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MINIMAL DYNAMICAL SYSTEM WITH GIVEN *D*-FUNCTION AND TOPOLOGICAL ENTROPY

D-function is a new topological invariant introduced by the author in [3] to classify the minimal dynamical system and to generalize Sharkovsky's theorem on coexistence of periodic orbits. We show that the D-function and the topological entropy are independent.

D-функція мінімальної множини — новий інваріант, запропонований автором [3] для класифікації мінімальних множин і розповсюдження на них теорем А. М. Шарковського про співіснування періодичних орбіт. Доведена незалежність *D*-функції та топологічної ентропії.

1. Introduction. Topological dynamics is used in the study of continuous transformations of a topological space onto itself. The randomness of such transformations can be measured by topological entropies. Hahn and Katznelson [1] and Grillenberger [2] assumed that minimality of topological dynamics has no influence on its topological entropies. In [3], the author introduces a new topological invariant, *D*-function, to classify minimal dynamical systems in compact Hausdorff spaces and to generalize Sharkovsky's theorem on coexistence of periodic orbit of interval mapping to minimal sets.

Let X be a compact Hausdorff space, $f \in C^0(X, X)$ and let A be a minimal set of f. Then *D-function of* A, f_A, is n if A is a periodic orbit with period n; it is $n' \in \mathrm{N}'$ if A is not a periodic orbit of f but the number of distinct minimal sets of f^k which are contained in A is (n, k) for all $k \in \mathrm{N}$; it is a function from N to N such that $f_A(k)$ is a number of distinct minimal sets of f^k which are contained in A, and f_A is not a bounded function. For convenience, we also write $f_A(k) = (n, k)$ for $k \in \mathrm{N}$ if $f_A = n'$. It was proved in [3] that $f_A \in \mathcal{Y}$, where $\mathcal{Y} = \mathcal{Z} \cup \mathrm{N}'$,

$$\mathcal{Z} = \{ s : \mathrm{N} \to \mathrm{N} \mid \quad 1)\ s(m \cdot n) = s(m) \cdot s(n), \text{ if } (m, n) = 1;$$
$$2) \text{ for every prime number } p \text{ either } s(p^l) = p^l \text{ for all } l \in \mathrm{N}$$
$$\text{or } s(p^l) = (p^l, p^{l_0}) \text{ for all } l \in \mathrm{N} \text{ and some } l_0 \in \mathrm{N} \cup \{0\} \}.$$

$\mathrm{N}' = \{ n' : n \in \mathrm{N} \}$. Let $E = \{ s \in \mathcal{Z} : s(n) = s(n, n_0)$ for all $n \in \mathrm{N}$ and some $n_0 \in \mathrm{N} \}$. Note that we identify $n \in \mathrm{N}$ with a function $s \in E$ defined by $s(n) = (n, n_0)$ for all $n \in \mathrm{N}$.

A natural question arises: Let X be a compact Hausdorff space, $f \in C^0(X, X)$, and A be a minimal set of f with some D-function $\in \mathcal{Y}$; is it possible for (A, f) to have positive topological entropy ?

In this paper, we first give fundamental notions and basic lemmas (sec. 2). In sec. 3, we construct the minimal dynamical system related to a number $0 \le \delta \le \log K$ and the D-function $s \in \mathcal{Z} \setminus E$ in the shift space with K alphabet and show that the minimal dynamical system which we constructed has the topological entropy equal to the given number δ and the D-function equal to s. In the last section, we prove the existence of the minimal dynamical system with a given topological entropy $0 \le \delta \le \log K$ and a given D-function $n' \in \mathrm{N}'$. The examples in the last section are strictly ergodic but not in a constructive way as in sect. 3.

Since any periodic orbit always has the topological entropy zero, we can conclude that the answer to the question above is positive for all $s \in \mathcal{Y} \setminus E$.

We mention that our construction of the minimal dynamical system with a given topological entropy (sect. 3) is simpler than those of [1] and [2], but we know under which conditions the constructed systems are strictly ergodic.

