Nonlinear Oscillations, Vol. 4, No. 3, 2001

EXPONENTIAL DICHOTOMY AND MEAN SQUARE BOUNDED SOLUTIONS OF LINEAR STOCHASTIC ITO SYSTEMS

O. M. Stanzhyts'ky

Kyiv National Taras Shevshenko University
Volodymyrs'ka St., 64, Kyiv, 01033, Ukraine

We prove that a sufficient condition for stochastic Ito systems to be exponentially dichotomous on the semiaxis is that the nonhomogeneous system havemean square bounden solutions.

AMS Subject Classification: 34C25, 34C29

Let's consider a system of linear stochastic Ito differential equations,

$$dx = \ A(t)xdt + \sum_{i=1}^{m} B_i(t)xdW_i(t), \tag{1}$$

where $t \geq 0, \ x \in \mathbf{R}^n, \ A(t), B_i(t)$ are matrices determinate, continuous, and bounded on the positive semiaxis, $W_i(t), i = \overline{1, \ldots}$, are jointly independent scalar Winner processes defined on a probability space (Ω, F, P). As implied by [1, p. 230], for $x_0 \in \mathbf{R}^n$ system (1) has a unique strong solution of the Cauchy problem, $x(t, x_0), \ x(0, x_0) = x_0$, defined for $t \geq 0$ and such that the second moment of the solution is finite for $t \geq 0$.

Definition 1. *System (1) is called exponentially dichotomous in mean square on the semiaxis $t \geq 0$, if the space $\mathbf{R}^n$ can be represented as a direct sum of two subspaces R^-, R^+ such that an arbitrary solution $x(t, x_0)$ of system (1), where $x_0 \in R^-$, satisfies the inequality*

$$\mathrm{M}\,|x(t, x_0)|^2 \leq K \exp\{-\gamma(t-\tau)\}\mathrm{M}\,|x(\tau, x_0)|^2, \tag{2}$$

for $t \geq \tau \geq 0$, and an arbitrary solution $x(t, x_0)$ of system (1), where $x_0 \in R^+$, satisfies the inequality

$$\mathrm{M}\,|x(t, x_0)|^2 \geq K_1 \exp\{\gamma_1(t-\tau)\}\mathrm{M}\,|x(\tau, x_0)|^2 \tag{3}$$

for $t \geq \tau \geq 0$ with an arbitrary $\tau > 0$. Here $K, K1, \gamma, \ \gamma_1$ are positive constants independent of τ and x_0.

An example of such a system is an exponentially stable in mean square system (1)(in this case $R^+ = \{0\}$, and $R^- = \mathbf{R}^n$).

As opposed to the ordinary differential equations, where conditions for dichotomy are well known [2, p. 230; 3], the problems in stochastic systems remain opened. The author knows only the results of [1, p. 296], where the conditions for exponential dichotomy in mean square are obtained for system of type (1) and for stochastic systems with delay in case where the matrices $A(t), \ B(t)$ are constant or periodic. But the results of [1, p. 296] are obtained with help of a system of ordinary differential equations written for second moments of solutions of system (1) or a system of matrix equations for correlation matrix of the solutions. It leads to an analysis of systems of dimension much greater than the dimension of the initial system, and moreover, this system may not be exponentially dichotomous, although the initial system is such.

That's why there is an interest in studying dichotomy conditions for system (1), when the matrices $A(t)$ and $B(t)$ are not obligatory constant or periodic, and the conditions can be obtained in terms of the initial system, without an analysis of an auxiliary system for other moments.

From the cited works it follows that for ordinary differential equations, the question of exponential dichotomy on the semiaxis is equivalent to the question of existence of solutions, bounded on the semiaxis, of a nonhomogeneous system. This work is devoted to a study of this problem for system (1). Another point of view on studying dichotomy with the use of quadratic forms is published in other work.

