}\gamma a^2}. \tag{21}$$

The amplitude curves, giving the dependence of a on ν, are presented in Fig. 29 for $Q = 1, \gamma = 0.005, \beta = 0.2$, and for different values of $\delta, \delta = 0.05$ (curve *1*), $\delta = 0.45$ (curve *2*), $\delta = 0.60$ (curve *3*) and $\delta = 0.75$ (curve *4*) and in Fig. 30 for $Q = 1, \gamma = 0.005, \beta = 0.04$ and for $\delta = 0.05$ (curve *1*), $\delta = 0.23$ (curve *2*), $\delta = 0.276$ (curve *3*) and $\delta = 0.65$ (curve *4*).

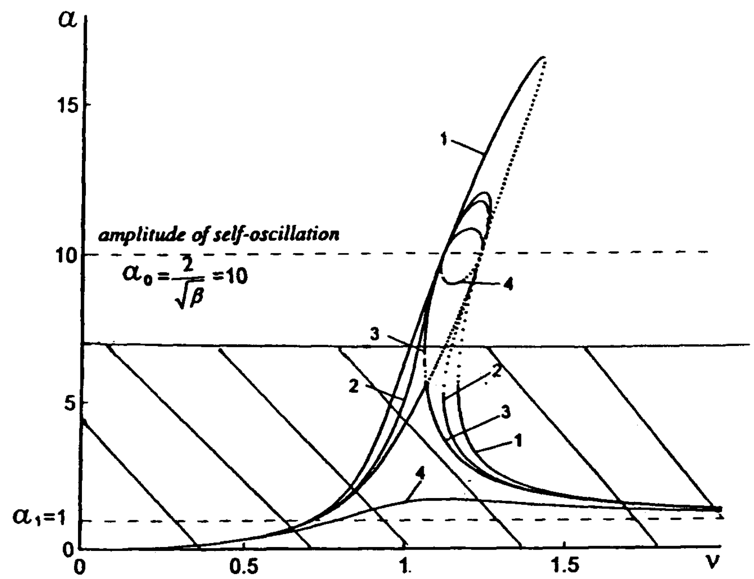

Fig. 30. The resonance curves for the case $Q = 1$, $\gamma = 0.005$, $\beta = 0.04$, and for various values of δ.

The value δ_* at which the resonance curve crosses itself can be calculated approximately as follows. The resonance curve (20) cuts the skeleton line

$$\nu^2 = 1 + \frac{3}{4}\gamma a^2$$

at the points with ordinate a_* satisfying the relationship

$$\delta^2 a_*^2 \nu^2 \left(1 - \frac{1}{4}\beta a_*^2\right)^2 = Q^2 \nu^4$$

or approximately ($\nu^2 \simeq 1$),

$$u = (1-u)^2 = F, \tag{22}$$

where

$$u = \frac{1}{4}\beta a_*^2, \quad F = \frac{\beta Q^2}{4\delta^2}.$$

Equation (22) has a double root relative to u, which corresponds to the crossing resonance curve on the skeleton line, when F is equal to $4/27$. So, the corresponding value δ_* is approximately,

$$\delta_* = \frac{Q}{4}\sqrt{27\beta}.$$

From Figs 29 and 30, one observes that with $Q = 1$ and with a small value of δ, the resonance curve has only one branch (see curve *1*) which reaches asymptotically the straight line $a = a_1 =$

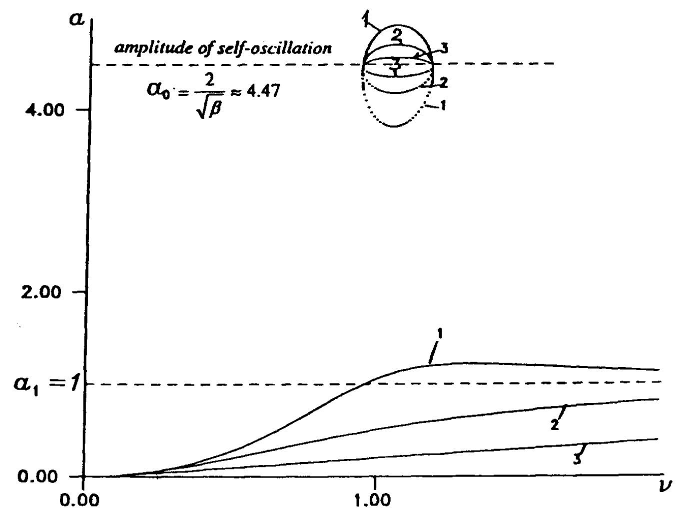

Fig. 31a. The resonance curves for the case $Q = 1, \gamma = 0.005, \beta = 0.2$, and very large values of δ.

