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ON EQUILIBRIUM EQUATIONS OF CYLINDRICAL SHELL WITH ATTACHED RIGID BODY

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The mechanical system consisting of a circular cylindrical shell and a rigid body attached to one of the shell ends is considered. In linear statements, the boundary-value problem on a stressedly-deformed state of this system under concentraited and distributed loads is formulated. The equations obtained can also be used for a study of free oscillations of the considered construction if one replaces the applied loads with forces of inertia and their moments.

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1. The Equations of Equilibrium and the Boundary Conditions for the System "Body-Shell"

Today, the necessity of calculation of the stressedly-deformed state and dynamical characteristics of the mechanical system consisting of a circular cylindrical shell and a rigid body attached to one of the shell ends arises in different fields of science and engineering. Below a linear mathematical model of equilibrium state of the considered mechanical system under loads of general form is consructed on the base of the shell theory proposed by V. Z. Vlasov. The equations of the disturbed state of a prestressed, flexible, rotational shell with a rigid concentric inclusion in the form of elastic disk are obtained in paper [1].

Let us consider a mechanical system consisting of a thin, circular, cylindrical shell and a perfactly rigid body which is rigidly attached to one of the shell ends. Suppose that the other shell end is fixed in a certain way. Let the body have two symmety planes whose line of intersection, Oz coincides with the shell longitudinal axis. The coordinate plane Oxz is combined with one of the symmetry planes of the rigid body, and the origin O of the coordinate system $Oxyz$ is placed in the plane of the end section of the shell which is free of the rigid body.

To describe the body displaycements let us introduce an orthogonal coordinate system $Cx_cy_cz_c$ with its origin C placed at the center of inertia of the body. Its axes, Cx_c and Cy_c, are parallel to the axes Ox and Oy respectivly. The unit vectors of the coordinate system $Cx_cy_cz_c$ are denoted by $\vec{i}_c, \vec{j}_c, \vec{k}_c$. A median surface of the cylindrical shell is referred to the orthogonal curvilinear coordinate system of z and φ, where φ is the polar angle counted from Ox axis. With this coordinate system we connect a local orthogonal basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ the unit vectors $\vec{e}_1$,$\vec{e}_2$ of which are tangent to the principal curvature lines of the shell's median surface and are directed in direction of increase of the coordinates z and φ. The vector $\vec{e}_3 = [\vec{e}_1 \times \vec{e}_2]$.

Suppose that the considered construction is under a small load of the most general form, namely the rigid body is under the force $\Delta\vec{F} = \Delta F_1\vec{i}_c + \Delta F_2\vec{j}_c + \Delta F_3\vec{k}_c$ concentraited in the point C and under the moment relative to the point C, $\Delta\vec{M} = \Delta M_1\vec{i}_c + \Delta M_2\vec{j}_c + \Delta M_3\vec{k}_c$. In turn, the shell is under a distributed load $\Delta\vec{Q} = \Delta Q_1\vec{e}_1 + \Delta Q_2\vec{e}_2 + \Delta Q_3\vec{e}_3$. As a result the system will come to a disturbed equilibrium state and be subjected to strains and displacements. We shall characterize this equilibrium state by the displacement vector of points of the shell's

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median surface, $\vec{u} = u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3$, by the displacement vector of the center of mass of the body, $\vec{u}_0 = u_{01}\vec{i}_c + u_{02}\vec{j}_c + u_{03}\vec{k}_c$, and by the vector of turning angle around this center, $\vec{\theta}_0 = \theta_{01}\vec{i}_c + \theta_{02}\vec{j}_c + \theta_{03}\vec{k}_c$. In addition, we suppose that the displacements of the rigid body and the shell are so small that one can neglect the terms of the second and higher order of smallness in comparison with the terms of the first order.