In the sequel, we'll assume that there is only the one scalar Winner process $W(t)$ in system (1), and system (1) is of the form

$$dx = A(t)xdt + B(t)xdW(t). \tag{4}$$

Let us consider a system of linear nonhomogeneous equation,

$$dx = [A(t)x + \alpha(t)]dt + B(t)xdW(t), \tag{5}$$

where $\alpha(t)$ is n-dimensional and measurable, and for each $t \geq 0$, F_t is a measurable stochastic process. Here, F_t is a flow of σ-algebras involued in the definition of the solution of the initial systems solution. We will assume that $\sup_{t\geq 0} \mathrm{M}\,|\alpha(t)|^2 < \infty$. With the norm $||\alpha||_2 = (\sup_{t\geq 0} \mathrm{M}\,|\alpha(t)|^2)^{1/2}$, the set of stochastic processes becomes a Banach space. Denote it by B.

Theorem 1. *Let the system (5) with an arbitrary stochastic process $\alpha(t) \in B$ have a bounded in mean square positive solution $x(t, x_0)$, $x_0 \in \mathbf{R}^n$. Then the system (4) is exponentially dichotomic in mean square on the positive semiaxis.*

Proof. Let $G_1 \subset R^n$ be the set of initial values of solutions of system (4), which are bounded in mean square on the semiaxis. It follows from linearity of system (4) that G_1 is a subspace of $\mathbf{R}^n$. Let's show that it plays the role of R^- in the definition of the exponential dichotomy. Let's prove the lemma.

Lemma 1. *Suppose that the conditions of Theorem 1 hold. Then, for each stochastic processes $\alpha(t) \in B$, there exists a unique bounded in mean square solution $x(t, x_0)$ of system (5) such that $x(0, x_0) \in G_1^{\perp} = G_2$. ($G_1^{\perp}$ is the orthogonal complement.) This solution satisfies the estimate*

$$||x||_2 \leq K||\alpha||_2, \tag{6}$$

where K is some positive constant, independent from $\alpha(t)$.

Proof. Let $\alpha(t)$ satisfy the condition of the Theorem. Then for this $\alpha(t)$, it follows from the conditions of the lemma that there is a bounded in mean square solution $x(t, x_0)$ of system (5).

Let P_1, P_2 be a pair of complement projectors on G_1, G_2. Let $x_1(t)$ be a solution of equation (4), corresponding to equation (5), with the initial condition $x_1(0) = P_1 x_0$. From the definition of the subspace G_1, it follows that such a solution is bounded in mean square on the semiaxis

$t \geq 0$. It is obvious that $x_2 = x(t, x_0) - x_1(t)$ is a solution of system (3). It is easy to see that it is bounded on the positive semiaxis. We have $x_2(0) = x_0 - P_1 x_0 = P - 2x_0 \in G_2$. So its initial condition belongs to G_2. The unicity of the solution follows from the fact that the difference of two such solutions is a solution of a homogenous equation, bounded in mean square and starts in G_2, which is possible only for the zero solution.

Let's prove inequality (6). Let's consider the space B_1 of all bounded in the norm $|| \; ||_2$ solutions of the stochastic equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha(s))ds + \int_0^t (B(s)x(s))dW(s) \tag{7}$$

with the condition that $x(0) \in G_2, \alpha(t) \in B$.

This equation defines a one-to-one linear operator $F : B_1 \to B$ which $\forall x \in B_1$ defines $\alpha \in B$ such that $x(t)$ is a bounded in mean square solution of equation (5). Indeed, if $x(t) \in B_1$, then it follows from the definition of this space that there exists $\alpha(t) \in B$ such that $x(t)$ is a solution of equation (7) with this $\alpha(t)$. Let there exist another $\alpha_1(t) \in B$ such that $x(t)$ is a solution of the equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha_1(s))ds + \int_0^t (B(s)x(s))dW(s). \tag{8}$$

Subtracting (7) from (8) we get

$$\int_0^t (\alpha(s) - \alpha_1(s))\, ds = 0. \tag{9}$$

Then $\alpha(t) = \alpha_1(t)$, for $t \geq 0$, with probability 1, from which it follows that $\alpha(t)$ and $\alpha_1(t)$ are equal as elements of the space B. It was showen above that for an arbitrary $\alpha(t) \in B$ there exists only one solution, $x(t)$, of equation (7) such that $x(0) \in G_2,\ x(t) \in B_1$. The linearity of the operator F is obvious.

