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SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS OF FUCHSIAN TYPE WITH FOUR SINGULARITIES

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We study a system of linear singularly perturbed functional differential equations by the method of integral manifolds. We construct a change of variables that decomposes this system into two subsystems, an ordinary differential equation on the center manifold and integral equations on the stable manifold.

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Consider a second order linear differential equation,

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

where $p(x)$ and $q(x)$ are arbitrary analytic functions. Given the initial conditions $x = x_0$, $y(x_0) = y_0$, $y'(x_0) = y'_0$, suppose we know a particular solution of the equation, $y_1(x)$. Let any other solution, which is linearly independent of y_1, be given by the formula

$$y = \xi(x)y_1. \tag{2}$$

By differentiating (2) along the solution y_1, we successively find that

$$2\xi' y'_1 + (p\xi' + \xi'')y_1 = 0, \tag{3}$$

$$(3\xi'' - p\xi')y'_1 + (p\xi'' + p'\xi' - 2q\xi' + \xi''')y_1 = 0. \tag{4}$$

Eliminating the variable $y_1(x)$ and its derivative from equations (3) and (4), we get the Schwarz equations for determining the function $\xi(x)$,

$$2\xi'\xi''' - 3\xi''^2 + (p^2 + 2p' - 4q)\xi'^2 = 0. \tag{5}$$

By setting

$$\xi' = \eta, \qquad \eta' = w\eta \tag{6}$$

in (5), to find the function $w(x)$, we get the Riccati equation

$$2w' = w^2 - (p^2 + 2p' - 4q). \tag{7}$$

It follows from (6) and (7) that, in order to find a general solution of equation (1), it is sufficient to find a particular solution of equation (7). In the sequel, we consider equation (1) as a

Fuchsian type equation with four singularities located in the points $x = 0, a_1, a_2$, and in $x = \infty$ $(a_1, a_2 \neq 0, a_1 \neq a_2)$ and written in the form

$$y'' + \frac{p_0x^2 + p_1x + p_2}{x(x-a_1)(x-a_2)}y' + \frac{q_0x^4 + q_1x^3 + q_2x^2 + q_3x + q_4}{x^2(x-a_1)^2(x-a_2)^2}y = 0. \tag{8}$$

The constant coefficients p_k and q_k, $k = \overline{0,4}$, must have the following form in this case [1]:

$$\begin{gathered} p_0 = \alpha_1 + \alpha_2 + \alpha_3, \\ p_1 = -(\alpha_1 a_2 + \alpha_2 a_1 + \alpha_3(a_1 + a_2)), \\ p_2 = \alpha_3 a_1 a_2, \quad \alpha_k = 1 - \rho_{k1} - \rho_{k2}, \quad k = 1, 2, 3, \end{gathered} \tag{9}$$

and

$$\begin{gathered} q_0 = \beta_4,\ q_1 = b - (a_1 + a_2)\beta_4,\ q_2 = \beta_1 + \beta_2 + \beta_3 + a_1a_2\beta_4 - (a_1 + a_2)b, \\ q_3 = -\beta_1 a_2 - \beta_2 a_1 - \beta_3(a_1 + a_2) + ba_1a_2,\ q_4 = \beta_3 a_1 a_2, \\ \beta_1 = \rho_{11}\rho_{12}a_1(a_1 - a_2),\ \beta_2 = \rho_{21}\rho_{22}a_2(a_2 - a_1), \\ \beta_3 = \rho_{31}\rho_{32}a_1a_2,\ \beta_4 = \rho_{01}\rho_{02}, \end{gathered} \tag{10}$$

where b is the accessor coefficient and the following Fuchsian condition holds:

$$\sum_{k=0}^{3}(1 - \rho_{k1} - \rho_{k2}) = 2, \tag{11}$$

where ρ_{01} and ρ_{02} are exponents with respect to the point $z = \infty$.