To describe the stressedly-deformed state of the cylindrical shell, we shall use the shell theory based on the Kirchhoff – Love hypotheses. The moment shell theory based on this hypotheses is successfully applied for solving statics and dynamics problems. But one must be careful in applying this theory, since the corresponding boundary-value problem may be not self-adjoint and thus an input problem may not be formulated in the form of the corresponding variational principle. In this case, when the problem of free oscillations is solved, one cannot guarantee the reality of the natural frequencies. In addition, the difficulties arise when one formulates the conditions of orthogonality of the natural modes which play an essential role in calculation of shell constructions and in investigation of a response of this constructions to arbitrary perturbations. These difficulties may be overcomed within the framework of the Kirchhoff – Love hypotheses as well by means of writing equations that would lead to a self-adjoint boundary problem. For this purpose it is necessary to choose a certain variant of the elastic relations that do not contradict the sixth equation of equilibrium [2]. But such an approach leads to a complication of equations of the shell theory that hampers the construction of their solutions. In this sense, a variant of the engineering shell theory worked out by V. Z. Vlasov [3] is more preferable. It is the simplest shell theory which, along with a satisfactory accuracy, leads to the self-adjoint boundary-value problems that permits to obtain equations of this theory from variational principle. The last circumstance opens perspectives in applications of the energy method for solving the formulated boundary-value problems.

Thus, according to the engineering thin shell theory the tangential components of the bend deformations is neglected and the equilibrium equations of a cylindrical shell element are represented as follows [2]:

$$\frac{\partial T_1}{\partial z} + \frac{1}{R}\frac{\partial S}{\partial \varphi} + \Delta Q_1 = 0,$$

$$\frac{\partial S}{\partial z} + \frac{1}{R}\frac{\partial T_2}{\partial \varphi} + \Delta Q_2 = 0, \tag{1}$$

$$D\Delta\Delta W + \frac{1}{R}T_2 - \Delta Q_3 = 0,$$

where

$$D = \frac{Eh^3}{12\,(1-\nu^2)}, \quad \Delta = R^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \varphi^2};$$

E, ν, h, and R is the elastic module, the Poisson's ratio, the thickness and the radius of the shell; T_1 , T_2, and S is the meridional force, the ring force, and the shering force, which is refered to the unit of length of the normal section of the shell's median surface.

The forces and the moments in the normal sections of shell are associated with the components of a median surface deformation and with the parameters of change of the median surface curvature by

$$
\begin{gathered}
T_1 = \frac{Eh}{1-\nu^2}\left(\varepsilon_1 + \nu\varepsilon_2\right), \quad T_2 = \frac{Eh}{1-\nu^2}\left(\varepsilon_2 + \nu\varepsilon_1\right), \quad S = \frac{Eh}{2\left(1+\nu\right)}\omega, \\
M_1 = D\left(\chi_1 + \nu\chi_2\right), \quad M_2 = D\left(\chi_2 + \nu\chi_1\right), \quad M_{12} = D\left(1-\nu\right)\chi_{12},
\end{gathered}
\tag{2}
$$

where M_1, M_2, M_{12} is the linear bending moment in the meridional plane, the linear peripheral moment, and the linear torque, respectively.

The linear lateral force must by calculated with by the formula

$$
Q_1 = D\left[\frac{(1-\nu)}{R}\frac{\partial \chi_{12}}{\partial \varphi} + \frac{\partial \chi_1}{\partial z} + \nu\frac{\partial \chi_2}{\partial z}\right]. \tag{3}
$$

In turn, six components of the median surface deformation of the shell are expressed in terms of its displacements in the following way:

$$
\begin{gathered}
\varepsilon_1 = \frac{\partial u}{\partial z}, \quad \varepsilon_2 = \frac{1}{R}\left(\frac{\partial v}{\partial \varphi} + w\right), \quad \omega = \frac{1}{R}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial z}, \\
\chi_1 = -\frac{\partial^2 w}{\partial z^2}, \quad \chi_2 = -\frac{1}{R^2}\frac{\partial^2 w}{\partial \varphi^2}, \quad \chi_{12} = -\frac{1}{R}\frac{\partial^2 w}{\partial z \partial \varphi}.
\end{gathered}
\tag{4}
$$