Let's introduce the norm

$$|||x||| = ||x|| + ||Fx||_2. \tag{10}$$

This at once implies continuity of the operator F. Let's prove completeness of the space B_1. Let $\{x_n(t)\}$ be a fundamental sequence. Then it is fundamentality in B, as follows from (10). Hence, there exists a limit $x(t) \in B$. So, $\forall t \geq 0,\ \mathrm{M}\,|x_n(t) - x(t)|^2 \to 0,\ n \to \infty$. And thus $|x_n(0) - x(0)| \to 0, n \to \infty$. Since $x_n(0) \in G_2$ and G_2 is a subspace of in $\mathbf{R}^n$, $x(0) \in G_2$.

It follows from the inequality $||F(x_n - x_m)||_2 \leq ||F||\,|||x_n - x_m|||$ that the sequence $Fx_n = \alpha_n$ is fundamental in B, and so there is a limit $\alpha(t)$ such that $\sup_{t \geq 0} \mathrm{M}\,|\alpha_n(t) - \alpha(t)|^2 \to 0$, as $n \to \infty$, and $\alpha(t) \in B$.

Let's show that $x(t)$ satisfies the equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha(s))ds + \int_0^t (B(s)x(s))dW(s). \tag{11}$$

Since $A(t)$ and $B(t)$ are continuous and bounded, $x(t) \in B$, we have that $x(t)$ is F_t-measurable and both integrals in (11) exist. Let's estimate, for each $t \geq 0$, the expression

$$\mathrm{M}\left|x(t) - x(0) - \int_0^t (A(s)x(s) + \alpha(s))ds - \int_0^t B(s)x(s)dW(s)\right|^2$$

$$\leq \mathrm{M}\left(|x(t) - x_n(t)| + \left|x_n(t) - x(0) - \int_0^t (A(s)x(s) + \alpha(s))\, ds\right.\right.$$

$$\left.\left.- \int_0^t (B(s)x(s))dW(s)\right|\right)^2 \leq 2\left[\mathrm{M}\,|x(t) - x_n(t)|^2 + \mathrm{M}\left|x_n(t) - x(0)\right.\right.$$

$$\left.\left.- \int_0^t (A(s)x(s) + \alpha(s))ds - \int_0^t (B(s)x(s))dW(s)\right|^2\right]. \tag{12}$$

The first summand in the last expression tends to zero when $n \to \infty$. Let's estimate the second summand. Since $x_n(t)$, for each n, belongs to B_1, it satisfies the equation

$$x_n(t) = x_n(0) + \int_0^t (A(s)x_n(s) + \alpha_n(s))ds + \int_0^t B(s)x_n(s)dW(s). \tag{13}$$

Let's substitute (13) into (12). We get that the second summand in (10) does not exceed the following expression

$$3\left[\mathrm{M}\,|x_n(0) - x(0)|^2 + \mathrm{M}\left(\int_0^t (\|A(s)\|\,|x_n(s) - x(s)| + |\alpha_n(s) - \alpha(s)|)ds\right)^2\right.$$

$$\left.+\ \mathrm{M}\left|\int_0^t B(s)(x_n(s) - x(s))dW(s)\right|^2\right]$$

$$\leq 3\left[\mathrm{M}\,|x_n(t) - x(0)|^2 + 2t\int_0^t \|A(s)\|^2 \mathrm{M}\,|x_n(s) - x(s)|^2 ds\right.$$

$$\left.+\ 2t\int_0^t \mathrm{M}\,|\alpha_n - \alpha(s)|^2 ds + \int_0^t \|B(s)\|^2 \mathrm{M}\,|x_n(s) - x(s)|^2 ds\right].$$

Each of the summands in the last expression tends to zero as $n \to \infty$. From (12) it follows that $x(t)$ satisfies (11) with probability of 1 for each $t \geq 0$. So, the space B_1 is complete. That's why the linear continuous operator F defines a one-to-one mapping of the Banach space B_1 onto the Banach space B. By the Banach theorem, the inverse operator B^{-1} is also continuous. Then, for the solution of equation (3), we have the estimate

$$||x||_2 \leq |||x||| \leq ||F^{-1}||||\alpha||_2,$$

what is the needed estimate (4). The Lemma is proved.