Let us look for a solution of (7) in the form

$$w = \frac{v_0x^2 + v_1x + v_2}{x(x-a_1)(x-a_2)}. \tag{12}$$

Substituting (12) into (7) we find

$$\begin{aligned} 2\bigl(&-v_0x^4 - 2v_1x^3 + (v_0a_1a_2 + v_1(a_1 + a_2) - 3v_2)x^2 \\ &+ 2v_2(a_1 + a_2)x - v_2a_1a_2\bigr) \\ = {}& (v_0x^2 + v_1x + v_2)^2 - (p_0x^2 + p_1x + p_2)^2 \\ &+ 4(q_0x^4 + q_1x^3 + q_2x^2 + q_3x + q_4) - 2\bigl(-p_0x^4 - 2p_1x^3 \\ &+ (p_0a_1a_2 + p_1(a_1 + a_2) - 3p_2)x^2 + 2p_2(a_1 + a_2)x - p_2a_1a_2\bigr). \end{aligned} \tag{13}$$

Using (13) we get the following system for finding the unknowns v_0, v_1, and v_2:

$$(v_0+1)^2 = p_0^2 - 4q_0 + 1 - 2p_0, \qquad (v_0+2)v_1 = (p_0-2)p_1 - 2q_1,$$
$$2(v_0a_1a_2 + v_1(a_1+a_2) - 3v_2)$$
$$= v_1^2 + 2v_0v_2 - p_1^2 - 2p_0p_2 + 4q_2 - 2(p_0a_1a_2 + p_1(a_1+a_2) - 3p_2), \tag{14}$$
$$2v_2(a_1+a_2) = v_1v_2 - p_1p_2 + 3q_3 - 2p_2(a_1+a_2),$$
$$v_2^2 + 2v_2a_1a_2 - p_2^2 + 2p_2a_1a_2 + 4q_4 = 0.$$

Using notations (9), (10) and identity (11) we find from the first equation of system (14) that

$$v_0 = \varepsilon_1(\rho_{01} - \rho_{02}) - 1, \qquad \varepsilon_1^2 = 1. \tag{15}$$

Similarly, from the fifth equation of system (14) we get

$$v_2 = (\varepsilon_2(\rho_{31} - \rho_{32}) - 1)a_1a_2, \qquad \varepsilon_2^2 = 1. \tag{16}$$

The second and the fourth equations of system (14), with the use of (15) and (16), become

$$\begin{aligned}(\varepsilon_1(\rho_{01}-\rho_{02})+1)v_1 + 2b &= \gamma_{11}a_1 + \gamma_{12}a_2,\\ (\varepsilon_2(\rho_{31}-\rho_{32})-1)v_1 + 2b &= \gamma_{21}a_1 + \gamma_{22}a_2,\end{aligned} \tag{17}$$

where

$$\gamma_{11} = \alpha_0(\alpha_2+\alpha_3) + 2\beta_4, \qquad \gamma_{12} = \alpha_0(\alpha_1+\alpha_3) + 2\beta_4,$$
$$\gamma_{21} = 2(\varepsilon_2(\rho_{31}-\rho_{32})-1) + \alpha_3(\alpha_0+\alpha_1) + 2(\rho_{31}\rho_{32} + \rho_{11}\rho_{12} - \rho_{21}\rho_{22}), \tag{18}$$
$$\gamma_{22} = 2(\varepsilon_2(\rho_{31}-\rho_{32})-1) + \alpha_3(\alpha_0+\alpha_2) + 2(\rho_{31}\rho_{32} - \rho_{11}\rho_{12} + \rho_{21}\rho_{22}).$$

Using system (17) we find that

$$[\varepsilon_1(\rho_{01}-\rho_{02}) - \varepsilon_2(\rho_{31}-\rho_{32}) + 2]v_1 = (\gamma_{11}-\gamma_{21})a_1 + (\gamma_{12}-\gamma_{22})a_2$$

and if

$$\delta \equiv \varepsilon_1(\rho_{01}-\rho_{02}) - \varepsilon_2(\rho_{31}-\rho_{32}) + 2 \neq 0, \tag{19}$$

then

$$v_1 = \frac{1}{\delta}[(\gamma_{11}-\gamma_{21})a_1 + (\gamma_{12}-\gamma_{22})a_2], \tag{20}$$

$$\begin{aligned} b &= \frac{1}{2\delta}[(\varepsilon_1(\rho_{01}-\rho_{02})+1)\gamma_{21} - (\varepsilon_2(\rho_{31}-\rho_{32})-1)\gamma_{11}]a_1\\ &\quad + \frac{1}{2\delta}[(\varepsilon_1(\rho_{01}-\rho_{02})+1)\gamma_{22} - (\varepsilon_2(\rho_{31}-\rho_{32})-1)\gamma_{12}]a_2. \end{aligned} \tag{21}$$