If in equations (1) one expresses the forces according to elastic equations (2) and replaces deformations for displacements (4), the equilibrium equations of the circular cylindrical shell in displacements will be obtained. It is convenient to reduce this system of equations to the following matrix form:

$$
L\vec{u} = \vec{g}. \tag{5}
$$

Here

$$
L = \begin{Vmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{Vmatrix}, \quad \vec{u} = \begin{Vmatrix} u \\ v \\ w \end{Vmatrix}, \quad \vec{g} = \frac{1-\nu^2}{Eh}\begin{Vmatrix} -\Delta Q_1 \\ -\Delta Q_2 \\ \Delta Q_3 \end{Vmatrix},
$$

$$
L_{11} = \frac{\partial^2}{\partial z^2} + \frac{\nu_1}{R^2}\frac{\partial^2}{\partial \varphi^2}, \quad L_{12} = L_{21} = \frac{\nu_2}{R}\frac{\partial^2}{\partial z \partial \varphi}, \quad L_{13} = L_{31} = \frac{\nu}{R}\frac{\partial}{\partial z},
$$

$$
L_{22} = \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2} + \nu_1\frac{\partial^2}{\partial z^2}, \quad L_{23} = L_{32} = \frac{1}{R^2}\frac{\partial}{\partial \varphi}, \quad L_{33} = \frac{1}{R^2}\left(c^2\Delta\Delta + 1\right),
$$

$$c^2 = \frac{h^2}{12R^2}, \quad \nu_1 = \frac{1-\nu}{2}, \quad \nu_2 = \frac{1+\nu}{2}.$$

Let us turn to deriving the equilibrium equations of the rigid body. To this end, let us calculate the forces and the moments, relative to the point C, which act on the body. The elastic forces and moments (referenced to the unit of length of the section of the shell's median surface) will act on the edge of the cylindrical shell; they are of the following form [2]:

$$\vec{T} = -\left(T_1\vec{e}_1 + S\vec{e}_2 + Q_1\vec{e}_3\right), \quad \vec{M} = M_{12}\vec{e}_1 - M_1\vec{e}_2.$$

However, by analogy with the bending plate theory and starting from the Kirchhoff kinematic hypothesis, one can establish that the torque M_{12} on the shell edge is statically equivalent to the lateral force distributed along the contour and its intensity is described by the expression $R^{-1}\partial M_{12}/\partial\varphi$ [4]. With regard to this circumstance, the linear forces and the moments acting on the body and coming from the shell will be equal to

$$\vec{T} = -\left(T_1\vec{e}_1 + S\vec{e}_2 + Q_1^*\vec{e}_3\right), \quad \vec{M} = -M_1\vec{e}_2. \tag{6}$$

Here Q_1^* is a generalized lateral force on the shell contour calculated, with regard to relations (3) and (4), according to the formula

$$Q_1^* = Q_1 + \frac{1}{R}\frac{\partial M_{12}}{\partial\varphi} = -c^2\frac{Eh}{1-\nu^2}\left[R^2\frac{\partial^3 w}{\partial z^3} + (2-\nu)\frac{\partial^3 w}{\partial z\partial\varphi^2}\right]. \tag{7}$$

Taking into account the relation between the Darboux unit vectors and the unit vectors of the coordinate system $Cx_cy_cz_c$, which are of the form

$$\vec{e}_1 = \vec{k}_c,$$

$$\vec{e}_2 = -\sin\varphi\vec{i}_c - \cos\varphi\vec{j}_c, \tag{8}$$

$$\vec{e}_3 = \cos\varphi\vec{i}_c - \sin\varphi\vec{j}_c,$$

we shall represent vectors (6) as expansions relatively to the unit vectors $\vec{i}_c, \vec{j}_c, \vec{k}_c$. In this case we shall have

$$\vec{T} = \left(S\sin\varphi - Q_1^*\cos\varphi\right)\vec{i}_c + \left(Q_1^*\sin\varphi + S\cos\varphi\right)\vec{j}_c - T_1\vec{k}_c,$$

$$\vec{M} = M_1\sin\varphi\vec{i}_c + M_1\cos\varphi\vec{j}_c.$$

Denote futher the distance along the axis Oz from the point C to the shell end section to which the rigid body is attached by l_c. Then for the resulting moment $\vec{M}_c^y$, which is relative to point C, of the elastic forces acting on the rigid body, we shall obtain the following expression

$$M_c^y = \oint\limits_L \left[\vec{r}_0 \times \vec{T}\right] ds = \vec{i}_c \oint\limits_L \left[T_1 R \sin\varphi + l_c \left(Q_1^* \sin\varphi + S\cos\varphi\right)\right] ds$$

$$+ \ \vec{j}_c \oint\limits_L \left[T_1 R \cos\varphi + l_c \left(Q_1^* \cos\varphi - S\sin\varphi\right)\right] ds + \vec{k}_c \oint\limits_L RSds, \tag{9}$$