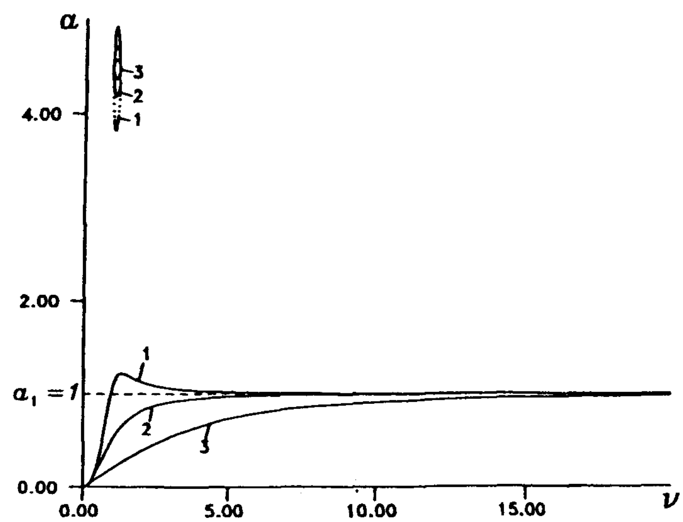

Fig. 31b. The resonance curves for the case $Q = 1$, $\gamma = 0.005, \beta = 0.2$, and very large values of δ (compressed form of Fig. 31a).

Q. When increasing δ, the resonance curve is deformed (curve *2*) then intersected (curve *3*). As δ increases further, the resonance curve separates into two branches (curves *4*); the upper branch has an oval form. The center of the oval has an ordinate which is equal to the amplitude a_0 of self-excitation, $a_0 = 2/\sqrt{\beta}$. The ratio γ/β in Fig. 30 is 5 times larger than in Fig. 29. The maximum of amplitudes in Fig. 30 is bigger than in Fig. 29 and the resonance curves bend to the right.

Figs 31a and 31b show resonance curves for $Q = 1$ for large values of δ. In these cases the lower branch goes down, approaching the straight line $a = a_1 = Q$ either from above (see

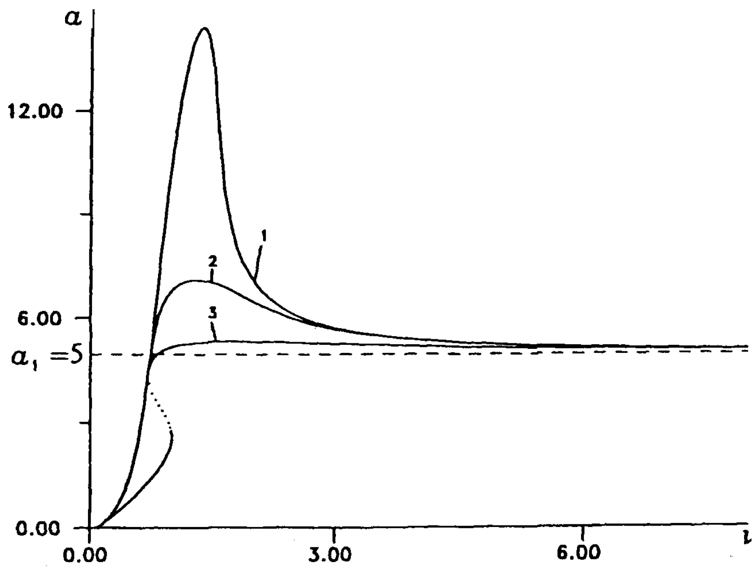

Fig. 32. The resonance curves for the case $\gamma = 0.005, \beta = 0.2$, very large values of the external force Q, $Q = 5$, and for various values of δ.