Let $x(t)$ be a nonzero solution of system (4), so that $x(0) \in G_1$. Let

$$y(t) = x(t) \int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\, ds, \tag{14}$$

where

$$\beta(t) = \begin{cases} 1, & 0 \leq t \leq t_0 + \tau, \\ 1 - (t - t_0 - \tau), & t_0 + \tau \leq t \leq t_0 + \tau + 1, \\ 0, & t \geq t_0 + \tau + 1. \end{cases}$$

It is obvious that $y(t)$ is F_t-dimensional and has a stochastic differential. Let's evaluate it,

$$\begin{aligned} dy &= \int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\, ds dx + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}} dt \\ &= \int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\, ds(A(t)xdt + B(t)xdW(t)) + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}} dt \\ &= A(t)ydt + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}} dt + B(t)ydW(t). \end{aligned}$$

So, $y(t)$ is a solution of equation (5) with $\alpha(t) = x(t)\dfrac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{1/2}}$. Obviously, $||y||_2 < \infty$ and $\alpha(t) \in B$. And since $y(0) = 0 \in G_2$, the previous Lemma gives that

$$||y||_2 \leq K||\alpha||_2.$$

Whence,

$$(M|y(t)|^2)^{\frac{1}{2}} \leq K(\sup_{t\geq 0} \mathrm{M}\,|\alpha(t)|^2)^{\frac{1}{2}} \leq K$$

for $t \geq 0$. In particular, if $t = t_0 + \tau$, then

$$(\mathrm{M}\,|y(t)|^2)^{\frac{1}{2}} = (\mathrm{M}\,|x(t_0+\tau)|^2)^{\frac{1}{2}} \leq K. \tag{15}$$

Let's consider the function

$$\psi(t) = \int\limits_{t_0}^{t} \frac{1}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,ds.$$

Since the second moments of system (4) satisfy a system of ordinary linear differential equations, see [1, p. 236], this function is continuously differentiable. Then (15) gives that

$$\frac{\psi'(t_0+\tau)}{\psi(t_0+\tau)} \geq \frac{1}{K}.$$

If we integrate the last inequality from 1 to τ , we get

$$\psi(t_0+\tau) \geq \psi(t_0+1)\exp\left\{\frac{\tau-1}{K}\right\} \tag{16}$$

for $\tau \geq 1$. Since $x(t)$ is a solution of system (4),

$$x(t) = x(t_0) + \int\limits_{t_0}^{t} A(s)x(s)\,ds + \int\limits_{t_0}^{t} B(s)x(s)\,dW(s). \tag{17}$$

And, hence, if $t \in [t_0\ t_0+1]$, we have

$$\mathrm{M}\,|x(t)|^2 \leq 3\left(\mathrm{M}\,|x(t_0)|^2 + \int\limits_{t_0}^{t_0+1} ||A(s)||^2 \mathrm{M}\,|x(s)|^2\,ds + \int\limits_{t_0}^{t_0+1} ||B(s)||^2 \mathrm{M}\,|x(s)|^2\,ds\right).$$

This and the Gronwall – Bellman inequality give

$$\mathrm{M}\,|x(t)|^2 \leq 3\mathrm{M}\,|x(t_0)|^2 \exp\{C\}, \tag{18}$$

where $C > 0$ is a constant independent of t_0. Therefore,

$$\psi(t_0+1) = \int\limits_{t_0}^{t_0+1} \frac{1}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,ds \geq \frac{1}{3^{\frac{1}{2}}}(\mathrm{M}\,|x(t_0)|^2)^{-\frac{1}{2}} \exp\left\{-\frac{C}{2}\right\}.$$

From this inequality using (15) and (16), we get for $\tau \geq 1$ that

$$(\mathrm{M}\,|x(t_0+\tau)|^2)^{\frac{1}{2}} \leq \frac{K}{\psi(t_0+\tau)} \leq N(\mathrm{M}\,|x(t_0)|^2)^{\frac{1}{2}} \exp\left\{-\frac{\tau}{K}\right\}, \tag{19}$$

where $N > 0$ is a constant independent of τ and t_0. If $\tau \leq 1$, it follows from inequality (18) that

$$\mathrm{M}\,|x(t_0+\tau)|^2 \leq 3\mathrm{M}\,|x(t_0)|^2 \exp\left\{\frac{2}{K} + C - \frac{2\tau}{K}\right\}. \tag{20}$$

