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## ON CONTINUITY OF THE INVARIANT TORUS FOR COUNTABLE SYSTEM OF DIFFERENCE EQUATIONS DEPENDENT ON PARAMETERS

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*By using the method of the Green – Samoilenko function, an invariant torus is constructed for a system of discrete equations which are defined on tori in the space of bounded number sequences. Sufficient conditions are established for continuous dependence of the invariant torus on the angular variable and the parameter contained in this system.*

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It is well known that in the theory of discrete dynamical systems, investigations of invariant tori play an important role, particularly for their continuity and smoothness. In recent time there were published some papers dedicated to an investigation of invariant tori for difference equations in spaces of bounded number sequences (countable systems of difference equations). Let us mention, for example, papers [1 – 5]. Closely related to the results obtained in them are theorems on continuity of invariant tori proved in the present article.

Let us consider the system of equations

$$\varphi_{n+1} = \varphi_n + a(\varphi_n, \mu), \quad x_{n+1} = P(\varphi_{n+p}, \mu)x_n + c(\varphi_{n+g+1}, \mu), \tag{1}$$

in which $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^m) \in R^m$, $x = (x^1, x^2, x^3, \dots) \in \mathfrak{M}$, where $\mathfrak{M}$ is the space of bounded number sequences with the norm $\|x\| = \sup_i\{|x^i|\}$; the functions

$$a(\varphi, \mu) = \{a_1(\varphi, \mu), a_2(\varphi, \mu), \dots, a_m(\varphi, \mu)\}, c(\varphi, \mu) = \{c_1(\varphi, \mu), c_2(\varphi, \mu), \dots\}$$

and the infinite matrix $P(\varphi, \mu) = [p_{ij}(\varphi, \mu)]_{i,j=1}^{\infty}$ are real and periodic with respect to $\varphi^i,\ i = 1, 2, \dots, m$, with period $2\pi$; $n \in Z$, $Z$ is the set of all integers; $p$ and $g$ are integer parameters which determine deviations of the discreet argument; $\mu \in [\mu_1, \mu_2] \subset R^1$ is a real parameter.

With $\varphi^i,\ i = \overline{1, m}$, being interpreted as the angular coordinates, we will consider the system of equations (1) as one which is defined on an $m$-dimensional torus, $\mathcal{T}_m$.

Let $\varphi_n(\varphi, \mu)$ be a solution of the first equation from (1)such that $\varphi_0(\varphi, \mu) = \varphi \in \mathcal{T}_m$ for all $\mu \in [\mu_1, \mu_2]$. Let $x_n(p, g, \mu) = x_n(p, g, \mu, \varphi, x_k)$ stands for a solution of the equation

$$x_{n+1} = P(\varphi_{n+p}(\varphi, \mu), \mu)x_n + c(\varphi_{n+1+g}(\varphi, \mu), \mu), \quad n \in Z, \tag{2}$$

such that $x_k(p, g, \mu) = x_k \in \mathfrak{M}, k \in Z$.

The next conditions we will call "conditions **A**":

1) for all $\mu \in [\mu_1, \mu_2]$ and $\varphi \in \mathcal{T}_m$, the matrix $P(\varphi, \mu)$ is invertible and for every $\mu \in [\mu_1, \mu_2]$, the mapping $\Phi(\varphi, \mu) = \varphi + a(\varphi, \mu) : R^m \to R^m$ is invertible as well;

2) $\|a(\varphi, \mu)\| \le A^0, \|c(\varphi, \mu)\| \le C^0, \|P(\varphi, \mu)\| = \sup_i \sum_{j=1}^{\infty} |p_{ij}(\varphi, \mu)| \le P^0, \|P^{-1}(\varphi, \mu)\| \le P_1$, where $A^0, P^0, C^0, P_1$ are positive constants independent of $\mu \in [\mu_1, \mu_2], \varphi \in \mathcal{T}_m$.

It is easy to see that under conditions **A** for each $x_k \in \mathfrak{M}, \varphi \in \mathcal{T}_m, \{p, g\} \subset Z, \mu \in [\mu_1, \mu_2]$, the solution $x_k(p, g, \mu)$ for equation (2) exists, is unique and contained in $\mathfrak{M}$ for all $n \in Z$.

