

KP-HIERARCHY AND (1+1)-DIMENSIONAL MULTICOMPONENT INTEGRABLE SYSTEMS

ІЕРАРХІЯ КП ТА (1+1)-МІРНІ БАГАТОКОМПОНЕНТНІ ІНТЕГРОВНІ СИСТЕМИ

New types of reduction of the Kadomtsev – Petviashvili (KP) hierarchy are considered on the basis of Sato's approach. As a result we obtain a new multicomponent nonlinear integrable system. Bi-Hamiltonian structures for the new equations are presented.

Основуючись на підході Сато до ієрархії рівнянь Кадомцева – Петвіашвілі (КР) розглянуто новий тип редукцій. В результаті одержані нові багатоконпонентні нелінійні інтегровні системи. Знайдено їх бігамільтонову структуру.

1. Introduction. In the Sato approach [1-3] the KP-hierarchy is described by the isospectral deformations of the eigenvalue problem $L\Psi = \lambda\Psi$ for the pseudodifferential operator $L = \delta + u_2\partial^{-1} + u_3\partial^{-2} + \dots$ which is given by

$$\begin{aligned}L_{l_l} &= [B_l, L], \\ \Psi_{l_l} &= B_l\Psi, \\ \Psi_{l_l}^* &= -B_l^*\Psi^*,\end{aligned}$$

where $l = 1, 2, 3, \dots$, $B_l = (L^l)_+$, $B_l^* = (L^{*l})_+$; L^* is the operator adjoint to L and Ψ^* is its eigenfunction.

The Lax-equations have the following form:

$$u_{k,l_m} = F_{k,m}(u_2, u_3, \dots, u_{k+m-1})$$

with differential polynomials $F_{k,m}$ in $u_2, u_3, \dots, u_{k+m-1}$ [3]. The usual KP-hierarchy arises as an infinite set of equations for $u = u_2$ after elimination of the variables u_3, u_4, \dots in terms of u . The quantities

$$\partial_x S_m(u) = \partial_x F_{2,m}(u),$$

are the usual Lie – Bäcklund symmetries of the KP-equation

$$u_{l_3} = \partial_x S_4.$$

The quantity $\Psi\Psi^*$ possesses an asymptotic expansion $\Psi\Psi^* = \sum_{n=0}^{\infty} S_n \lambda^n$ and was shown to represent a symmetry of the KP-hierarchy [4–8]. From this fact it is easy to see that $\left(\sum_{i=1}^n \Psi_i \Psi_i^*\right)_x$ is a symmetry where Ψ_i is an eigenfunction and Ψ_i^* is an adjoint eigenfunction with eigenvalue λ_i , $i = 1, \dots, n$.

The KP-hierarchy includes a number of well-known soliton equations. The so called l -reduction, for which the operator L^l is forced to become a purely differential operator, leads to hierarchies of 1+1-dimensional equations [1–3]. It can be shown that the Hamiltonian structure of these hierarchies is obtained directly from the condition of l -reduction by using the properties of pseudodifferential operators [9]. Recently, it was pointed out that the KP-hierarchy admits another type of reductions with constraints relating the potentials u_n with the eigenfunctions Ψ and Ψ^* [10–15]. The corresponding reduced equations form the commuting 1+1-dimensional hierarchies.

The idea to study these new constraints goes back to the reduction of 1+1-dimen-

sional integrable soliton equations to finite-dimensional integrable equations [16–19]. The well-known example is the restriction of the KdV flow to the pure multi-soliton submanifold [17]. In this case, we impose the constraint $u = \sum_{i=1}^N c_i \Psi_i^2$ on the KdV potential u and eigenfunctions Ψ_i . This leads to the finite-dimensional integrable system $\Psi_{i,xx} + 2 \left(\sum_{k=1}^N c_k \Psi_k^2 \right) \Psi_i = \lambda_i^2 \Psi_i$; $i = 1, \dots, N$. The further motivation for the new constraints lies in the methods of solving integrable equations in 1+1- or 2+1-dimensions by the nonlinearization of linear problems [13–15].