2. Preliminaries. Let X be a topological space, $f \in C^0(X, X)$. We call $O(x, f) = \{x, f(x), f^2(x), \ldots\}$ the orbit of x under f. ω-limit set of x under f, $\omega(x, f)$, is the set $\bigcap_{i=1}^{\infty} O(f^i(x), f)$. A subset A of X is called minimal under f provided that A is nonvacuous closed and invariant $(f(A) \subseteq A)$ under f, and any proper subset of A does not have these properties. A point $x \in X$ is called almost periodic under f provided that for each neighborhood U of x there exists a corresponding $m \in \mathbb{N}$ with the properties that in every set of m consecutive positive integers one can find an integer n such that $f^n(x) \in U$. Denote the set of almost periodic points by $AP(f)$.

Let $P(K) = \{0, 1, \ldots, K-1\}$, $\Sigma_K = P(K)^{\mathbb{N}} = \{(x_1, x_2, \ldots) | x_i \in P(K), i \in \mathbb{N}\}$.

If we equip $P(K)$ with the discrete topology, then Σ_K with the product topology becomes a compact metrizable space. One of the metrics is

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 2^{-k}, & \text{if } x_i = y_i, 1 \le i\ k-1, x_k \ne y_k, \end{cases}$$

for $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in \Sigma_K$. A shift σ is a transformations from Σ_k to Σ_k satisfying $\sigma(x) = (x_3, x_4, \ldots)$ if $x = (x_1, x_2, \ldots)$. Furthermore, we use the following notations : $\Sigma_K^f = \bigcup_{n \in N} P(K)^n$. Every element of Σ_K^f is called a block over $P(K)$. For $A \in P(K)^n$, we call $l(A) = n$ the length of A. If $A_i = \left(a_i^1, a_i^2, \ldots, a_i^{l(A_i)}\right) \in \Sigma_K^f$, we write

$$A_i A_j = \left(a_i^1, \ldots, a_i^{l(A_i)}, a_j^1, \ldots, b_j^{l(A_j)}\right) \in \Sigma_K^f, \quad \lim_{k \to \infty} A_1 A_2 \ldots, A_k = A_1 A_2, \ldots, \in \Sigma_K.$$

Let $A \in \Sigma_K$, $B \in \Sigma_k \cup \Sigma_K^f$. We write $A < B$, if there exist i_0 such that $A = (b_{i_0}, \ldots \ldots, b_{i_0+n-1})$.

In what follows, we suppose that X is a compact metric space, $f \in C^0(X, X)$, and A is a minimal set of f.

Lemma 1. [4]. *For every minimal set A, $A \subset AP(f)$; for every $x \in AP(f)$, $O(x, f)$ is a minimal set of f.*

***Remark* 1.** We denote $f_{\overline{O(x,f)}}$ by f_x for every $x \in A$.

Lemma 2. [3]. $f_A \in \mathcal{Y}$.

Lemma 3. [3]. 1) *$x \in \omega(f(x), f^n)$ for some $n \in \mathbb{N}$ and some $x \in A$ if and only if $f_x(n) = 1$;* 2) *$x \notin \omega(f^i(x), f^n)$ for all $1 \le i \le n-1$ and some $x \in A$ if and only if $f_x(n) = n$;* 3) *$f_x(m\,n) = (f^m)_x(n) f_x(m)$ for every $m, n \in \mathbb{N}, x \in A$.*

Lemma 4. *Let X, Y be two compact Hausdorff spaces, $f \in C^0(X, X)$, $g \in C^0(Y, Y)$ and h be a homeomorphism from X onto Y satisfying $h \circ f = g \circ h$. Then $f_A = g_{h(A)}$, where A is an arbitrary minimal set of f.*

The proof of this lemma is easy, we omit it.