The third equation of (14) becomes

$$(v_1 - a_1 - a_2)^2 - (p_1 + a_1 + a_2)^2 = 2(a_1a_2 - v_2)v_0 - 6v_2 + 2p_0(p_2 + a_1a_2) - 6p_2 - 4q_2,$$

or using notations (9), (10) and identities (11), (20), and (21) we get

$$k_0a_1^2 + 2k_1a_1a_2 + k_2a_2^2 = 0, \tag{22}$$

where

$$\begin{aligned}
k_0 \equiv\ & (\gamma_{11} - \gamma_{21} - \delta)^2 - (\alpha_2 + \alpha_3 - 1)^2\delta^2 + 4\rho_{11}\rho_{12}\delta^2 \\
& - 2\delta[(\varepsilon_1(\rho_{01} - \rho_{02}) + 1)\gamma_{21} - (\varepsilon_2(\rho_{31} - \rho_{32}) - 1)\gamma_{11}], \\
k_1 \equiv\ & (\gamma_{11} - \gamma_{12} - \delta)(\gamma_{12} - \gamma_{22} - \delta) - (\alpha_2 + \alpha_3 - 1)(\alpha_1 + \alpha_3 - 1)\delta^2 \\
& + 2(\rho_{31}\rho_{32} + \rho_{01}\rho_{02} - \rho_{11}\rho_{12} - \rho_{21}\rho_{22})\delta^2 - \delta[(\varepsilon_1(\rho_{01} - \rho_{02}) + 1)(\gamma_{21} + \gamma_{22}) \\
& - (\varepsilon_2(\rho_{31} - \rho_{32}) - 1)(\gamma_{11} + \gamma_{12})] \\
& - [2\delta - \varepsilon_1\varepsilon_2(\rho_{01} - \rho_{02})(\rho_{31} - \rho_{32}) - \alpha_0\alpha_3 - \alpha_0 - \alpha_3 - 1]\delta^2, \qquad (23) \\
k_2 \equiv\ & (\gamma_{12} - \gamma_{22} - \delta)^2 - (\alpha_1 + \alpha_3 - 1)^2\delta^2 \\
& + 4\rho_{21}\rho_{22}\delta^2 - 2\delta[(\varepsilon_1(\rho_{01} - \rho_{02}) + 1)\gamma_{22} - (\varepsilon_2(\rho_{31} - \rho_{32}) - 1)\gamma_{12}].
\end{aligned}$$

Equation (22) is a condition imposed on the coefficients of equation (8) so that the function given by (12) is a partial solution of equation (7). Considering (22) as a quadratic equation for the unknowns a_k, $k = 1, 2$, we should keep in mind that its roots, λ_k, $k = 1, 2$, as follows from the sense of the problem, must be distinct and nonzero. Suppose we found from (22) that

$$a_1 = \lambda_k a_2, \qquad k = 1, 2, \quad \lambda_k \neq 1. \tag{24}$$

Represent the particular solution (12) of the Riccati equation (7) as

$$\frac{v_0x^2 + v_1x + v_2}{x(x - a_1)(x - a_2)} = \frac{r_1}{x} + \frac{r_2}{x - a_1} + \frac{r_3}{x - a_2}. \tag{25}$$

To evaluate the unknowns r_k, $k = 1, 2, 3$, (25) gives the system

$$\begin{aligned}
r_1 + r_2 + r_3 &= \varepsilon_1(\rho_{01} - \rho_{01}) - 1, \\
(r_1 + r_3)a_1 + (r_1 + r_2)a_2 &= \frac{1}{\delta}[(\gamma_{21} - \gamma_{11})a_1 + (\gamma_{22} - \gamma_{12})a_2], \qquad (26) \\
r_1 &= \varepsilon_2(\rho_{31} - \rho_{32}) - 1.
\end{aligned}$$