The invariant torus $\mathcal{T}(p, g, \mu)$ for the system of equations (1) stands for the set of points $x \in \mathfrak{M}$ such that

$$x = u(p, g, \mu, \varphi) = (u_1(p, g, \mu, \varphi), u_2(p, g, \mu, \varphi), \dots), \quad \varphi \in \mathcal{T}_m,$$

if the function $u(p, g, \mu, \varphi)$ is defined for each $\{p, g\} \subset Z, \mu \in [\mu_1, \mu_2], \varphi \in R^m$, is $2\pi$-periodic with respect to $\varphi^i$, is bounded with respect to the norm $\|\cdot\|$, and for every $\varphi \in \mathcal{T}_m$ satisfies the equality

$$u(p, g, \mu, \varphi_{n+1}(\varphi, \mu)) = P(\varphi_{n+p}(\varphi, \mu), \mu) u(p, g, \mu, \varphi_n(\varphi, \mu)) + c(\varphi_{n+g+1}(\varphi, \mu), \mu).$$

The torus $\mathcal{T}(p, g, \mu)$ will called continuous with respect to $\varphi, \mu$, if this is true for its generating function $u(p, g, \mu, \varphi)$.

Let us assume that the uniform equation

$$x_{n+1} = P(\varphi_{n+p}(\varphi, \mu), \mu) x_n, \quad n \in Z, \tag{3}$$

for $p = 0$ has a GSF (Green – Samoilenko function), $G_0(l, \mu, \varphi)$, for the invariant torus problem, i.e., there exist an infinite matrix $C(\varphi, \mu)$, $2\pi$-periodic with respect to $\varphi^i,\ i = \overline{1, m}$, and bounded with respect to the norm for all $\varphi \in \mathcal{T}_m, \mu \in [\mu_1, \mu_2]$, and constants $M > 0$ and $0 < \lambda < 1$ such that the function

$$G_0(l, \mu, \varphi) = \begin{cases} \Omega_l^0(\varphi, \mu) C(\varphi_l(\varphi, \mu), \mu) & \text{for} \quad l \le 0; \\ \Omega_l^0(\varphi, \mu)[C(\varphi_l(\varphi, \mu), \mu) - E] & \text{for} \quad l > 0 \end{cases}$$

satisfies the inequality $\|G_0(l, \mu, \varphi)\| \le M\lambda^{|l|}$ uniformly with respect to $\varphi$ and $\mu$.

In paper [1] it is shown that the function

$$G_0(l, p, \mu, \varphi) = \begin{cases} \Omega_l^0(\varphi_p(\varphi, \mu), \mu) C(\varphi_l(\varphi_p(\varphi, \mu)), \mu) & \text{for} \quad l \le 0; \\ \Omega_l^0(\varphi_p(\varphi, \mu), \mu)[C(\varphi_l(\varphi_p(\varphi, \mu)), \mu) - E] & \text{for} \quad l > 0 \end{cases}$$

is a GSF for equation (3) for every $p \in Z$. Note that, in the same paper, it is proved that a GSF $G_0(l, \mu, \varphi)$ exists in the case of $Z$-dichotomic equation (3) and necessary and sufficient conditions of its $Z$-dichotomicity are given. Considering equation (3) as a linear extension for discrete dynamical system on the torus and using the weighted shift operators it is possible to tie together the conditions on existence and uniqueness of the GSF for this linear extension with the conditions on its hyperbolicity [6].

The next proposition takes place.

**Proposition 1.** *Let conditions* **A** *hold and for* $p = 0$ *there exists a GSF* $G_0(l, \mu, \varphi)$ *for equation (3). Then for every* $\{p, g\} \subset Z, \mu \in [\mu_1, \mu_2]$, *system of equations (1) has an invariant torus*

*$\mathcal{T}(p, g, \mu)$ defined by the function*

$$u(p, g, \mu, \varphi) = \sum_{l=-\infty}^{+\infty} G_0(l, p, \mu, \varphi) c(\varphi_{l+p}(\varphi, \mu), \mu). \tag{4}$$

*This torus is covered by the family of bounded solutions,*

$$x_n = \sum_{l=-\infty}^{+\infty} G_n(l, p, \mu, \varphi) c(\varphi_{l+p}(\varphi, \mu), \mu)$$

*for equation (2). This family depends on the parameters $p, g, \mu, \varphi$.*

The proof for this proposition does not present difficulties, so we omit it.