In this paper, we study the KP-hierarchy under the following constraints:

$$\left(\sum_{i=1}^n \Psi_i \Psi_i^* \right)_x = S_{2,x} = u_x, \quad \left(\sum_{i=1}^n \Psi_i \Psi_i^* \right)_x = S_{3,x} = u_y$$

and

$$\left(\sum_{i=1}^n \Psi_i \Psi_i^* \right)_x = S_{4,x} = \frac{1}{4} u_{xxx} + 3uu_x + \frac{1}{4} \partial_x^{-1} u_{yy}.$$

The first case is well-known and leads to the multicomponent AKNS-hierarchy. We use it as an introductory example to demonstrate the methods. In the second case, we obtain a multicomponent version of the Yajima – Oikawa hierarchy. In the third case, the reduction yields a multicomponent Melnikov-hierarchy which contains as a special case a multicomponent version of the Drinfeld – Sokolov system. The Hamiltonian and bi-Hamiltonian structures of the new hierarchies are discussed.

2. Description of the KP-hierarchy. In this section, we describe the linear problems associated with the KP-hierarchy and the higher order analogues of the KP-equation which arise as its symmetries or conserved covariants. Let us consider the following pseudodifferential operator [1–3]

$$L = \sum_{l \geq 1} u_{l-1} \partial_x^l, \quad \partial_x^l = \frac{\partial^l}{\partial x^l}, \quad (1)$$

where $u_0 = 1$; $u_1 = 0$; $u_k = u_k(t_1, t_2, t_3, t_4, \dots)$. For convenience we shall often use the abbreviations $t_1 \stackrel{\text{df}}{=} x$; $t_2 \stackrel{\text{df}}{=} y$; $t_3 \stackrel{\text{df}}{=} t$. Next, we introduce the nonnegative, i.e., the purely differential part of L^n :

$$B_n = (L^n)_+, \quad (2)$$

The first four of the operators (2) have the form

$$\begin{aligned} B_1 &= \partial_x, \\ B_2 &= \partial_x^2 + 2u_2, \\ B_3 &= \partial_x^3 + 3u_2 \partial_x + 3u_3 + 3u_{2,x}, \\ B_4 &= \partial_x^4 + 4u_2 \partial_x^2 + (4u_3 + 6u_{2,x}) \partial_x + 4u_4 + 6u_{3,x} + 4u_{2,xx} + 6u_2^2. \end{aligned} \quad (3)$$

Consider the following set of linear equations for the eigenfunction Ψ :

$$L\Psi = \lambda\Psi, \quad (4)$$

$$\Psi_{t_l} = B_l \Psi, \quad l = 2, 3, \dots \quad (5)$$

As a compatibility condition between (4) and (5), we obtain the Lax-equations

$$L_{t_l} = [B_l, L], \quad l = 2, 3, \dots \quad (6)$$

The first set of the Lax-equations (6), i.e., $l = 2$ can be used for expressing the functions u_l , $l > 2$, in terms of $u_2 \stackrel{\text{df}}{=} u$ [3]. This gives, for instance,

$$\begin{aligned} u_3 &= -\frac{1}{2} u_x + \frac{1}{2} \partial_x^{-1} u_y; \\ u_4 &= \frac{1}{4} u_{xx} - \frac{1}{2} u_y - \frac{1}{2} u^2 + \frac{1}{4} \partial_x^{-2} u_{yy}. \end{aligned} \quad (7)$$