Using (24) we find from system (26) that

$$r_2 = \delta - 2 - r_3, \tag{27}$$

where

$$\begin{aligned} r_3 \quad = \quad & \frac{1}{\lambda_k - 1}\Bigg[\frac{1}{\delta}(\gamma_{21} - \gamma_{11})\lambda_k + \frac{1}{\delta}(\gamma_{22} - \gamma_{12}) + 2 - \delta \\ & - (1 + \lambda_k)(\varepsilon_2(\rho_{31} - \rho_{32}) - 1)\Bigg]. \end{aligned}$$

Let us set, in equation (7),

$$W = \frac{r_1}{x} + \frac{r_2}{x - a_1} + \frac{r_3}{x - a_2} + V. \tag{28}$$

To find the function V, we have the following equation:

$$2V' = V^2 + \Big(\frac{r_1}{x} + \frac{r_2}{x - a_1} + \frac{r_3}{x - a_2}\Big)V,$$

from which we find that

$$V = \frac{2x^{r_1}(x - a_1)^{r_2}(x - a_2)^{r_3}}{C_1 - \int x^{r_1}(x - a_1)^{r_2}(x - a_2)^{r_3}\,dx},$$

and, consequently,

$$W = \frac{r_1}{x} + \frac{r_2}{x - a_1} + \frac{r_3}{x - a_2} + \frac{2x^{r_1}(x - a_1)^{r_2}(x - a_2)^{r_3}}{C_1 - \int x^{r_1}(x - a_1)^{r_2}(x - a_2)^{r_3}\,dx}. \tag{29}$$

By substituting (29) into formulas (6), we find

$$\begin{aligned} \eta(x) &= C_2\frac{x^{r_1}(x - a_1)^{r_2}(x - a_2)^{r_3}}{\big[C_1 - \int x^{r_1}(x - a_1)^{r_2}(x - a_2)^{r_3}\,dx\big]^2}, \\ \xi(x) &= C_3 + C_2\frac{1}{-C_1 + \int x^{r_1}(x - a_1)^{r_2}(x - a_2)^{r_3}\,dx}. \end{aligned} \tag{30}$$

Now, using equation (3) find $y_1(x)$. Namely,

$$\begin{aligned} y_1(x) \quad = \quad & \frac{C_4}{C_2}x^{-\frac{1}{2}(r_1+\alpha_1)}(x - a_1)^{-\frac{1}{2}(r_2+\alpha_2)}(x - a_2)^{-\frac{1}{2}(r_3+\alpha_3)} \\ & \times\left[C_1 - \int x^{r_1}(x - a_1)^{r_2}(x - a_2)^{r_3}\,dx\right]. \end{aligned} \tag{31}$$

Substituting (30) and (31) into formula (2) we finally find that

$$\begin{aligned} y(x) &= \xi(x)y_1(x) \\ &= x^{-\frac{1}{2}(r_1+\alpha_1)}(x-a_1)^{-\frac{1}{2}(r_2+\alpha_2)}(x-a_2)^{-\frac{1}{2}(r_3-\alpha_3)} \\ &\quad \times \left[C + C_1 \int x^{r_1}(x-a_1)^{r_2}(x-a_2)^{r_3}\,dx\right], \end{aligned} \tag{32}$$

where C and C_1 are new arbitrary constants.

The preceding gives the following theorem.

Theorem. *For equation (8) to have a general solution of the form (32), it is sufficient that 1) the accessor coefficient* $\mathfrak{b}$ *have the form (21) and 2) its coefficients satisfy the condition (22).*

Together with equation (8), consider the related Heun equation

$$\begin{aligned} y'' + \frac{(\alpha+\beta+1)x^2 - [a(\gamma+\delta)+\alpha+\beta-\delta+1]x + a\gamma}{x(x-1)(x-a)} y' \\ + \frac{(\alpha\beta x - q)}{x(x-1)(x-a)} y = 0, \end{aligned} \tag{33}$$

the coefficients of which, as opposed to the coefficients of (9) and (10), have the form

$$p_0 = \alpha+\beta+1,\ p_1 = -[a(\gamma+\delta)+\alpha+\beta-\delta+1],\ p_2 = a\gamma,\ a_1 = 1,\ a_2 = a, \tag{34}$$

$$q_0 = \alpha\beta,\ q_1 = -(a+1)\alpha\beta - q,\ q_2 = a\alpha\beta + (a+1)q,\ q_3 = -aq,\ q_4 = 0. \tag{35}$$