Let $\omega(z)$ be some scalar function, continuous and nondecreasing on the segment $[0; \mu_2 - \mu_1]$, such that $\omega(0) = 0$.

**Theorem 1.** *Let the conditions of Proposition 1 hold and, moreover, for all $\{\varphi, \bar{\varphi}\} \subset \mathcal{T}_m$, $\{\mu, \bar{\mu}\} \subset [\mu_1, \mu_2]$,*

*1)* $\|a(\varphi, \mu) - a(\bar{\varphi}, \bar{\mu})\| \le \alpha_1 \|\varphi - \bar{\varphi}\| + \alpha_2 \omega(|\mu - \bar{\mu}|)$,

$\|P(\varphi, \mu) - P(\bar{\varphi}, \bar{\mu})\| \le \beta_1 \|\varphi - \bar{\varphi}\| + \beta_2 \omega(|\mu - \bar{\mu}|)$,

$\|c(\varphi, \mu) - c(\bar{\varphi}, \bar{\mu})\| \le \gamma_1 \|\varphi - \bar{\varphi}\| + \gamma_2 (|\mu - \bar{\mu}|)$,

$\|\Phi^{-1}(\varphi, \mu) - \Phi^{-1}(\bar{\varphi}, \bar{\mu})\| \le \xi_1 \|\varphi - \bar{\varphi}\| + \xi_2 \omega(|\mu - \bar{\mu}|)$,

*where $\alpha_i, \beta_i, \gamma_i, \xi_i,\ i = 1, 2$, are positive constants independent of $\varphi, \mu, \bar{\varphi}, \bar{\mu}$;*

*2) for all $\varphi \in \mathcal{T}_m$, $\mu \in [\mu_1, \mu_2]$, and $p = 0$, equation (3) has a unique solution bounded on $Z$;*

*3)* $\xi_1 \ge 1, \quad \lambda < \min\left\{\dfrac{1}{1 + \alpha_1}; \dfrac{1}{\xi_1}\right\}$.

*Then the function $u(p, g, \mu, \varphi)$, which determines the invariant torus for the system of equation (1), is continuous with respect to the set of variables $\varphi, \mu$ and, moreover, since at some moment in the process $\|\varphi - \bar{\varphi}\| \to 0, |\mu - \bar{\mu}| \to 0$, the inequality*

$$\|u(p, g, \mu, \varphi) - u(p, g, \bar{\mu}, \bar{\varphi})\| \le M_* \{\|\varphi - \bar{\varphi}\| + \omega(|\mu - \bar{\mu}|)\}^{\frac{1}{2}} \tag{5}$$

*holds, where $M_*$ is a positive constant independent of $\varphi, \mu, \bar{\varphi}, \bar{\mu}$.*

***Proof.*** Having in mind (4) and the condition 2) of the theorem, we can write the equality

$$G_0(l, p, \mu, \varphi) - G_0(l, p, \bar{\mu}, \bar{\varphi}) = \sum_{k=-\infty}^{+\infty} G_0(k, p, \mu, \varphi)$$

$$\times \{P(\varphi_{k+p-1}(\varphi, \mu), \mu) - P(\varphi_{k+p-1}(\bar{\varphi}, \bar{\mu}), \bar{\mu})\} G_{k-1}(l, p, \bar{\mu}, \bar{\varphi}). \tag{6}$$

It is easy to see that the next inequalities hold:

$$\|\varphi_n(\varphi,\mu)-\varphi_n(\bar{\varphi},\bar{\mu})\| \le (1+\alpha_1)^n\{\|\varphi-\bar{\varphi}\|+\frac{\alpha_2}{\alpha_1}\omega(|\mu-\bar{\mu}|)\}, n\ge 0,$$
$$\|\varphi_n(\varphi,\mu)-\varphi_n(\bar{\varphi},\bar{\mu})\| \le \xi_1^{-n}\{\|\varphi-\bar{\varphi}\|+\frac{\xi_2}{\xi_1-1}\omega(|\mu-\bar{\mu}|)\}, n< 0. \tag{7}$$