The linear system (4), (5) possesses a formal solution of the form

$$\Psi = \chi \exp \left(\sum_{j=1}^{\infty} t_j \lambda^j \right). \quad (8)$$

Instead of the linear problem for Ψ we can consider the problem for the adjoint eigenfunction

$$\Psi^* = \chi^* \exp \left(- \sum_{j=1}^{\infty} t_j \lambda^j \right) \quad (9)$$

which is given by

$$L^* \Psi^* = \lambda \Psi^*, \quad (10)$$

$$\Psi_{t_l}^* = -\hat{B}_l^* \Psi^*, \quad l = 2, 3, \dots, \quad (11)$$

where L^* is the adjoint of L and B_n^* is the adjoint of B_n . Obviously, we have an equivalent formulation of the Lax-equations (12) in terms of the adjoint operators

$$L_l^* = [L^*, B_l^*]. \quad (12)$$

It has been pointed out [7, 8] that the product $\Psi \Psi^*$ acts as a generator of conserved densities and symmetries of the KP-hierarchy. Indeed, from the expansions (8) and (9) we obtain the expansion $\Psi \Psi^* = \sum_{n=0}^{\infty} S_n \lambda^{-n}$ with conserved densities S_n and symmetries $(S_n)_x$.

3. Reductions by symmetry constraints. Since we are concerned in multicomponent reductions, we introduce the following vectors of eigenfunctions $\bar{\Psi} = (\Psi_1, \Psi_2, \dots, \Psi_n)$ and $\bar{\Psi}^* = (\Psi_1^*, \Psi_2^*, \dots, \Psi_n^*)$, where Ψ_i solves the system (4), (5) and Ψ_i^* solves the system (10), (11) with the eigenvalue λ_i , $i = 1, \dots, n$. It is assumed that $\lambda_i \neq \lambda_m$ for $i \neq m$. Using (5) and (11), we obtain linear equations for $\bar{\Psi}$ and $\bar{\Psi}^*$:

$$\hat{A} \bar{\Psi}_{t_l} = \hat{B}_l \bar{\Psi}, \quad \bar{\Psi}_{t_l}^* = -\hat{B}_l^* \bar{\Psi}^*, \quad (13)$$

where $\hat{A} \stackrel{\text{df}}{=} \text{diag}(\underbrace{A, A, \dots, A}_n)$ means a $n \times n$ diagonal matrix with A in the diagonal.

The system of linear equations (13) takes the following Hamiltonian form:

$$\begin{pmatrix} \bar{\Psi} \\ \bar{\Psi}^* \end{pmatrix}_{t_l} = M \nabla H_l, \quad (14)$$

where

$$M = \begin{pmatrix} 0 & \hat{1} \\ -\hat{1} & 0 \end{pmatrix}, \quad H_l = \int \langle \bar{\Psi}^*, \hat{B}_l \bar{\Psi} \rangle dx. \quad (15)$$

Here, the gradient is defined as $\nabla = \frac{\partial}{\partial \langle \bar{\Psi}, \bar{\Psi}^* \rangle} = \left(\frac{\delta}{\delta \Psi_1}, \dots, \frac{\delta}{\delta \Psi_n}, \frac{\delta}{\delta \Psi_1^*}, \dots, \frac{\delta}{\delta \Psi_n^*} \right)^T$ and

$\langle \bar{\Omega}^*, \bar{\Omega} \rangle \stackrel{\text{df}}{=} \sum_{i=1}^n \Omega_i^* \Omega_i$ means the usual product between two arbitrary vectors $\bar{\Omega}$ and $\bar{\Omega}^*$. For the later convenience, we shall list the first three of the systems (13):

$$\bar{\Psi}_y = \bar{\Psi}_{xx} + 2u \bar{\Psi}, \quad (16)$$

$$\bar{\Psi}_y^* = -\bar{\Psi}_{xx}^* - 2u \bar{\Psi}^*,$$

$$\bar{\Psi}_t = \bar{\Psi}_{xxx} + 3u\bar{\Psi}_x + \frac{3}{2}u_x\bar{\Psi} + \frac{3}{2}(\partial_x u_y)\bar{\Psi}, \quad (17)$$

$$\bar{\Psi}_t^* = \bar{\Psi}_{xxx}^* + 3u\bar{\Psi}_x^* + \frac{3}{2}u_x\bar{\Psi}^* - \frac{3}{2}(\partial_x u_y)\bar{\Psi}^*,$$

$$\bar{\Psi}_{t_4} = \bar{\Psi}_{xxxx} + 4u\bar{\Psi}_{xx} + (4u_x + 2\partial_x^{-1}u_y)\bar{\Psi}_x + (2u_{xx} + u_y + 4u^2 + \partial_x^{-2}u_{yy})\bar{\Psi}, \quad (18)$$

$$\bar{\Psi}_{t_4}^* = -\bar{\Psi}_{xxxx}^* - 4u\bar{\Psi}_{xx}^* - (4u_x - 2\partial_x^{-1}u_y)\bar{\Psi}_x^* - (2u_{xx} - u_y + 4u^2 + \partial_x^{-2}u_{yy})\bar{\Psi}^*.$$