where $\vec{r}_0 = (R\cos\varphi)\,\vec{i}_c - (R\sin\varphi)\,\vec{j}_c - l_c\vec{k}_c$ is the radius vector of points of the shell's edge contour in the coordinate system $Cx_cy_cz_c$; L is the contour formed by the cross-section of the median shell surface at $z = l$; s is the length of the contour arc; l is the length of the cylindrical shell. After calculation of the resulting vector of all forces acting on the rigid body we arrive at the following three scalar relations:

$$\oint\limits_L \left(Q_1^* \cos\varphi - S\sin\varphi\right) ds = \Delta F_1,$$

$$\oint\limits_L \left(Q_1^* \sin\varphi + S\cos\varphi\right) ds = -\Delta F_2, \tag{10}$$

$$\oint\limits_L T_1 ds = \Delta F_3.$$

Similarly, with regard to expression (9) and using the condition that the resulting moment relative to the center of mass of the rigid body equals zero, we shall obtain another equations, namely,

$$\oint\limits_L \left[T_1 R \sin\varphi + M_1 \sin\varphi + l_c \left(Q_1^* \sin\varphi + S\cos\varphi\right)\right] ds = -\Delta M_1,$$

$$\oint\limits_L \left[M_1 \cos\varphi + T_1 R \cos\varphi + l_c \left(Q_1^* \cos\varphi - S\sin\varphi\right)\right] ds = -\Delta M_2, \tag{11}$$

$$\oint\limits_L SRds = -\Delta M_3.$$

The boundary conditions on the shell contour at $z = 0$ should be added to equations (5), (10), (11), as well as the relations connecting the displacements and the angles of rotation of the shell with the corresponding generalized coordinates of the rigid body in the place of the shell and body binding.

The equality of the displaycements of the shell and the rigid body on the contour L leads to the relation

$$\vec{u} = \vec{u}_0 + \left[\vec{\theta}_0 \times \vec{r}_0\right]. \tag{12}$$

Let us express the right-hand side of expression (12) in the form an expansion with respect to the unit vectors of the Darboux trihedron, taking into account their connection with the unit vectors of the coordinate system $Cx_cy_cz_c$, in the form of

$$\vec{i}_c = -\sin\varphi\vec{e}_2 + \cos\varphi\vec{e}_3,$$
$$\vec{j}_c = -\cos\varphi\vec{e}_2 - \sin\varphi\vec{e}_3, \tag{13}$$
$$\vec{k}_c = \vec{e}_1.$$

After equating the vectors components in expression (12) with regard to (13) we shall obtain

$$u = u_{03} - \theta_{01}R\sin\varphi - \theta_{02}R\cos\varphi,$$
$$v = (\theta_{02}l_c - u_{01})\sin\varphi - (\theta_{01}l_c + u_{02})\cos\varphi - \theta_{03}R, \tag{14}$$
$$w = -(\theta_{01}l_c + u_{02})\sin\varphi + (u_{01} - \theta_{02}l_c)\cos\varphi.$$

Boundary conditions on the shell end are imposed on the displacements $u,\ v,\ w$ and on the angle of rotation, $\theta_1 = -\partial w/\partial z$, of the vector $\vec{e}_1$ around the vector $\vec{e}_2$ as a result of deformation of the shell's median surface. In order to determine the corresponding angle of rotation of the rigid body, we shall calculate the vector product $\left[\vec{k}_c \times \vec{k}^*\right]$, where $\vec{k}^*$ is a unit vector of the body coordinate system $Cx_cy_cz_c$ directed along the axis C^*z^*. Within the accuracy of linear terms, this unit vector is equal to

$$\vec{k}^* = \theta_{02}\vec{i}_c - \vec{\theta}_{01}\vec{j}_c + \vec{k}_c. \tag{15}$$

With regard for relations (15) and (13) we shall have

$$\left[\vec{k}_c \times \vec{k}^*\right] = \theta_{01}\vec{i}_c + \theta_{02}\vec{j}_c = (-\theta_{02}\cos\varphi - \theta_{01}\sin\varphi)\,\vec{e}_2 + (-\theta_{02}\sin\varphi + \theta_{01}\cos\varphi)\,\vec{e}_3.$$

Equating the angle of rotation of the vector $\vec{k}_c$ around the direction $\vec{e}_2$ to the corresponding angle of rotation of the shell edge, we shall obtain

$$\left.\frac{\partial w}{\partial z}\right|_{z=l} = \theta_{01}\sin\varphi + \theta_{02}\cos\varphi. \tag{16}$$

Thus determination of the disturbed state of the considered system is reduced to solving shell equations (5) together with the body equilibrium equations (10), (11) subject to conditions (14), (16) on the contour L. Conditions of binding of the shell end, which is free of the rigid body, should be added to this relations.