Let us introduce the following notations for convenience:

$$\omega(|\mu-\bar{\mu}|)=\omega, \quad \|G_0(l,p,\mu,\varphi)-G_0(l,p,\bar{\mu},\bar{\varphi})\|=G,$$

$$\|c(\varphi_{l+g}(\varphi,\mu),\mu)-c(\varphi_{l+g}(\bar{\varphi},\bar{\mu}),\bar{\mu})\|=\bar{c},$$

$$\|u(p,g,\mu,\varphi)-u(p,g,\bar{\mu},\bar{\varphi})\|=\bar{u}, \quad \frac{1}{1-\lambda\xi_1}+\frac{\lambda}{\xi_1-\lambda}=\lambda_\xi,$$

$$\frac{1}{1+\alpha_1-\lambda}+\frac{1}{1-\lambda(1+\alpha_1)}=\lambda_\alpha.$$

Relations (6) and (7) imply the inequality

$$G\le M^2(I_1+I_2), \tag{8}$$

where

$$I_1=\sum_{k=-\infty}^{-p}\lambda^{|k|}\left\{\beta_1\xi_1^{-(k+p-1)}\|\varphi-\bar{\varphi}\|+\Big[\frac{\beta_1\xi_2\xi_1^{-(k+p-1)}}{\xi_1-1}+\beta_2\Big]\omega\right\},$$

$$I_2=\sum_{k=-p+1}^{+\infty}\lambda^{|k|}\left\{\beta_1(1+\alpha_1)^{k+p-1}\|\varphi-\bar{\varphi}\|+\Big[\frac{\beta_1(\alpha_2)(1+\alpha_1)^{k+p-1}}{\alpha_1}+\beta_2\Big]\omega\right\}.$$

For $I_1$ we have

$$I_1\le\beta_1\xi_1^{1-p}\|\varphi-\bar{\varphi}\|\sum_{k=-\infty}^{+\infty}\lambda^{|k|}\xi_1^{-k}+\frac{\beta_2(1+\lambda)\omega}{1-\lambda}$$

$$+\ \frac{\beta_1\xi_2\xi_1^{1-p}}{\xi_1-1}\omega\sum_{k=-\infty}^{+\infty}\lambda^{|k|}\xi_1^{-k},$$

and, with condition 3) of the theorem, it implies the inequality

$$I_1 \le K^1\|\varphi - \bar{\varphi}\| + \bar{K}^1\omega, \tag{9}$$

where

$$K^1 = \beta_1\xi_1^{1-p}\lambda_\xi, \quad \bar{K}^1 = \beta_2\frac{1+\lambda}{1-\lambda} + \frac{\beta_1\xi_2\xi_1^{1-p}}{\xi_1 - 1}\lambda_\xi.$$

The inequality for $I_2$ is obtained in a similar way,

$$I_2 \le K^2\|\varphi - \bar{\varphi}\| + \bar{K}^2\omega, \tag{10}$$

where

$$K^2 = \beta_1(1+\alpha_1)^{p-1}\lambda_\alpha, \quad \bar{K}^2 = \beta_2\frac{1+\lambda}{1-\lambda} + \frac{\beta_1\alpha_2(1+\alpha_1)^{p-1}}{\alpha_1}\lambda_\alpha.$$

Using (8) – (10) we obtain the inequality

$$G \le \psi_1\|\varphi - \bar{\varphi}\| + \psi_2\omega, \tag{11}$$

where $\psi_1 = M^2(K^1 + K^2), \psi_2 = M^2(\bar{K}^1 + \bar{K}^2)$.

Having in mind (11) it is easy to see that

$$G \le (2M)^{\frac{1}{2}}\lambda^{\frac{|l|}{2}}(\psi_1\|\varphi - \bar{\varphi}\| + \psi_2\omega)^{\frac{1}{2}},$$

therefore,

$$\bar{u} \le M^0\{\psi_1\|\varphi - \bar{\varphi}\| + \psi_2\omega\}^{\frac{1}{2}} + M\sum_{l=-\infty}^{+\infty}\lambda^{|l|}\bar{c}, \tag{12}$$

where $M^0 = C^0(2M)^{\frac{1}{2}}(1+\sqrt{\lambda})/(1-\sqrt{\lambda})$.