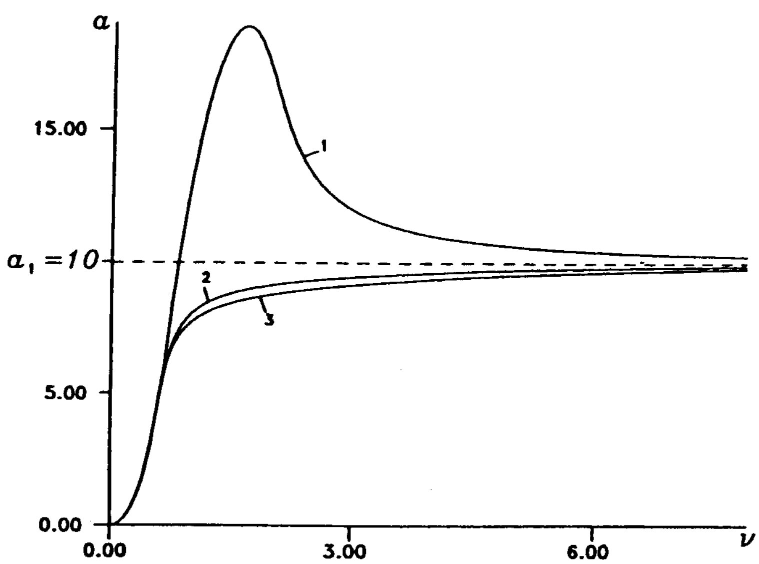

Fig. 33. The resonance curves for the case $\gamma = 0.005, \beta = 0.2$, very large values of the external force Q, $Q = 10$, and for various values of δ.

curve *1* for $\delta = 1$) or from below (see curve *2* for $\delta = 2$ and curve *3* for $\delta = 5$). The upper branch is an oval which is compressed as δ increases (see oval 1 for $\delta = 1$, oval 2 for $\delta = 2$, and oval 3 for $\delta = 5$).

Figure 32 represents resonance curves for large values of external force ($Q = 5$). In this case the oval no longer exists, the curves in Fig. 32 correspond to different values of the friction coefficient $\delta : \delta = 0.05$ (curve *1*), $\delta = 0.55$ (curve *2*) and $\delta = 2.9$ (curve *3*). These curves approach the straight line $a_1 = Q = 5$ from above.

Figure 33 gives resonance curves for an external force twice as large as that of Fig. 32 ($Q =$

10). The resonance curve has only one branch which approaches the straight line $a_1 = Q = 10$ either from above (curve *1* for $\delta = 0.05$) or from below (curve *2* for $\delta = 0.55$ and curve *3* for $\delta = 0.65$).

Stability of Stationary Oscillations. The stability of solution (18) can be determined by means of an equation in variation which is obtained by substituting $x = x_0 + \xi, \quad \dot{x} = \dot{x}_0 + \dot{\xi}$ into equation (17), where x_0 and $\dot{x}_0$ are of the form (18) with a and φ satisfying (19). We have the following equation in variation (the high degree terms relative to ξ are neglected):

$$\ddot{\xi} + F(t)\dot{\xi} + G(t)\xi = 0,$$

where

$$F(t) = -\delta(1 - \beta x_0^2) = -\delta\Big[1 - \frac{a^2}{2}\beta(1 + \cos 2\psi)\Big],$$

$$G(t) = 1 + 2\delta\beta x_0\dot{x}_0 + 3\gamma x_0^2 = 1 - \delta\beta a^2 \sin 2\psi + \frac{3\gamma a^2}{2}(1 + \cos 2\psi).$$

Using the transformation to the new variable η by the formula

$$\xi = \Big\{\exp\Big[-\frac{1}{2}\int F(\tau)d\tau\Big]\Big\}\eta,$$

we get

$$\ddot{\eta} + (\theta_0 + 2\theta_{1c}\cos 2\psi + 2\theta_{1s}\sin 2\psi + 2\theta_{2c}\cos 4\psi)\eta = 0, \tag{23}$$

where

$$\theta_0 = 1 + \frac{3}{2}\gamma a^2 - \frac{\delta^2}{4}\left[\Big(1 - \frac{1}{2}\beta a^2\Big)^2 + \frac{1}{8}\beta^2 a^4\right],$$

$$\theta_{1c} = \frac{3}{4}\gamma a^2 + \frac{1}{8}\delta^2\beta a^2\Big(1 - \frac{1}{2}\beta a^2\Big),$$

$$\theta_{1s} = -\frac{1}{4}\delta\beta\nu a^2,$$

$$\theta_{2c} = -\frac{\delta^2}{64}\beta^2 a^4.$$

Equation (23) belongs to Hill's differential equation. As is already well known [3], the stability conditions for periodic solution (18) of equation (17) are