In what follows, we shall also use the t_5 -flow which is too lengthy for being written down here.

We list the first few of the conserved densities S :

$$S_0 = 1,$$

$$S_1 = 0,$$

$$S_2 = u,$$

$$S_3 = \partial_x^{-1}u_y,$$

$$S_4 = \frac{1}{4}u_{xx} + \frac{3}{2}u^2 + \frac{3}{4}\partial_x^{-2}u_{yy},$$

$$S_5 = \frac{1}{2}u_{xy} + 2\partial_x^{-1}uu_y + \frac{1}{2}\partial_x^{-3}u_{yyy} + 2u\partial_x^{-1}u_y, \quad (19)$$

$$S_6 = u_{xxx} + 10u_{yy} + 20uu_{xx} + 10u_x^2 + 40u^3 + 5\partial_x^{-4}u_{yyyy} + 20u\partial_x^{-2}u_{yy} + \\ + 20\partial_x^{-1}(u\partial_x^{-1}u_{yy}) + 20(\partial_x^{-1}u_y)^2 + 10\partial_x^{-2}(u^2)_{yy},$$

$$S_7 = u_{xxy} + 16uu_{xy} + 8u_xu_y + 4\partial_x^{-1}u_yu_{xx} + 4u_{xx}\partial_x^{-1}u_y + 16u\partial_x^{-1}uu_y + \\ + 24\partial_x^{-1}u^2u_y + 24u^2\partial_x^{-1}u_y + \partial_x^{-5}u_{yyyyy} + 4u\partial_x^{-3}u_{yyy} + 4\partial_x^{-1}(u\partial_x^{-2}u_{yyy}) + \\ + 4\partial_x^{-2}(u\partial_x^{-1}u_{yyy}) - 4\partial_x^{-2}(u_{yy}\partial_x^{-1}u_y) + 2\partial_x^{-3}(u^2)_{yyy} + 8(\partial_x^{-1}u_y)(\partial_x^{-2}u_{yy}) + \\ + 8\partial_x^{-1}[(\partial_x^{-1}u_y)(\partial_x^{-1}u_{yy})].$$

In what follows, we shall also use S_8 which is too lengthy to be written down explicitly. Note that the quantities S_n may be calculated by using the master symmetries approach [21] and that the KP itself becomes $u_t = \partial_x S_4$.

We now consider the following constraint:

$$S_k = \langle \bar{\Psi}^*, \bar{\Psi} \rangle \quad (20)$$

and impose it onto the system of equations

$$u_{t_m} = \partial_x S_{m+1}, \\ \Psi_{t_m} = B_m \bar{\Psi}, \\ \bar{\Psi}_{t_m}^* = -B_m^* \bar{\Psi}^*. \quad (21)$$

Furthermore, we shall impose the constraint (20) onto the densities S_n . Our claim is that under the constraint the system (21) becomes an integrable system with the conserved densities ρ_n , where $S_i \rightarrow \rho_i$; $i = 2, 3, \dots$ under (20).