The relation

$$\bar{c} \le \begin{cases} \gamma_1\xi_1^{-(l+g)}\Big(\|\varphi - \bar{\varphi}\| + \dfrac{\xi_2\omega}{\xi_1 - 1}\Big) + \gamma_2\omega & \text{for} \quad l+g < 0; \\[2ex] \gamma_1(1+\alpha_1)^{l+g}\Big(\|\varphi - \bar{\varphi}\| + \dfrac{\alpha_2}{\alpha_1}\omega\Big) + \gamma_2\omega & \text{for} \quad l+g \ge 0 \end{cases}$$

implies

$$M\sum_{l=-\infty}^{+\infty}\lambda^{|l|}\bar{c} \le M(I_3 + I_4), \tag{13}$$

where

$$I_3 = \sum_{l=-\infty}^{-g-1} \lambda^{|l|} \left\{ \gamma_1 \xi_1^{-(l+g)} \Big( \|\varphi - \bar{\varphi}\| + \frac{\xi_2 \omega}{\xi_1 - 1} \Big) + \gamma_2 \omega \right\},$$

$$I_4 = \sum_{l=-g}^{+\infty} \lambda^{|l|} \left\{ \gamma_1 (1+\alpha_1)^{l+g} \Big( \|\varphi - \bar{\varphi}\| + \frac{\alpha_2}{\alpha_1} \omega \Big) + \gamma_2 \omega \right\}.$$

Similarly to inequalities (9), (10), we obtain the inequalities for $I_3$ and $I_4$,

$$I_3 \le K^3 \|\varphi - \bar{\varphi}\| + \bar{K}^3 \omega, \quad I_4 \le K^4 \|\varphi - \bar{\varphi}\| + \bar{K}^4 \omega, \tag{14}$$

where

$$K^3 = \gamma_1 \xi_1^{-g} \lambda_\xi, \quad \bar{K}^3 = \gamma_2 \frac{1+\lambda}{1-\lambda} + \frac{\gamma_1 \xi_2 \xi_1^{-g}}{\xi_1 - 1} \lambda_\xi,$$

$$K^4 = \gamma_1 (1+\alpha_1)^g \lambda_\alpha, \quad \bar{K}^4 = \gamma_2 \frac{1+\lambda}{1-\lambda} + \frac{\gamma_1 \alpha_2 (1+\alpha_1)^g}{\alpha_1} \lambda_\alpha.$$

Having in mind (12)–(14) and denoting $M(K^3 + K^4)$ and $M(\bar{K}^3 + \bar{K}^4)$ by $\eta_1$ and $\eta_2$, respectively, we obtain the inequality

$$\bar{u} \le M^0 \{\psi_1 \|\varphi - \bar{\varphi}\| + \psi_2 \omega(|\mu - \bar{\mu}|)\}^{\frac{1}{2}} + \eta_1 \|\varphi - \bar{\varphi}\| + \eta_2 \omega(|\mu - \bar{\mu}|),$$

which, since at some moment in the process $\|\varphi - \bar{\varphi}\| \to 0$, $|\mu - \bar{\mu}| \to 0$, gives us inequality (5), where $M_*$ stands for the expression $\max\{M^0(\max\{\psi_1, \psi_2\})^{\frac{1}{2}}, \eta_1, \eta_2\}$. The theorem is proved.

Now we prove a theorem that allows us to omit the condition 3) in Theorem 1.