1) $\quad 0 < h = 1/2\langle F(t)\rangle = -\delta(1 - \beta a^2/2)/2, \qquad (24)$

where $\langle F\rangle$ is the averaged value of the function $F(t)$,

2) $\quad (\theta_0 - n^2\nu^2)^2 + 2(\theta_0 + n^2\nu^2)h^2 + h^4 > \theta_n^2 = \theta_{nc}^2 + \theta_{ns}^2, \quad n = 1, 2, \qquad (25)$

or in an open form,

$$\left\{1 - n^2\nu^2 + \frac{3}{2}\gamma a^2 - \frac{\delta^2}{4}\left[\left(1 - \frac{1}{2}\beta a^2\right)^2 + \frac{1}{8}\beta^2 a^4\right]\right\}^2$$

$$+ \frac{1}{2}\delta^2\left\{1 + n^2\nu^2 + \frac{3}{2}\gamma a^2 - \frac{\delta^2}{4}\left[(1 - \frac{1}{2}\beta a^2)^2 + \frac{1}{8}\beta^2 a^4\right]\right\}(1 - \frac{\beta}{2}a^2)^2$$

$$+ \frac{\delta^4}{16}\left(1 - \frac{\beta}{2}a^2\right)^4 > \theta_n^2, \quad n = 1, 2.$$

***N*ote.**

1. Condition (24) gives

$$a > \sqrt{\frac{2}{\beta}}. \tag{26}$$

2. When the parameters γ, δ are sufficiently small for the terms with smallness higher than two relative to γ, δ to be neglected, the stability condition (25) is satisfied for $n = 2$. This condition, in the case $\nu \simeq 1$ and $n = 1$, can be presented in the form

$$\left(1 - \nu^2 + \frac{3}{2}\gamma a^2\right)^2 + \delta^2\nu^2\left[\left(1 - \frac{1}{2}\beta a^2\right)^2 - \frac{1}{16}\beta^2 a^4\right] - \frac{9}{16}\gamma^2 a^4 > 0,$$

or

$$\frac{\partial W}{\partial a^2} > 0, \text{ or the same, } \frac{\partial W}{\partial a} > 0, \text{ because } \quad a > 0,$$

where $W = 0$ is the equation for the resonance curve (20). Using the rule stated in [1] we can easily identify stable branches of the resonance curve. In the figures presented, the stable branches are shown by heavy solid lines, while the unstable branches are shown by dash lines. The hatching in the figures shows the unstable region in which the condition (26) is not satisfied.

Because the problem is interesting for large values of δ and a, the stability condition (25) should therefore be considered for concrete values of the parameters.

Transitional Oscillation. In the oscillatory system described by equation (17) the phenomenon of synchronization of harmonic oscillation occurs when the initial point is given inside the region containing the stable branches of the resonance curve. The oscillation starting from the unstable zone of the resonance curve will be in a transitional regime. After some instants, this oscillation reaches the periodic stationary and stable regime.

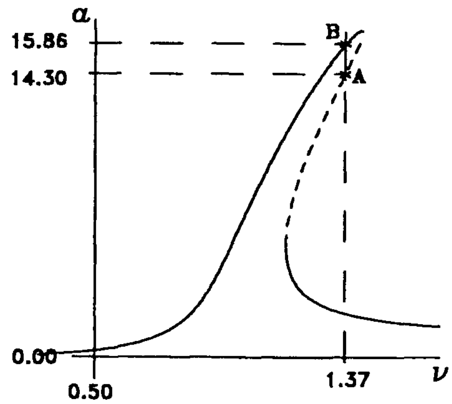

Fig. 34. The resonance curve for $Q = 1$, $\gamma = 0.005, \beta = 0.04, \delta = 0.05$.