2. Derivation of the Equilibrium Equations from the Variational Virtual Displaycements Principle

In previous section an integro-differential statement of the problem on determining the equilibrium state of a cylindrical shell with an attached rigid body is given under the most general small load. In the present section, the equilibrium equations of this mechanical system will be derived with the use of the variational principles of mechanics. Such an approach will serve as an additional criterion of reliability of the constructed mathematical model and besides will permit to formulate an equivalent variational statement of the considered problem which can later on be used for constructing an approximate solution of this problem.

To obtain the equilibrium equations of the considered system and natural boundary conditions, we use the virtual displycements principle according to which

$$\delta\Pi = \delta A, \tag{17}$$

where $\delta\Pi$ is a variation of the potential energy of the system; δA is a variation of the work of external forces.

The work of external forces acting on the body and the shell is equal to

$$A = \iint\limits_{\Sigma} \Delta\vec{Q}\cdot\vec{u}\,d\Sigma + \Delta\vec{F}\cdot\vec{u}_0 + \Delta\vec{M}\cdot\vec{\theta}_0, \tag{18}$$

where Σ is the median surface of shell.

The potential strain energy of a thin cylindrical shell may be represented in the form [2]

$$\Pi = \frac{Eh}{2\,(1-\nu^2)} \iint\limits_{\Sigma} \left[(\varepsilon_1+\varepsilon_2)^2 - 2\,(1-\nu)\left(\varepsilon_1\varepsilon_2 - \frac{\omega^2}{4}\right)\right] d\Sigma$$
$$+ \frac{D}{2} \iint\limits_{\Sigma} \left[(\chi_1+\chi_2)^2 - 2\,(1-\nu)\left(\chi_1\chi_2 - \chi_{12}^2\right)\right] d\Sigma. \tag{19}$$

The first term in formula (19) is the potential energy of elongation and shear and the second is the potential energy of bending and torsion. Substituting the expressions of the components of the strain of the shell median surface (4) into (19) we obtain the following expression of the potential energy in displaycements

$$\Pi = \frac{Eh}{2\,(1-\nu^2)} \iint\limits_{\Sigma} \left[\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{R^2}\left(\frac{\partial v}{\partial\varphi}+w\right)^2 + \frac{2\nu}{R}\frac{\partial u}{\partial z}\left(\frac{\partial v}{\partial\varphi}+w\right) + \frac{1-\nu}{2}\left(\frac{1}{R}\frac{\partial u}{\partial\varphi}+\frac{\partial v}{\partial z}\right)^2\right] d\Sigma$$
$$+ \frac{D}{2} \iint\limits_{\Sigma} \left[\left(\frac{\partial^2 w}{\partial z^2}\right)^2 + \left(\frac{1}{R^2}\frac{\partial^2 w}{\partial\varphi^2}\right)^2 + \frac{2\nu}{R^2}\frac{\partial^2 w}{\partial z^2}\frac{\partial^2 w}{\partial\varphi^2} + 2\,(1-\nu)\left(\frac{1}{R}\frac{\partial^2 w}{\partial z\partial\varphi}\right)^2\right] d\Sigma. \tag{20}$$

Denote the displaycement variations of points of the shell's median surface by $\delta u,\ \delta v,\ \delta w$. Then the variation of potential energy of the shell elastic strain takes the following form:

$$\delta\Pi = \frac{Eh}{1-\nu^2}\iint\limits_{\Sigma}\left\{\left[\frac{\partial u}{\partial z}+\frac{\nu}{R}\left(\frac{\partial v}{\partial\varphi}+w\right)\right]\frac{\partial\delta u}{\partial z}+\left[\frac{\nu}{R}\frac{\partial u}{\partial z}+\frac{1}{R^2}\left(\frac{\partial v}{\partial\varphi}+w\right)\right]\frac{\partial\delta v}{\partial z}\right.$$