Let us demonstrate this procedure by the example $k = 2$ [11–13], where the constraint (20) takes the form

$$u = \langle \bar{\Psi}^*, \bar{\Psi} \rangle. \quad (22)$$

The constraint (22) can be easily imposed on the last two equations of the system (21). In the case $m = 2$, this yields

$$\tilde{\Psi}_y = \tilde{\Psi}_{xx} + 2\langle \tilde{\Psi}, \tilde{\Psi}^* \rangle \tilde{\Psi}, \tag{23}$$

$$\tilde{\Psi}_y^* = -\tilde{\Psi}_{xx}^* - 2\langle \tilde{\Psi}, \tilde{\Psi}^* \rangle \tilde{\Psi}^*.$$

Clearly, (23) represents the multicomponent AKNS-system. In the case $m = 3$, we obtain

$$\tilde{\Psi}_t = \tilde{\Psi}_{xxx} + 3\langle \tilde{\Psi}, \tilde{\Psi}^* \rangle \tilde{\Psi}_x + 3\langle \tilde{\Psi}_x, \tilde{\Psi}^* \rangle \tilde{\Psi}, \tag{24}$$

$$\tilde{\Psi}_t^* = \tilde{\Psi}_{xxx}^* + 3\langle \tilde{\Psi}, \tilde{\Psi}^* \rangle \tilde{\Psi}_x^* + 3\langle \tilde{\Psi}, \tilde{\Psi}_x^* \rangle \tilde{\Psi}^*.$$

The system (24) represents the first higher order multicomponent AKNS-system.

Let us impose the constraint (21) on the densities S_2, \dots, S_5 which are converted into the densities ρ_2, \dots, ρ_5 of the multicomponent AKNS-system (22)

$$\begin{aligned} \rho_2 &= \langle \tilde{\Psi}, \tilde{\Psi}^* \rangle, \\ \rho_3 &= \langle \tilde{\Psi}_x, \tilde{\Psi}^* \rangle - \langle \tilde{\Psi}, \tilde{\Psi}_x^* \rangle, \\ \rho_4 &= \frac{3}{4} (\langle \tilde{\Psi}, \tilde{\Psi}_{xx}^* \rangle - 2\langle \tilde{\Psi}_x, \tilde{\Psi}_x^* \rangle + \langle \tilde{\Psi}_{xx}, \tilde{\Psi}^* \rangle) + 3\langle \tilde{\Psi}, \tilde{\Psi}^* \rangle^2, \\ \rho_5 &= 2\langle \tilde{\Psi}_x, \tilde{\Psi}_{xx}^* \rangle + 3(\langle \tilde{\Psi}_x, \tilde{\Psi}^* \rangle + \langle \tilde{\Psi}, \tilde{\Psi}_x^* \rangle) \langle \tilde{\Psi}, \tilde{\Psi}^* \rangle. \end{aligned} \tag{25}$$

4. The multicomponent Yajima – Oikawa hierarchy. We now consider the constraint (20) with $k = 3$, i.e.,

$$\partial_x^{-1} u_y = \langle \tilde{\Psi}^*, \tilde{\Psi} \rangle. \tag{26}$$

Imposing (26) onto the system (21), we obtain for $m = 2$:

$$\begin{aligned} \tilde{\Psi}_y &= \tilde{\Psi}_{xx} + 2u\tilde{\Psi}, \\ u_y &= \langle \tilde{\Psi}^*, \tilde{\Psi} \rangle_x, \\ \tilde{\Psi}_y^* &= -\tilde{\Psi}_{xx}^* - 2u\tilde{\Psi}^*, \end{aligned} \tag{27}$$

and for $m = 3$:

$$\begin{aligned} \tilde{\Psi}_t &= \tilde{\Psi}_{xxx} + 3u\tilde{\Psi}_x + \frac{3}{2}u_x\tilde{\Psi} + \frac{3}{2}\langle \tilde{\Psi}, \tilde{\Psi}^* \rangle \tilde{\Psi}, \\ u_t &= \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}(\langle \tilde{\Psi}_x, \tilde{\Psi}^* \rangle - \langle \tilde{\Psi}, \tilde{\Psi}_x^* \rangle)_x, \\ \tilde{\Psi}_t^* &= \tilde{\Psi}_{xxx}^* + 3u\tilde{\Psi}_x^* + \frac{3}{2}u_x\tilde{\Psi}^* - \frac{3}{2}\langle \tilde{\Psi}, \tilde{\Psi}^* \rangle \tilde{\Psi}^*. \end{aligned} \tag{28}$$