Let us consider some cases of oscillations originating from the unstable branch of the resonance curve. Figure 34 represents the resonance curve for the case $Q = 1,\ \gamma = 0.005,\ \beta = 0.04,\ \delta = 0.05$. The point $A(\nu_0 = 1.37,\ a_0 = 14.30)$ lies on the unstable branch. Now, we construct the phase trajectory, which departs from A, of the equations

$$\begin{aligned} \frac{dx}{dt} &= \dot{x}, \\ \frac{d\dot{x}}{dt} &= \delta(1 - \beta x^2)\dot{x} - x - \gamma x^3 + Q\nu^2 \cos \nu t, \end{aligned} \tag{27}$$

which are equivalent to equation (17). The behaviour of the system under consideration is described by the movement of a representative point $(x(t), \dot{x}(t))$ along the solution curves of Eqs. (27) in the $(x, \dot{x})$ plane.

The phase φ of oscillation is determined by formulas (21). The corresponding initial conditions $t_0 = 0,\ x_0 = x(0),\ \dot{x}_0 = \dot{x}(0)$ are found and the Cauchy problem relative to the nonlinear differential equation (17) is solved by using the Runge – Kutta method. The Maple and Fortran power station software have been used.

Figure 35 gives the phase trajectory of oscillation with the initial amplitude and frequency which correspond to point A of Fig. 34. When $t \to \infty$ this trajectory asymptotically approaches the limit cycle with a stable amplitude $a = 15.86$. This limit cycle is called **an attractor**. Point S in the Fig. 36 is the stroboscopic point at the instants $t = nT(n = 1, 2 \ldots$, T is the period of the external force). All stroboscopic points coincide at S because the motion is periodic. The point S is a sink (by Ueda's concept [4]). This means that there exists a neighbourhood of the point S, around which all stroboscopic points should be attracted.

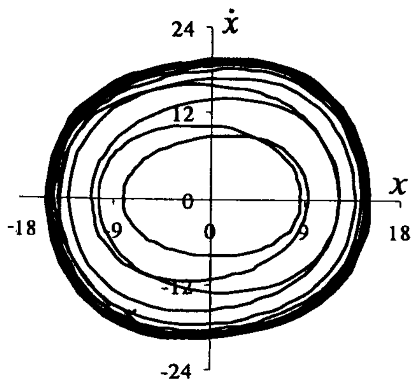

Fig. 35. The phase trajectory corresponding to the point A of Fig. 24.

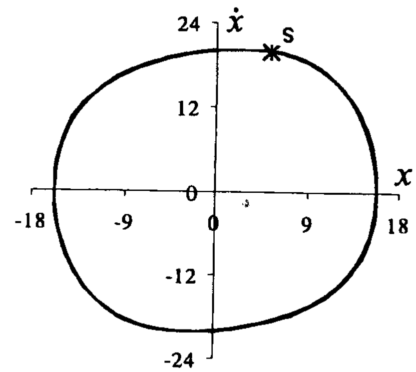

Fig. 36. The limit cycle and the stroboscopic point S at the instants nT.

Chaotic motion. In the case considered above with the chosen initial conditions, we have regular motions. The largest Lyapunov's exponent is negative.

Now, let us study the motion with the initial conditions $t_0 = 0, x_0 = 1, \dot{x}_0 = 0$, keeping the same parameters of Fig. 35. The largest Lyapunov's exponent is positive ($\lambda \backsimeq 0.008$) and therefore we have a chaotic motion [5].

Figs 37 and 38 give phase trajectories in the first seventy periods and the first one hundred and fifty periods, respectively.

The chaotic attractor is shown in Fig. 39. It is the set of stroboscopic points at the instants $t = nT$ $(T = 2\pi/1.37)$. The enlargement of a part of this chaotic attractor can be observed in Fig. 40.

Figs 41 and 42 are time dependents of the motion corresponding to the phase trajectories 37 and 38.

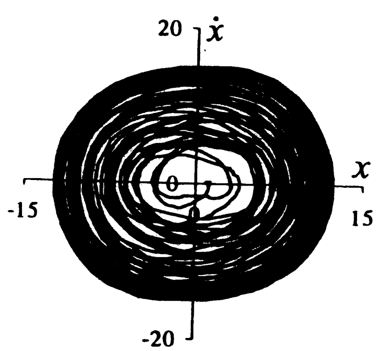

Fig. 37. Phase trajectories in the first seventy periods.