3. Construction of Minimal Dynamical System with D-function $\in \mathfrak{Z} \setminus E$.

3.1. ***The construction.*** We construct our system in the shift space with finite alphabet, Σ_K. Let $0 \le \delta \le \log K$, s be an element of $\mathfrak{Z} \setminus E$ and let $p_1 < p_2, \ldots$ be all prime numbers. Define $T: \mathbb{N} \to \mathbb{N}$ by

$$T(i) = s\left(p_1^i p_2^{i-1}, \ldots, p_i\right), \quad \forall i \in \mathbb{N}.$$

It is easy to see that $T(i) \mid T(i+1)$, $i \in \mathrm{N}$, and $\lim_{i\to\infty} T(i) = \infty$.

We construct the minimal dynamical system as follows.

Let $i_1 = 1$, $a = (a_1, \ldots, a_{T(i_1)}) \in P(K)^{T(i_1)}$ and $M_1 = P(K)$. In what follows, we construct $M_j, M'_j \subset \Sigma_K^f$, $j \geq 2$. Note that we endow M_j, M'_j with the lexicographic ordering and denote by $|M_j|, |M'_j|$ the number of elements in M_j and M'_j, respectively; m_j^i is the i-th element of M_j in the lexicographic ordering.

Define

$$M_2 = \{ a\, m_2 : m_2 \in M_1^{T(i_2)-T(i_1)} \},$$

where i_2 satisfies

$$(\log(|M_2| - 1)) / T(i_2) > \delta \text{ and } T(i_2) > T(i_1).$$

Let

$$k(2) = 1, \ M'_2 = M_2 \setminus \{ m_2^{k(2)} \},$$

$$M_3 = \{ m_2^{k(2)} m_3 : m_3 \in M_2'^{(T(i_3)-T(i_2))/T(i_2)} \},$$

where i_3 satisfies

$$(1/T(i_2) - 1/T(i_2)) \log(|M_2| - 1) > \delta \text{ and } T(i_3) \geq (|M_2| - 1)\, T(i_2).$$

For $j \geq 4$, we define M_j, M'_j inductively. Namely, if we have already defined M_{j-1}, then we can define

$$M_j = \{ m_{j-1}^{k(j-1)} m_j : m_j \in M_{j-1}'^{(T(i_j)-T(i_{j-1}))/T(i_{j-1})} \},$$

where $m_{j-1}^{k(j-1)}$ satisfies $\forall a \in M'_{j-2}, a < m_{j-1}^{k(j-1)}$, $m_{j-1}^{k(j-1)} \notin M'_{j-1}$, and M'_{j-1} satisfies

$$\delta < (\log(|M_2| + 1)) / T(i_{j-1}) < \delta + 1/(j-1).$$

Finally, we choose i_j such that :

1) $((1/T(i_{j-1}) - 1/T(i_j)) \log(|M'_{j-1}| > \delta$;

2) $T(i_j) \geq |M'_{j-1}| \cdot T(i_{j-1})$;

3) $(\log(T(i_{j-1})|M'_{j-1}|)) / T(i_j) < 1/j$.

***Remarks.* 2**. For every $A \in M_j$ or M'_j, we have $l(A) = T(i_j)$.

3. For every $A, B \in M_j$ or M'_j, we have

$$a_1 a_2, \ldots, a_{T(i_j-1)} = b_1 b_2, \ldots, b_{T(i_j-1)} = m_{j-1}^{k(j-1)}.$$

4. $m_2^{k(2)} < m_3^{k(3)} < , \ldots$

We now define $x(s, \delta) = \lim_{i\to\infty} m_j^{k(j)}$ and claim that $\overline{O(x(s,\delta),\sigma)}$ is the minimal dynamical system with topological entropy equal the to given number δ and with the given D-function s.

2. The Proofs. We denote $x = x(s, \delta)$. First, we show :

Theorem 1. $\sigma_x = s$.

Proof. We split our proof in to several steps :

a) $x \in AP(\sigma)$. This statement can be verified directly by using Remarks 2 – 4.

b) $\sigma_x(T(i_j)) = T(i_j)$ for all $j \in \mathrm{N} \setminus \{1\}$. We prove this by induction. First, we show that $\sigma_x(T(i_2)) = T(i_2)$.