Using the structure of the general solution of equation (8) in the form (32), a particular solution of (33) is sought in the form

$$y_1 = x^{s_1}(x-1)^{s_2}(x-a)^{s_3}, \tag{36}$$

where the constants s_1, s_2, s_3 are to be found. From (36) we get

$$\begin{aligned} y' &= \left(\frac{s_1}{x} + \frac{s_2}{x-1} + \frac{s_3}{x-a}\right) y, \\ y'' &= \left[\left(\frac{s_1}{x} + \frac{s_2}{x-1} + \frac{s_3}{x-a}\right)^2 - \left(\frac{s_1}{x^2} + \frac{s_2}{(x-1)^2} + \frac{s_3}{(x-a)^2}\right)\right] y. \end{aligned} \tag{37}$$

Substituting (37) into (33) we get the system

$$(s_1+s_2+s_3)^2+(p_0-1)(s_1+s_2+s_3)+q_0=0,$$

$$2s_1(s_1-1)(a+1)+2as_2(s_2-1)+2s_3(s_3-1)+2s_1s_2(2a+1)$$

$$+2s_2s_3(a+1)+2s_1s_3(a+2)+p_0[(a+1)s_1+as_2+s_3]$$

$$-p_1(s_1+s_2+s_3)-q_1=0,$$

$$s_1(s_1-1)(a^2+4a+1)+s_2(s_2-1)a^2+s_3(s_3-1)+2s_1s_2(a^2+2a)$$

$$+2s_2s_3a+2s_1s_3(1+2a)+p_0as_1-p_1[(a+1)s_1+as_2+s_3]$$

$$+p_2(s_1+s_2+s_3)+q_2=0, \tag{38}$$

$$2s_1(s_1-1)(a^2+2a)+2s_1s_2a^2+2s_1s_3a-p_1as_1$$

$$+p_2[(a+1)s_1+as_2+s_3]-q_3=0,$$

$$s_1(s_1-1)a^2+p_2as_1=0.$$

It follows from the first and the fifth equations of system (38) that

1) either $s_1+s_2+s_3=-\alpha$,
2) or $s_1+s_2+s_3=-\beta$ and (39)
3) either $s_1=0$, 4) or $s_1=1-\gamma$.

The fourth equation of system (38) defines the accessor coefficient q,

$$q=-[2s_1(s_1-1)(a+2)+2s_1s_2a+2s_1s_3-s_1p_1+\gamma((a+1)s_1+as_2+s_3)]. \tag{40}$$

Substituting (40) into the second equation of (38) and setting

$$s_2=h-s_1-s_3, \tag{41}$$

where h equals either $-\alpha$ or $-\beta$ we find that

$$(2s_1-2h+\gamma-\alpha-\beta+1)(a-1)s_3=[2(s_1-h)(h-1)+h(\gamma-p_0)]a$$

$$+3s_1^3-(2h+1)s_1+(\gamma-p_0)s_1+p_1(h-s_1). \tag{42}$$

Assume that, for any choice of s_1 and h, the quantity

$$2s_1-2h+\gamma-\alpha-\beta+1\neq 0. \tag{43}$$

Note that, if $a \neq 1$, then assuming that the condition (43) holds, the quantities s_1, s_2, and s_3 can be uniquely expressed in terms of the parameters α, β, γ, δ, and a using formulas (39), (41), and (42). Substituting their values into the third equation of system (38), the condition implies that equation (33) has a particular solution of the form (36). Then the general solution of equation (33) will be

$$y = x^{s_1}(x-1)^{s_2}(x-a)^{s_3}$$

$$\times \left[C_1 + C_2 \int x^{-2s_1}(x-1)^{-2s_2}(x-a)^{-2s_3} \exp\left(-\int p(x)\,dx\right)\,dx\right]. \qquad (44)$$

The cases where the condition (19) or (43) is violated and the comparison of general solutions of the forms (32) and (44) are not considered in this paper.

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