The system (27) represents a multicomponent version of the Yajima – Oikawa system [22] while the system (28) is its first higher order symmetry. If we impose the constraint (26) onto the conserved densities (19), then we obtain the following densities of system (27) ($S_i \rightarrow \rho_i; i = 2, 3, \dots$):

$$\begin{aligned} \rho_2 &= u, \\ \rho_3 &= \langle \tilde{\Psi}, \tilde{\Psi}^* \rangle, \\ \rho_4 &= u^2 + \frac{1}{2}(\langle \tilde{\Psi}_x, \tilde{\Psi}^* \rangle - \langle \tilde{\Psi}, \tilde{\Psi}_x^* \rangle), \end{aligned}$$

$$\rho_5 = \frac{1}{4} \left(\langle \bar{\Psi}, \bar{\Psi}_{xx}^* \rangle - 2 \langle \bar{\Psi}_x, \bar{\Psi}_x^* \rangle + \langle \bar{\Psi}_{xx}, \bar{\Psi}_x^* \rangle \right) + 2u \langle \bar{\Psi}, \bar{\Psi}^* \rangle, \quad (29)$$

$$\rho_6 = \langle \bar{\Psi}_x, \bar{\Psi}_{xx}^* \rangle + \frac{3}{2} u \left(\langle \bar{\Psi}, \bar{\Psi}_x \rangle - \langle \bar{\Psi}, \bar{\Psi}_x^* \rangle \right) + \frac{3}{4} \langle \bar{\Psi}, \bar{\Psi}^* \rangle^2 - \frac{1}{4} u_x^2 + u^3.$$

Let us discuss the Hamiltonian structure of the multicomponent Yajima – Oikawa system (27). For this purpose, we first introduce the Poisson brackets

$$\{F, G\}_i = \int dx (\nabla F) M_i (\nabla G)$$

where M_i is a Hamiltonian operator and

$$\nabla F = \frac{\partial F}{\partial(\bar{\Psi}, u, \bar{\Psi}^*)} = \frac{\partial F}{\partial(\Psi_1, \Psi_2, \dots, \Psi_n, u, \Psi_1^*, \dots, \Psi_n^*)}.$$

Now, (27) possesses two Hamiltonian structures. The first structure is given by

$$M_1 = \begin{pmatrix} \bar{0} & \bar{0}^T & \hat{1} \\ \bar{0} & \frac{1}{2} \partial_x & \bar{0} \\ -\hat{1} & \bar{0}^T & \bar{0} \end{pmatrix}, \quad (30)$$

where $\bar{0} = \underbrace{(0, 0, \dots, 0)}_n$. The second structure is given by