**Theorem 2.** *Let all the conditions of Theorem 1 hold, excluding the third one. Then the function $u(p, g, \mu, \varphi)$ is continuous with respect to the set of variables $\varphi, \mu$ and, moreover, since at some moment in the process $\|\varphi - \bar{\varphi}\| \to 0$, $|\mu - \bar{\mu}| \to 0$, the inequality*

$$\|u(p, g, \mu, \varphi) - u(p, g, \bar{\mu}, \bar{\varphi})\| \le M^* \{\|\varphi - \bar{\varphi}\| + \omega(|\mu - \bar{\mu}|)\}^{\frac{\nu}{2(\nu+1)}} \tag{15}$$

*holds, where $M^*$ is a positive constant independent of $\{\varphi, \bar{\varphi}\} \subset \mathcal{T}_m$ and $\{\mu, \bar{\mu}\} \subset [\mu_1, \mu_2]$, and $\nu$ is an arbitrarily chosen positive real number satisfying the condition*

$$\frac{\nu}{\nu+1} < \min\{-\log_{\xi_1} \lambda; -\log_{(1+\alpha_1)} \lambda\}, \quad \xi_1 > 1. \tag{16}$$

***Proof.*** Let us put $\xi_1 > 1$ and denote

$$\|P(\varphi_n(\varphi,\mu),\mu) - P(\varphi_n(\bar\varphi,\bar\mu),\bar\mu)\|$$

by $\bar P$. Using (7) we get the inequality

$$\bar P \leq \begin{cases} P_1 \xi_1^{-\frac{n\nu}{\nu+1}} (\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}} & \text{for} \quad n < 0; \\ \\ P_2 (1+\alpha_1)^{\frac{n\nu}{\nu+1}} (\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}} & \text{for} \quad n \geq 0, \end{cases} \tag{17}$$

where $\nu$ is an arbitrarily chosen positive number, and $P_1, P_2$ stand for

$$(2P^0)^{\frac{1}{\nu+1}} \left( \max\{\beta_1; \frac{\beta_1 \xi_2}{\xi_1 - 1} + \beta_2\} \right)^{\frac{\nu}{\nu+1}},$$

$$(2P^0)^{\frac{1}{\nu+1}} \left( \max\{\beta_1; \frac{\beta_1 \alpha_2}{\alpha_1} + \beta_2\} \right)^{\frac{\nu}{\nu+1}},$$

respectively.

Relations (6) and (17) give the inequality

$$G \leq M^2 (I_1^0 + I_2^0), \tag{18}$$

where

$$I_1^0 = \sum_{k=-\infty}^{-p} \lambda^{|k|} P_1 \xi_1^{-\frac{\nu(k+p-1)}{\nu+1}} (\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}},$$

$$I_2^0 = \sum_{k=-p+1}^{+\infty} \lambda^{|k|} P_2 (1+\alpha_1)^{\frac{\nu(k+p-1)}{\nu+1}} (\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}}.$$

Because of condition (16), $\lambda \xi_1^{\frac{\nu}{\nu+1}} < 1$ and $\lambda(1+\alpha_1)^{\frac{\nu}{\nu+1}} < 1$, which allows us to write the inequalities

$$I_i^0 \leq \bar P_i (\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}}, \quad i = 1, 2, \tag{19}$$

where

$$\bar P_1 = P_1 \lambda^{(1)} \xi_1^{-\frac{\nu(p-1)}{\nu+1}}, \quad \bar P_2 = P_2 \lambda^{(2)} (1+\alpha_1)^{\frac{\nu(p-1)}{\nu+1}},$$

$$\lambda^{(1)} = \left(1 - \lambda \xi_1^{\frac{\nu}{\nu+1}}\right)^{-1} + \lambda \xi_1^{-\frac{\nu}{\nu+1}} \left(1 - \lambda \xi_1^{-\frac{\nu}{\nu+1}}\right)^{-1},$$

$$\lambda^{(2)} = \left(1 - \lambda(1+\alpha_1)^{-\frac{\nu}{\nu+1}}\right)^{-1} + \lambda(1+\alpha_1)^{\frac{\nu}{\nu+1}} \left(1 - \lambda(1+\alpha_1)^{-\frac{\nu}{\nu+1}}\right)^{-1}.$$

It is easy to see that (18) and (19) imply continuity of the GSF for equation (3) with respect to the set of variables $\varphi, \mu$, because

$$G \le M^2(\bar P_1 + \bar P_2)\{\|\varphi - \bar\varphi\| + \omega\}^{\frac{\nu}{\nu+1}}.$$