By the construction of x, we can write $x = A_1 A_2, \ldots$, where $A_i = a A_i'$, $|A_i| = T(i_2)$, $\forall i \in \mathbb{N}$ such that $\forall b \in M_2'$, $b < A_1 A_2, \ldots, A_{T(i_3)} = m_3^{k(3)}$.

Suppose $\sigma_x(T(i_2)) \neq T(i_2)$. By Lemma 3 there exists $1 \le i_0 \le T(i_2) - 1$ such that $x \in \omega(\sigma^{i_0}(x), \sigma^{T(i_2)})$. Then there exists j_0 such that

$$\rho(x, \sigma^{j_0 T(i_2) + i_0}(x)) < 2^{-T(i_4)}.$$

So, if $1 \le i_0 \le T(i_1)$, then

$$x_n = (\sigma^{j_0 T(i_2) + i_0}(x))_n = x_{(j_0 T(i_2) + i_0 + n)}, \quad 1 \le n \le T(i_4).$$

Let $n = j T(i_2) - i_0 + 1$. We get

$$x_{(jT(i_2) + 1 - i_0)} = x_{(j + j_0) T(i_2) + 1} = a_1, \quad 1 \le j \le T(i_4) / T(i_2). \tag{1}$$

If $T(i_1) + 1 \le i_0 \le T(i_2) - 1$, let $n = j T(i) + T(i_1) - i_0$, then we obtain

$$x_{(jT(i_2) + T(i_1) - i_0)} = x_{(j + j_0) T(i_2) + T(i_1)} = a_{T(i_1)}, \quad 1 \le j \le T(i_4) / T(i_2). \tag{2}$$

Since $\forall b < x$, if $l(b) = T(i_4)$, we have $m_3^{k(3)} < b$; hence,

$$\{ x_{jT(i_2) + 1 - i} : 1 \le j \le T(i_4) / T(i_2) \} = P(K) \text{ for all } 1 \le i \le T(i_1)$$

and

$$\{ x_{jT(i_2) + T(i_1) - i} : 1 \le j \le T(i_4) / T(i_2) \} = P(K) \text{ for all } T(i_1) + 1 \le i \le T(i_1) - 1 \}.$$

This contradicts (1) and (2). This means that $(T(i_2)) = T(i_2)$.

Now let θ_* be a bijection from $P(K)^n$ onto $P(K^n)$, then the following diagram is commutative:

$$\begin{array}{ccc} \omega(x, \sigma^n) & \xrightarrow{\sigma^n} & \omega(x, \sigma^n) \\ \downarrow \theta & & \downarrow \theta \\ \omega(x', \sigma') & \xrightarrow{\sigma'} & \omega(x', \sigma'). \end{array}$$

Here, $x' = \theta(x)$ and θ is induced by θ_* in the following way: $\theta(z) = \theta_*(Z_1)\theta_*(Z_2), \ldots$, if $z = Z_1 Z_2, \ldots \in \omega(x, \sigma^n)$ and $l(Z_i) = n$, $\forall i \in \mathbb{N}$. By Lemma 3, we know that $\sigma_x(n m) = (\sigma^n)_x(m)\, \sigma_x(n)$. Thus, $\sigma_x(n m) = (\sigma^n)_x(m)\, \sigma_x(n) = (\sigma')_{\theta(x)}(m)\, \sigma_x(n)$ by Lemma 4. Let $n = T(j)$, $m = T(j+1) / T(j)$, then we obtain

$$\sigma_x(T(j+1)) = (\sigma')_{\theta(x)}(T(j+1) / T(j))\, \sigma_x(T(j)).$$

By the construction of x, it is easy to see that $\theta(x) \notin \omega(\sigma^i x, \sigma^{T(j+1)/T(j)})$ $(1 \le i \le T(j+1) / T(j))$, because $\theta_*(m_j^{k(j)} \notin \theta_*(M_j)$. In other words,

$$(\sigma')_{\theta(x)}(T(j+1) / T(j)) = T(j+1) / T(j).$$

Hence,

$$\sigma_x(T(j+1)) = (T(j+1) / T(j))\, \sigma_x(T(j)), \quad j \ge 2.$$

From this equality, we immediately get $\sigma_x(T(j)) = T(j)$, $j \ge 2$.

c) If $s(p^{l+1}) = p^l$ for some prime number p and some $l \in \mathbb{N}$, then $\sigma_x(p^{l+1}) = p^l$.