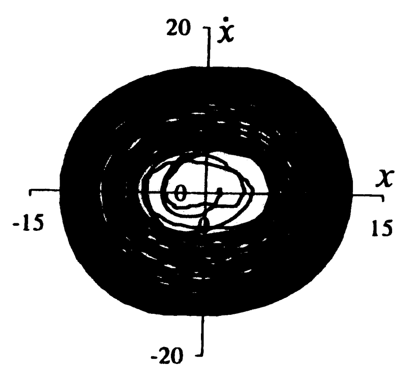

Fig. 38. Phase trajectories in the first one hundred and fifty periods.

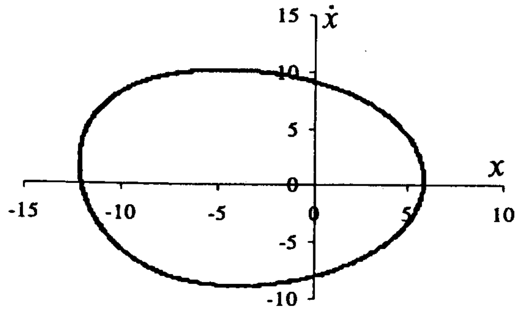

Fig. 39. The chaotic attractor.

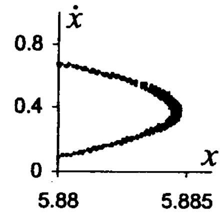

Fig. 40. The enlargement of a small region of the chaotic attractor.

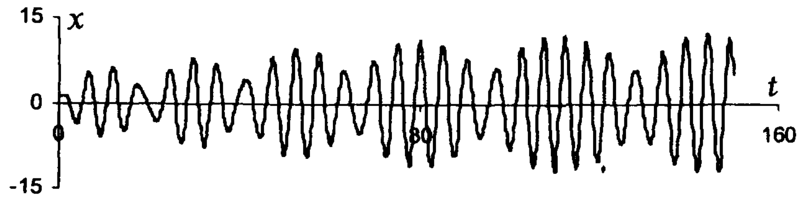

Fig. 41. The transitional process.

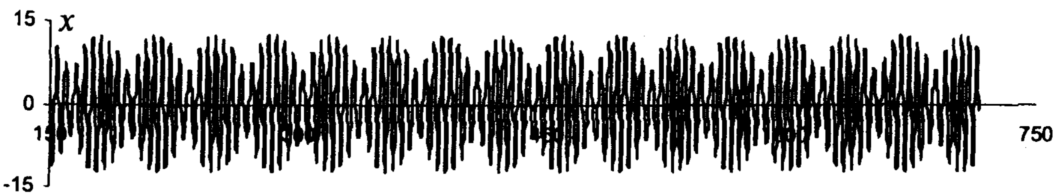

Fig. 42. The steady state after transition.

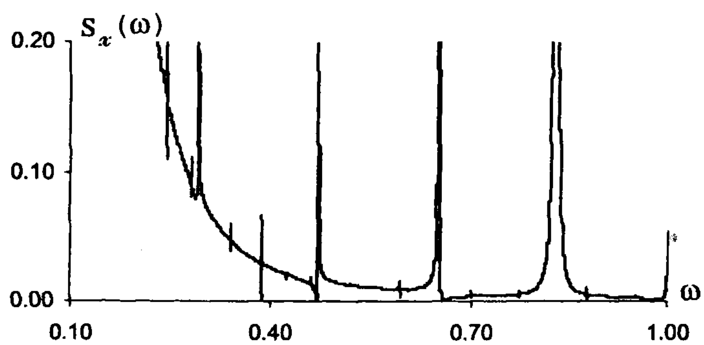

Fig. 43. The average power spectrum of the chaotic motion.

Figure 43 shows the average power spectrum in which there are continuous power spectra representing the chaotic nature of the motion.