$$+\left[\frac{\nu}{R}\frac{\partial u}{\partial z}+\frac{1}{R^2}\left(\frac{\partial v}{\partial\varphi}+w\right)\right]\delta w+\nu_1\left(\frac{1}{R^2}\frac{\partial u}{\partial\varphi}+\frac{1}{R}\frac{\partial v}{\partial z}\right)\frac{\partial\delta u}{\partial\varphi}$$

$$\left.+\nu_1\left(\frac{1}{R}\frac{\partial u}{\partial\varphi}+\frac{\partial v}{\partial z}\right)\frac{\partial\delta v}{\partial z}\right\}d\Sigma+D\iint\limits_{\Sigma}\left[\left(\frac{\partial^2 w}{\partial z^2}+\frac{\nu}{R^2}\frac{\partial^2 w}{\partial\varphi^2}\right)\frac{\partial^2\delta w}{\partial z^2}\right.$$

$$\left.+\left(\frac{1}{R^4}\frac{\partial^2 w}{\partial\varphi^2}+\frac{\nu}{R^2}\frac{\partial^2 w}{\partial z^2}\right)\frac{\partial^2\delta w}{\partial\varphi^2}+\frac{2(1-\nu)}{R^2}\frac{\partial^2 w}{\partial z\partial\varphi}\frac{\partial^2\delta w}{\partial z\partial\varphi}\right]d\Sigma. \quad (21)$$

In what follows we shall suppose that the shell end, which is free of the rigid body, is rigidly fixed. For the functions $f(z,\varphi)$ and $g(z,\varphi)$ which are 2π-periodic with respect to φ one can establish, on condition that $g(0,\varphi) = 0$, the following formulae of integration by parts for the surface integrals

$$\iint\limits_{\Sigma} f\frac{\partial g}{\partial z}d\Sigma = -\iint\limits_{\Sigma} g\frac{\partial f}{\partial z}d\Sigma + \oint\limits_{L} fg\,ds,$$
$$\iint\limits_{\Sigma} f\frac{\partial g}{\partial\varphi}d\Sigma = -\iint\limits_{\Sigma} g\frac{\partial f}{\partial\varphi}d\Sigma. \quad (22)$$

Applying formula (22) to integrals in (21), we get rid of derivatives of the variations $\delta u,\ \delta v,\ \delta w$ in them. Taking into account that the displacement variations satisfy the principal boundary conditions on the shell contour at $z = 0$, expression (21) may be reduced to the form

$$\delta\Pi = -\frac{Eh}{1-\nu^2}\iint\limits_{\Sigma}\left[\left(\frac{\partial^2 u}{\partial z^2}+\frac{\nu_2}{R}\frac{\partial^2 v}{\partial z\partial\varphi}+\frac{\nu}{R}\frac{\partial w}{\partial z}+\frac{\nu_1}{R^2}\frac{\partial^2 u}{\partial\varphi^2}\right)\delta u-\left(\frac{1}{R^2}\frac{\partial v}{\partial\varphi}+\frac{1}{R^2}w+\frac{\nu}{R}\frac{\partial u}{\partial z}\right)\delta w\right.$$

$$+\left(\frac{1}{R^2}\frac{\partial^2 v}{\partial\varphi^2}+\frac{1}{R^2}\frac{\partial w}{\partial\varphi}+\frac{\nu_2}{R}\frac{\partial^2 u}{\partial z\partial\varphi}+\nu_1\frac{\partial^2 v}{\partial z^2}\right)\delta v$$

$$\left.-\frac{c^2}{R^2}\left(R^4\frac{\partial^4 w}{\partial z^4}+\frac{\partial^4 w}{\partial\varphi^4}+2R^2\frac{\partial^4 w}{\partial z^2\partial\varphi^2}\right)\delta w\right]d\Sigma$$

$$+\frac{Eh}{1-\nu^2}\oint\limits_L\left\{\left[\frac{\partial u}{\partial z}+\frac{\nu}{R}\left(\frac{\partial v}{\partial\varphi}+w\right)\right]\delta u+\nu_1\left(\frac{1}{R}\frac{\partial u}{\partial\varphi}+\frac{\partial v}{\partial z}\right)\delta v\right\}ds$$

$$+\frac{D}{R^2}\oint\limits_L\left\{\left[-R^2\frac{\partial^3 w}{\partial z^3}-(2-\nu)\frac{\partial^3 w}{\partial z\partial\varphi^2}\right]\delta w+\left(R^2\frac{\partial^2 w}{\partial z^2}+\nu\frac{\partial^2 w}{\partial\varphi^2}\right)\frac{\partial\delta w}{\partial z}\right\}ds. \quad (23)$$