$$M_2 = \begin{pmatrix} -\frac{3}{2} \Psi_1 \partial_x^{-1} \Psi_1 & \dots & -\frac{3}{2} \Psi_1 \partial_x^{-1} \Psi_n & \frac{1}{2} \partial_x \Psi_1 + \frac{1}{4} \Psi_1 \partial_x & \partial_x^2 + 2u + & \dots & \frac{3}{2} \Psi_1 \partial_x^{-1} \Psi_n^* \\ & & & & + \frac{3}{2} \Psi_1 \partial_x^{-1} \Psi_1^* & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{3}{2} \Psi_n \partial_x^{-1} \Psi_1 & \dots & -\frac{3}{2} \Psi_n \partial_x^{-1} \Psi_n & \frac{1}{2} \partial_x \Psi_n + \frac{1}{4} \Psi_n \partial_x & \frac{3}{2} \Psi_n \partial_x^{-1} \Psi_1^* & \dots & \partial_x^2 + 2u + \\ & & & & + \frac{3}{2} \Psi_n \partial_x^{-1} \Psi_n^* & & \\ \frac{1}{2} \Psi_1 \partial_x + \frac{1}{4} \partial_x \Psi_1 & \dots & \frac{1}{2} \Psi_n \partial_x + \frac{1}{4} \partial_x \Psi_n & \frac{1}{2} \partial_x^3 + \frac{1}{2} \partial_x u + & \frac{1}{2} \Psi_1^* \partial_x + \frac{1}{4} \partial_x \Psi_1^* & \dots & \frac{1}{2} \Psi_n^* \partial_x + \frac{1}{4} \partial_x \Psi_n^* \\ & & & + \frac{1}{2} u \partial_x & & & \\ -\partial_x^2 - 2u + & \dots & \frac{3}{2} \Psi_1^* \partial_x^{-1} \Psi_n & \frac{1}{2} \partial_x \Psi_1^* + \frac{1}{4} \Psi_1^* \partial_x & -\frac{3}{2} \Psi_1^* \partial_x^{-1} \Psi_1^* & \dots & -\frac{3}{2} \Psi_n^* \partial_x^{-1} \Psi_n^* \\ + \frac{3}{2} \Psi_1^* \partial_x^{-1} \Psi_1^* & & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{3}{2} \Psi_n^* \partial_x^{-1} \Psi_1^* & \dots & -\partial_x^2 - 2u + & \frac{1}{2} \partial_x \Psi_n^* + \frac{1}{4} \Psi_n^* \partial_x & -\frac{3}{2} \Psi_n^* \partial_x^{-1} \Psi_1^* & \dots & -\frac{3}{2} \Psi_n^* \partial_x^{-1} \Psi_n^* \\ & & + \frac{3}{2} \Psi_n^* \partial_x^{-1} \Psi_n^* & & & & \end{pmatrix} \quad (31)$$

From the Hamiltonian operators (30) and (31), we obtain the recursion operator

$$\Lambda = M_1^{-1} M_2,$$

which generates conserved densities and symmetries of (27) by

$$\begin{pmatrix} \bar{\Psi} \\ u \\ \bar{\Psi} \end{pmatrix}_{t_m} = M_1 \Lambda^m \nabla \rho_2,$$

$$\nabla \rho_{m+1} = \Lambda \nabla \rho_m, \quad m=2, 3, \dots$$

5. The multicomponent Melnikov hierarchy. We shall now consider the constraint (20) with $k=4$, i.e.,

$$\frac{1}{4}u_{xx} + \frac{3}{2}u^2 + \frac{3}{4}\partial_x^{-2}u_{yy} = \langle \bar{\Psi}^*, \bar{\Psi} \rangle. \quad (32)$$

Imposing (32) onto the system (21), we obtain for $m=2$:

$$\begin{aligned} u_{yy} + \left(\frac{1}{3}u_{xxx} + 6uu_x - \frac{4}{3}\langle \bar{\Psi}^*, \bar{\Psi} \rangle_x \right)_x &= 0, \\ \bar{\Psi}_y &= \bar{\Psi}_{xx} + 2u\bar{\Psi}, \\ \bar{\Psi}_y^* &= -\bar{\Psi}_{xx}^* - 2u\bar{\Psi}^*. \end{aligned} \quad (33)$$

The system (33) can be written in the following form:

$$\begin{aligned} \bar{\Psi}_y &= \bar{\Psi}_{xx} + 2u\bar{\Psi}, \\ u_y &= \frac{2}{3}v_x, \end{aligned} \quad (34)$$

$$\begin{aligned} v_y &= -\frac{1}{2}u_{xxx} - 6uu_x + 2\langle \bar{\Psi}, \bar{\Psi}^* \rangle_x, \\ \bar{\Psi}_y^* &= -\bar{\Psi}_{xx}^* - 2u\bar{\Psi}^*. \end{aligned}$$