We choose j_0 such that $p^l | T(j)$, $\forall j \ge j_0$. By the definition of T, we get

$(p, T(i_j)/p^l) = 1$, $\forall j \geq j_0$. Hence, there exist $k(j)$ such that $(T(j)/p^l) | (k(j)p + 1)$, $j \geq j_0$. This means that $T(j) | (k(j)p^{l+1} + p^l)$. Thus, $x \in \omega(\sigma^{p^l}(x), \sigma^{p^{l+1}})$, because the beginning of each block in M_j is $m_{j-1}^{k(j-1)}$. By Lemma 3 and step b), we get $\sigma_x(p^{l+1}) < p^{l+1}$, $\sigma_x(p^l) = p^l$, because $\sigma_x(T(i_j)) = T(j)$, $j \geq 2$. On the other hand, by Lemma 2 we know that $\sigma_x(p^{l+1}) | p^{l+1}$, therefore, $\sigma_x(p^{l+1}) = p^l$.

d) $\sigma_x = s$. Since $s \in \mathcal{Z} \setminus E$, for every prime number p either 1) $s(p^l) = p^l$ for all $l \in \mathrm{N}$; or 2) $s(p^{l+1}) = (p^l, p^{l_0})$ for all $l \in \mathrm{N}$ and some $l_0 \in \mathrm{N} \cup \{0\}$. In the first case, by step b), $\sigma_x(p^l) = p^l = s(p^l)$ for all $l \in \mathrm{N}$, in the second case, by steps b), c),

$$\sigma_x(p^l) = \begin{cases} p^l, & l \leq l_0; \\ p^{l_0}, & l \geq l_0. \end{cases} = \left(p^l, p^{l_0}\right) = s(p^l).$$

Thus, $\sigma_x = s$.

Topological entropy was introduced by Adler, Konheim and McAndrew [5] in 1965. In the shift space Σ_K, we can define it as follows. Let $\Sigma \subset \Sigma_K$ be a nonempty closed invariant subset and

$$\theta_n = \theta_n(\Sigma) = |\{A \in P^n \mid A < x, x \in \Sigma\}|.$$

Then $1 \leq \theta_{n+m} \leq \theta_n \cdot \theta_m$, therefore, $\lim_{n\to\infty}(1/n)\log\theta_n$ exists. This limit is just the topological entropy of σ restricted to the subset Σ; it is denoted by $h(\sigma|_\Sigma)$.

Theorem 2. $h(\sigma|_{\omega(x,\sigma)}) = \delta$.

Proof. Note that the length of every block in M_j' is $T(i_j)$, and for every $m_j \in M_j'$, $m_j < x$, hence,

$$h \geq \lim_{j\to\infty} (1/T(i_j)) \log(|M_j'| + 1) \geq \delta.$$

We now choose $A < x$ such that $l(a) = T(i_{j+1})$, then there exists

$$Q_2, \ldots, Q_{(T(i_{j+1})/T(i_j)+1)},$$

where $Q_i \in M_i'$, $1 \leq i \leq (T(i_{j+1})/T(i_j)) + 1$ such that $A < Q$. Hence,

$$\theta_{T(i_{j+1})}(x) \leq T(i_j)|M_i'|^{(T(i_{j+1})/T(i_j))+1}.$$

From this inequality, we get

$$h \leq \underline{\lim}_{j\to\infty}(1/T(i_{j+1}))\log(|M_i'|^{(T(i_{j+1})/T(i_j))+1} \cdot T(i_j)) =$$

$$= \underline{\lim}_{j\to\infty}(((1/T(i_{j+1}))\log(T(i_j)|M_j'|)) + ((1/T(i_j))\log(|M_j'|)) \leq \delta.$$

Thus, we have proved that $h(\sigma|_{\omega(x,\sigma)}) = \delta$.