From the last inequality it, is easy to go to the next one,

$$G \le [2M^3(\bar P_1 + \bar P_2]^{\frac12} \lambda^{\frac{|l|}{2}} \{\|\varphi - \bar\varphi\| + \omega\}^{\frac{\nu}{2(\nu+1)}}. \tag{20}$$

And, at last, let us write the relation

$$\bar c \le \begin{cases} C_1 \xi_1^{-\frac{\nu(l+g)}{\nu+1}} (\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}} & \text{for} \quad l+g < 0; \\ C_2(1+\alpha_1)^{\frac{\nu(l+g)}{\nu+1}} (\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}} & \text{for} \quad l+g \ge 0, \end{cases}$$

in which

$$C_1 = (2C^0)^{\frac{1}{\nu+1}} \left( \max\Big\{\gamma_1; \frac{\gamma_1 \xi_2}{\xi_1 - 1} + \gamma_2\Big\}\right)^{\frac{\nu}{\nu+1}},$$

$$C_2 = (2C^0)^{\frac{1}{\nu+1}} \left( \max\Big\{\gamma_1; \frac{\gamma_1 \alpha_2}{\alpha_1} + \gamma_2\Big\}\right)^{\frac{\nu}{\nu+1}}.$$

Hence,

$$M \sum_{l=-\infty}^{+\infty} \lambda^{|l|} \bar c \le M(\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}} \Bigg\{ C_1 \xi_1^{-\frac{\nu g}{\nu+1}} \sum_{l=-\infty}^{-g-1} \lambda^{|l|} \xi_1^{-\frac{\nu l}{\nu+1}}$$

$$+ C_2(1+\alpha_1)^{\frac{\nu g}{\nu+1}} \sum_{l=-g}^{+\infty} \lambda^{|l|} (1+\alpha_1)^{\frac{\nu l}{\nu+1}} \Bigg\} \le C_3(\|\varphi - \bar\varphi\| + \omega)^{\frac{\nu}{\nu+1}}, \tag{21}$$

where

$$C_3 = M\{C_1 \xi_1^{-\frac{\nu g}{\nu+1}} \lambda^{(1)} + C_2(1+\alpha_1)^{\frac{\nu g}{\nu+1}} \lambda^{(2)}\}.$$

From (20) and (21) we obtain the inequality

$$\bar u \le \sum_{l=-\infty}^{+\infty} \{C^0 G + M\lambda^{|l|}\bar c\} \le C_4(\|\varphi - \bar\varphi\| + \omega(|\mu - \bar\mu|))^{\frac{\nu}{2(\nu+1)}}$$

$$+ C_3(\|\varphi - \bar\varphi\| + \omega(|\mu - \bar\mu|))^{\frac{\nu}{\nu+1}},$$

where $C_4 = C^0[2M^3(\bar{P}_1 + \bar{P}_2)]^{\frac{1}{2}}(1+\sqrt{\lambda})/(1-\sqrt{\lambda})$.

Having denoted $\max\{C_3, C_4\}$ by $M^*$ and since at some moment in the process $\|\varphi - \bar{\varphi}\| \to 0$, $\|\mu - \bar{\mu}\| \to 0$, we obtain inequality (15) which finishes the proof of the theorem.

***Remark 1.*** For any real $\nu \geq 0$ from inequality (15) it is not possible to obtain an inequality of form (5) with the indicator of power equal to $1/2$.

***Remark 2.*** Under conditions of Theorems 1 and 2, the function $u(p, g, \mu, \varphi)$, with $\mu$ fixed, satisfies the Gölder condition with respect to $\varphi$ with indicators of power $1/2$ and $\nu/(2(\nu+1))$, respectively.

In the case where the function $a(\varphi, \mu)$ does not depend on $\mu$, i.e., $a(\varphi, \mu) = a(\varphi)$, the continuity conditions for the function $u(p, g, \mu, \varphi)$ (that defines the invariant torus for the system of equations (1)) with respect to the parameter $\mu$ are much more simple.