Since $t_0 \geq 0$ is arbitrary, (19) and (20) the first inequality in Definition 1 with

$$\gamma = \frac{2}{K},\ K_1 = \max\left\{N^2; 3\exp\left\{\frac{2}{K} + C\right\}\right\}.$$

Let us prove the second inequality in Definition 1. Let $x(t)$ be a nonzero solution of equation (4) with $x(0) \in G_2$. It's easy to see that

$$y(t) = x(t)\int_t^\infty \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,ds$$

is a solution of equation (5) with

$$\alpha(t) = -\frac{x(t)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\beta(t)$$

and, since $t \geq t_0+\tau$, $\sup_{t\geq 0}\mathrm{M}\,|y(t)|^2 < \infty$. It is obvious that $y(0) \in G_2$. Therefore, because of the Lemma,

$$(\mathrm{M}\,|y(t)|^2)^{\frac{1}{2}} = (\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}\int_t^\infty \frac{\beta(s)}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,ds \leq K.$$

So, $\forall\tau \geq 0$ and $\forall t \geq 0$,

$$\int_t^\infty \frac{\beta(s)}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,ds \leq \frac{K}{\sqrt{\mathrm{M}\,|x(t)|^2}}. \tag{21}$$

The left-hand side of this inequality is monotone increasing for $\tau \geq 0$ and is bounded, so it has a limit for $\tau \to \infty$. But

$$\int_t^\infty \frac{\beta(s)}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,ds = \int_t^{t_0+\tau} \frac{1}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,ds + \int_{t_+\tau}^{t_0+\tau+1} \frac{\beta(s)}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,ds,$$

where the second summand tends to zero as $\tau \to \infty$ (since the integral is convergent), and so if $\tau \to \infty$, we get the inequality

$$\int_t^\infty \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,ds \leq \frac{K}{\sqrt{\mathrm{M}\,|x(t)|^2}}. \tag{22}$$

Let

$$\psi(t) = \int_t^\infty \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,ds.$$

Then it follows from (22) that

$$\psi'(t) \le -\frac{1}{K}\psi(t).$$

Whence we get

$$\psi(t) \le \psi(t_0)\exp\left\{-\frac{1}{K}(t-t_0)\right\}. \tag{23}$$

Since $x(t)$ is a solution of system (4),we have for $\tau \ge t$ that

$$\mathrm{M}\,|x(\tau)|^2 \le C_1\mathrm{M}|x(t)|^2\exp\{L(\tau-t)\},$$

where L, C_1 are positive constants, independent of τ and t. From the last inequality, it follows that

$$\mathrm{M}\,|x(\tau)|^2 \le 3\mathrm{M}\,|x(t)|^2\exp\{3L(\tau-t+1)(\tau-t)\}.$$

Therefore,

$$(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}\psi(t) = (\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}\int_t^\infty \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,ds$$

$$\ge \int_t^\infty \frac{1}{\sqrt{3}}\exp\left\{-\frac{3}{2}L(s-t+1)(s-t)\right\}ds = l.$$

Here L is a positive constant. Then (22) and (23) give

$$(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}} \ge \frac{l}{\psi(t)} \ge \frac{l}{\psi(t_0)}\exp\left\{\frac{1}{K}(t-t_0)\right\} \ge \frac{l}{K}\exp\left\{\frac{1}{K}(t-t_0)\right\}(\mathrm{M}\,|x(t_0)|^2)^{\frac{1}{2}}.$$

This estimate is the second inequality, which figures in the definition of the exponential dichotomy. The theorem is proved.

In theory of ordinary differential equation one proves a converse result that exponential dichotomy of a homogeneous system implies existence of a bounded solution of the nonhomogeneous system and that

$$y(t) = \int_0^\infty G(t,\tau)f(\tau)\,d\tau, \tag{24}$$

where $G(t,\tau)$ is Green's function

$$G(t,\tau) = \begin{cases} \Phi(0,t)P_1(\Phi(0,\tau))^{-1}, & t \geq \tau, \\ -\Phi(0,t)P_2(\Phi(0,\tau))^{-1}, & t < \tau, \end{cases} \tag{25}$$