4. Existence of Minimal Dynamical System with *D*-function $\in \mathrm{N}'$. In sect. 3, we have obtained the minimal dynamical system with a given topological entropy and a given *D*-function $\in \mathcal{Z} \setminus E$ in a constructive way. Here, we use the well-known results in ergodic theory to show the existence of the minimal dynamical system with a given topological entropy and a given *D*-function $\in \mathrm{N}'$. This is not a constructive way. For the necessary definitions from the ergodic theory we refer to [5].

Lemma 5. *If (X, f, μ) is strongly mixing and strictly ergodic, then $f_X = 1'$.*

Proof. Suppose that there exists $1 < n \in \mathbb{N}$ such that $f_X(n) = n$. Then there are minimal sets $A_0, A_1, \ldots, A_{n-1}$ of f^n such that $A = \bigcup_0^{n-1} A_i, f(A_i) = A_{i+1}$. Hence,

$$0 = \lim_{k \to \infty} \mu(f^{-kn}(A_0) \cap A_1) \neq 1/n^2 = \mu(f^{-kn}(A_0)) \times \mu(A_0),$$

we arrive at a contradiction.

Theorem 3. *For* $0 \le \delta \le \log K$, *there exists a strictly ergodic system* $(X, \sigma) \subset \subset (\Sigma_K, \sigma)$ *such that* $f_X = 1'$ *and* $h(\sigma|_X) = \delta$.

Proof. Let $p = (p_0, p_1, \ldots, p_{K-1})$ be a probability vector with nonzero entries and let $(P(K), 2^{P(K)}, \mu)$ denote the measure space with $\mu(i) = p_i$. Let $(\Sigma_K, \mathcal{B}, \mu_p) = \prod_{\infty}^{-\infty} \left(P(K), 2^{P(K)}, \mu\right)$ and σ be the shift. It is known that $(\Sigma_K, \sigma, \mu_p)$ is strongly mixing and has the metric entropy $-\sum_0^{K-1} p_i \log p_i$ [6]. By the theorem of Jewett and Krieger $(\Sigma_K, \sigma, \mu_p)$ is isomorphic to a strictly ergodic system $(X, \sigma) \subset (\Sigma_K, \sigma)$. Hence, (X, σ) is strongly mixing, and consequently, has the D-function $1'$. On the other hand, $h(\sigma|_X) = h_\beta(\sigma) = h_{\mu_p}(\sigma) = -\sum_0^{K-1} p_i \log p_i$, where β is a uniquely σ-invariant probability measure on $\mathcal{B}(X)$.

Choose $p = (p_0, p_1, \ldots, p_{K-1})$ such that $\delta = -\sum_0^{K-1} p_i \log p_i$. Then the following theorem holds true.

Theorem 4. *For* $0 \le \delta \le \log K$, *there exists a strictly ergodic system* (X_n, f_n) *such that* $(f_n)_{X_n} = n'$ *and* $h(f_n|_{X_n}) = \delta$.

Proof. Assume that (X, σ) is the strictly ergodic system we obtained above. Let $X_n = = \bigcup_0^{n-1} X \times \{i\}$ and let

$$f_n : X_n \to X_n, x \times \{i\} \mapsto \begin{cases} x \times \{i+1\}, & \text{if } i < n-1; \\ \sigma(x) \times \{0\}, & \text{if } i = n-1. \end{cases}$$

Since an orbit of every point $\in X_n$ is dense in X_n, (X_n, f_n) is minimal. On the other hand, $(X \times \{0\}, (f_n)^n)$ is topologically conjugate to (X, σ). So, by Lemmas 3, 4, it is easy to get $(f_n)_{X_n} = n'$. The strict ergodicity of (X_n, f_n) is stated in [6].

Furthermore, by the properties of topological entropy, we get

$$h(f_n) = h((f_n)^n|_{X \times \{0\}}) = h(\sigma|_X) = \delta.$$

The proof is completed.

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