On the contour L the variations $\delta u,\ \delta v,\ \delta w$, and $\partial\delta w/\partial z$ are not independent, since the shell displacements are connected with parameters of the rigid body motion by the conjuction conditions. Taking into account the expressions of the shell displacement variations on the contour L, we represent the variational equation (17) in the form

$$-\frac{Eh}{1-\nu^2}\Bigg\{\iint\limits_\Sigma\Bigg[\left(\frac{\partial^2 u}{\partial z^2}+\frac{\nu_2}{R}\frac{\partial^2 v}{\partial z\partial\varphi}+\frac{\nu}{R}\frac{\partial w}{\partial z}+\frac{\nu_1}{R^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{1-\nu^2}{Eh}\Delta Q_1\right)\delta u$$

$$+\left(\frac{1}{R^2}\frac{\partial^2 v}{\partial\varphi^2}+\frac{1}{R^2}\frac{\partial w}{\partial\varphi}+\frac{\nu_2}{R}\frac{\partial^2 u}{\partial z\partial\varphi}+\nu_1\frac{\partial^2 v}{\partial z^2}+\frac{1-\nu^2}{Eh}\Delta Q_2\right)\delta v$$

$$-\left(\frac{1}{R^2}\frac{\partial v}{\partial\varphi}+\frac{1}{R^2}w+\frac{\nu}{R}\frac{\partial u}{\partial z}+\frac{c^2}{R^2}\Delta\Delta w-\frac{1-\nu^2}{Eh}\Delta Q_3\right)\delta w\Bigg]d\Sigma\Bigg\}$$

$$+\left[\oint\limits_L\left(Q_1^*\cos\varphi-S\sin\varphi\right)ds-\Delta F_1\right]\delta u_{01}$$

$$+\left[\oint\limits_L\left(Q_1^*\sin\varphi+S\cos\varphi\right)ds+\Delta F_2\right]\delta u_{02}+\left[\oint\limits_L T_1 ds-\Delta F_3\right]\delta u_{03}$$

$$+\left[\oint\limits_L\left(RT_1\sin\varphi+M_1\sin\varphi+l_cS\cos\varphi+l_cQ_1^*\sin\varphi\right)ds+\Delta M_1\right]\delta\theta_{01}$$

$$+\left[\oint\limits_L\left(M_1\cos\varphi+RT_1\cos\varphi-l_cS\sin\varphi+l_cQ_1^*\cos\varphi\right)ds+\Delta M_2\right]\delta\theta_{02}$$

$$+\left[\oint\limits_L RSds+\Delta M_3\right]\delta\theta_{03}=0. \quad (24)$$

Equating the coefficients of $\delta u,\ \delta v,\ \delta w$ in the surface integrals to zero and using the independence of the shell displacement variations and the variations of parameters of the rigid body motion, we shall obtain the equilibrium equations of a cylindrical shell (5). In turn, setting the coefficients of the variations of the rigid body parameters equal to zero gives us the equilibrium equations (10), (11). As a result the boundary-value problem for the considered system which was formulated in the previous section turned out to be equivalent to the variational equation (17).

It should be noted separatly that the equilibrium equations (10) and (11) are the natural boundary conditions for the functional $I = \Pi - A$ on the class of functions which satisfy the conjugation conditions (14), (16) and the fixing conditions of the shell end which is free of the rigid body. It means that in minimization of the functional I on the mentioned function class the necessity of a priori realization of rather complicated boundary conditions (10) and (11) is eliminated. This gives a certain advantage to the energy method of construction of an approximate solution for the considering problem in comparison with other methods of mathematical physics.

The equations obtained can be used to determine the small strains of the cylindrical shell and displacements of the rigid body under a small load. This equations can also be used to study free oscillations of the considered system if according to the d'Alambert principle, one carries out the change of the load components in the equilibrium equations for the corresponding forces of inertia and their moments.

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