***Corollary 1.*** *Let the conditions of Proposition 1 and condition 2) of Theorem 1 hold and $a(\varphi, \mu)$ do not depend on $\mu$. Then the inequalities*

$$\sup_{\varphi} \|P(\varphi, \mu) - P(\varphi, \bar{\mu})\| \leq \omega_1(|\mu - \bar{\mu}|),$$

$$\sup_{\varphi} \|c(\varphi, \mu) - c(\varphi, \bar{\mu})\| \leq \omega_2(|\mu - \bar{\mu}|),$$

*in which the functions $\omega_1(z), \omega_2(z)$ have properties of $\omega(z)$, guarantee continuity of the function $u(p, g, \mu, \varphi)$ with respect to the parameter $\mu$.*

***Proof.*** The equality (6) implies the inequality

$$G \leq \sum_{k=-\infty}^{+\infty} M^2 \lambda^{|k|} \|P(\varphi_{k+p-1}(\varphi), \mu) - P(\varphi_{k+p-1}(\varphi), \bar{\mu})\| \leq M^2 \frac{1+\lambda}{1-\lambda} \omega_1(|\mu - \bar{\mu}|),$$

which, in its turn, implies the inequality

$$G \leq \left(M^3 \frac{1+\lambda}{1-\lambda}\right)^{\frac{1}{2}} \lambda^{\frac{|l|}{2}} \omega_1^{\frac{1}{2}}(|\mu - \bar{\mu}|).$$

Similarly,

$$M \sum_{l=-\infty}^{+\infty} \lambda^{|l|} \|c(\varphi_{l+g}(\varphi), \mu) - c(\varphi_{l+g}(\varphi), \bar{\mu})\| \leq M \frac{1+\lambda}{1-\lambda} \omega_2(|\mu - \bar{\mu}|).$$

Then we have

$$\|u(p,g,\mu,\varphi) - u(p,g,\bar{\mu},\varphi)\| \le$$

$$\le \sum_{l=-\infty}^{+\infty} C^0 \left(M^3 \frac{1+\lambda}{1-\lambda}\right)^{\frac{1}{2}} \lambda^{\frac{|l|}{2}} \omega_1^{\frac{1}{2}}(|\mu-\bar{\mu}|) + M\frac{1+\lambda}{1-\lambda}\omega_2(|\mu-\bar{\mu}|)$$

$$\le C^0 \frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}} \left(M^3 \frac{1+\lambda}{1-\lambda}\right)^{\frac{1}{2}} \omega_1^{\frac{1}{2}}(|\mu-\bar{\mu}|) + M\frac{1+\lambda}{1-\lambda}\omega_2(|\mu-\bar{\mu}|).$$

The last inequality finishes the proof.

Finally, let us note that the results obtained here are preserved for the case where the torus, on which the input system of equations (1) is being considered, is infinite-dimensional, i.e., when $\varphi = \{\varphi^1, \varphi^2, \varphi^3, \dots\}$.

## REFERENCES


1. *Samoilenko A. M. and Teplinsky Yu. V.* "The invariant tori for linear countable systems of discrete equations defined on the infinite-dimensional torus," Ukr. Mat. Zh., **50**, No. 2, 244 – 251 ( 1998).
2. *Samoilenko A. M. and Teplinsky Yu. V.* The Limit Theorems in the Theory of Systems of Difference Equations [in Russian], Preprint, No. 98.3, Institute of Mathematics, Ukrainian Academy of Sciences, Kyiv (1998).
3. *Teplinsky Yu. V. and Marchuk N. A.* "The method of reduction in the investigation of smoothness of the invariant torus for a countable system of difference equations with parameters," Zb. Nauk. Prats' Kam'yanets'-Podil's'kogo Pedagog. Univ. Ser. Mat., No. 5, 117 – 126 (2000).
4. *Martynuk D. I. and Veriovkina Gh. V.* "The invariant sets for countable systems of difference equations," Visn. Kyiv. Univ. Ser. Fys.&Mat., No. 1, 117 – 127 (1997).
5. *Veriovkina Gh. V.* "On the introduction of local coordinates for a countable discreet system in a neighbourhood of the invariant torus," Visn. Kyiv. Univ. Ser. Fys.&Mat., No. 4, 23 – 29 (1997).
6. *Latushkin Yu. D. and Stiopin A. M.* "The weighted shift operators and linear extensions for dynamical systems," Usp. Mat. Nauk, **48**, No. 2, 85 – 143 (1991).


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