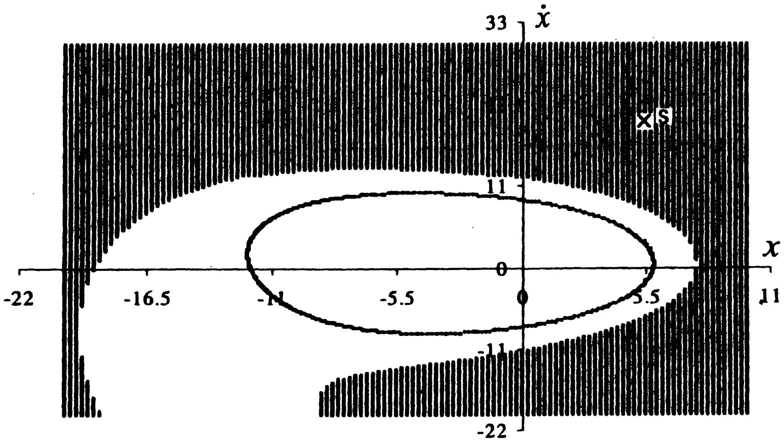

Fig. 44. The basins of atraction in the space of initial conditions.

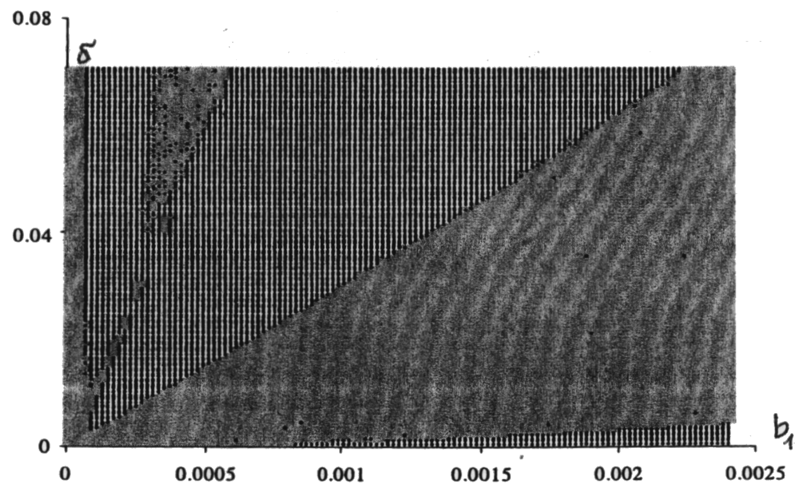

Fig. 45. The diagram of districts in the plane of parameters δ and $b_1 = \delta\beta$ with the initial conditions $t_0 = 0, x_0 = 1, \dot{x}_0 = 0$. The shaded area corresponds to regular motions, the grey area to chaotic motions.

Figure 44 shows basins of attraction in the space of initial conditions, in which the set A is a chaotic attractor. This set, together with the surrounding region M, makes a basin of attraction, corresponding to chaotic motions. The shaded region is the basin of attraction, corresponding to regular motions. Note that Fig. 44 could not be called an "attractor — basin phase portrait"[4] because we have not yet exhaustively studied all motions occuring in equations (27). Here we simply described qualitatively the basins of attraction.

Figure 45 is the diagram of districts in the plane of parameters δ and $b_1 = \delta\beta$ with the initial conditions $t_0 = 0, x_0 = 1, \dot{x}_0 = 0$. The shaded area (bold vertical lines) corresponds to regular motions and the grey area, except for some isolated black points, corresponds to the chaotic motion. Isolated black points in the grey area are doubt points, which correspond to either regular or chaotic motions and should be investigated very carefully. The results of investigation depend strongly on the exactness and the convergence of the computational process.

4. Concluding Remarks

The nonlinear phenomena observed through the resonance curves are more diverse and interesting than those of a simple classical oscillating systems. These curves were obtained by using analytical methods and numerical calculations.

The resonance curves for a strongly nonlinear Van der Pol's oscillator in a forced harmonic regime have been drawn for rather large values of the parameter δ, characterizing the nonlinear friction. In comparison with the case of a small value of δ [6], the resonance curve for $Q = 1$ and for a large value of δ consists of two parts, one of which has an oval form with a center at ordinate a_0 of self-oscillation. For large values of the external force (Q) there no longer exists an oval branch of the resonance curve. It has only one branch approaching the straight line $a_1 = Q$. The transitional motion from an unstable regime to a stable one is examined.

The chaotic phenomenon in the forced strongly nonlinear Van der Pol's oscillator, described by equation (17), has been studied for the parameters $\delta = 0.05, \beta = 0.04, \nu = 1.37, Q = 1$, and $\gamma = 0.005$.

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