$\Phi(0,t)$ is the matriciant of the homogeneous system. For stochastic nonhomogeneous systems,

$$dx = (A(t)x + \alpha(t))dt + (B(t)x + \beta(t))]dW(t), \tag{26}$$

one can also write the representation

$$y(t) = \int_0^\infty G(t,\tau)\alpha(\tau)d\tau + \int_0^\infty G(t,\tau)\beta(\tau)dW(\tau), \tag{27}$$

however, in such a case, $y(t)$ will not be F_t-measurable any more. Thus, by using Green's function, one succeeds in getting a similar result only in the case where the homogeneous system is exponentially stable and the nonhomogeneous system has the form

$$dx = [\,A(t)x + \alpha(t)]dt + \beta(t)dW(t). \tag{28}$$

Theorem 2. *Let the homogeneous system*

$$dx = A(t)xdt \tag{29}$$

be exponentially stable on the positive semiaxis. Then, for arbitrary $\alpha(t), \beta(t) \in B$, system (28) has a solution bounded in mean square on the positive semiaxis. In addition, all bounded solutions of system (28) are given by

$$x = \psi(t) + \int_0^t \Phi(t,\tau)\alpha(\tau)\,d\tau + \int_0^t \Phi(t,\tau)\beta(\tau)\,dW(\tau), \tag{30}$$

where $\psi(t)$ is an arbitrary solution of system (29), and $\Phi(t,\tau)$ is the matriciant of system (29), $\Phi(\tau,\tau) = E$.

Proof. Since system (29) is exponentially stable, its matriciant satisfies the estimate

$$||\Phi(t,\tau)|| \leq K \exp\{-\gamma(t-\tau)\}, \tag{31}$$

for $t \geq \tau \geq 0$, with some positive K and γ. Let's show that $x(t)$, defined by (30), is bounded in mean square for $t \geq 0$. To do that, it is sufficient to prove the boundedness of each of its summands. Indeed $\psi(t)$ is a function bounded on the semiaxis. Let's estimate the second summand. From the Cauchy – Bunyakovskii inequality, we have

$$\mathrm{M}\,|\int\limits_0^t \Phi(t,\tau)\alpha(\tau)\,d\tau|^2 \le \mathrm{M}\left(\int\limits_0^t ||\Phi(t,\tau)|||\alpha(\tau)|\,d\tau|\right)^2$$

$$\le K^2\mathrm{M}\left(\int\limits_0^t \exp\left\{\frac{-\gamma(t-\tau)}{2}\right\}\exp\left\{\frac{-\gamma(t-\tau)}{2}\right\}|\alpha(\tau)\,d\tau\right)^2$$

$$\le K^2\int\limits_0^t \exp\{-\gamma(t-\tau)\}\,d\tau\int\limits_0^t \exp\{-\gamma(t-\tau)\}\mathrm{M}\,|\alpha(\tau)|^2\,d\tau) < C,$$

where $C > 0$ is a constant, since $\alpha(t) \in B$. Let's use the properties of the stochastic integral

$$\mathrm{M}\,|\int\limits_0^t \Phi(t,\tau)\beta(\tau)\,dW(\tau)|^2 \le \int\limits_0^t ||\Phi(t,\tau)||^2\mathrm{M}\,|\beta(\tau)|^2\,d\tau$$

$$\le K^2\int\limits_0^t \exp\{-2\gamma(t-\tau)\}\,d\tau\sup_{t\ge0}\mathrm{M}\,|\beta(t)|^2 < C_1, \qquad C_1 > 0.$$

So, the function $x(t)$ in (30) is bounded in mean square. It is obvious that it is F_t-dimensional, which follows from [1, p. 234], and gives a solution of system (28). The theorem is proved.

REFERENCES

1. *Tsar'kov E. F.* Random Perturbations of Functional-Differential Equations [in Russian], Zinatne, Riga (1989).
2. *Daletsky Yu. L. and Krein M. G.* Stability of Solutions of Differential Equations in a Banach Space [in Russian], Nauka, Moscow (1970).
3. Mitropol'sky Yu. A., Samoilenko A. M., and Kulik V. L. A Study of Dichotomy of Linear Differential Systems by Using Lyapunov Functions [in Russian], Naukova Dumka, Kiev (1990).

Received 12.12.2000