If we subject (21) with $m=3$ to the constraint (32), then we obtain

$$\begin{aligned} \bar{\Psi}_t &= \bar{\Psi}_{xxx} + 3u\bar{\Psi}_x + \frac{3}{2}u_x\bar{\Psi} + v\bar{\Psi}, \\ u_t &= \langle \bar{\Psi}, \bar{\Psi}^* \rangle_x, \\ v_t &= \frac{3}{2}(\langle \bar{\Psi}_{xx}, \bar{\Psi}^* \rangle - \langle \bar{\Psi}, \bar{\Psi}_{xx}^* \rangle). \end{aligned} \quad (35)$$

$$\bar{\Psi}_t^* = \bar{\Psi}_{xxx}^* + 3u\bar{\Psi}_x^* + \frac{3}{2}u_x\bar{\Psi}^* - v\bar{\Psi}^*.$$

If we subject (21) with $m=4$ to the constraint (32) we obtain

$$\begin{aligned} \bar{\Psi}_{t_4} &= \bar{\Psi}_{xxxx} + 4u\bar{\Psi}_{xx} + \left(4u_x + \frac{4}{3}v \right) \bar{\Psi}_x + \left(\frac{5}{3}u_{xx} + \frac{2}{3}v_x + 2u^2 + \frac{4}{3}\langle \bar{\Psi}, \bar{\Psi}^* \rangle \right) \bar{\Psi}, \\ u_{t_4} &= \frac{2}{9}v_{xxx} + \frac{2}{3}(\langle \bar{\Psi}_{xx}, \bar{\Psi}^* \rangle - \langle \bar{\Psi}, \bar{\Psi}_{xx}^* \rangle) + \frac{4}{3}(uv)_x, \\ v_{t_4} &= \frac{1}{3} \left[-\frac{1}{2}u_{xxx} - 6uu_x + 2\langle \bar{\Psi}, \bar{\Psi}^* \rangle_x \right]_{xx} + (\langle \bar{\Psi}, \bar{\Psi}^* \rangle_{xx} - 2\langle \bar{\Psi}_x, \bar{\Psi}_x^* \rangle + \langle \bar{\Psi}_{xx}, \bar{\Psi}^* \rangle)_x + \\ &\quad + \frac{4}{3}v v_x - uu_{xxx} - 12u^2u_x + 4(u\langle \bar{\Psi}, \bar{\Psi}^* \rangle)_x, \\ \bar{\Psi}_{t_4}^* &= -\bar{\Psi}_{xxxx}^* - 4u\bar{\Psi}_{xx}^* - \left(4u_x - \frac{4}{3}v \right) \bar{\Psi}_x^* - \left(\frac{5}{3}u_{xx} - \frac{2}{3}v_x + 2u^2 + \frac{4}{3}\langle \bar{\Psi}, \bar{\Psi}^* \rangle \right) \bar{\Psi}^*. \end{aligned} \quad (36)$$

If we subject (21) with $m = 5$ to the constraint (32), we obtain

$$\begin{aligned}
 \bar{\Psi}_{t_5} &= \bar{\Psi}_{5x} + 5u\bar{\Psi}_{xx} + \left(\frac{15}{2}u_x + \frac{5}{3}v\right)\bar{\Psi}_{xx} + \left(5u^2 + \frac{35}{6}u_{xx} + \frac{5}{3}v_x + \frac{2}{3}\langle\bar{\Psi}, \bar{\Psi}^*\rangle\right)\bar{\Psi}_x + \\
 &+ \left(\frac{5}{3}u_{xxx} + 5uu_x + \frac{4}{3}\langle\bar{\Psi}, \bar{\Psi}^*\rangle_x + \frac{10}{9}v_{xx} + \frac{10}{3}uv + \frac{4}{3}(\langle\bar{\Psi}_x, \bar{\Psi}^*\rangle - \langle\bar{\Psi}, \bar{\Psi}_x^*\rangle)\right)\bar{\Psi}, \\
 u_{t_5} &= \frac{2}{9}\left[-\frac{1}{2}u_{xxx} - 7uu_x + 2\langle\bar{\Psi}, \bar{\Psi}^*\rangle_x\right]_{xx} - \frac{2}{3}u_x u_{xx} - \frac{1}{3}uu_{xxx} - 5u^2u_x + \\
 &+ \frac{2}{3}(\langle\bar{\Psi}, \bar{\Psi}_{xx}^*\rangle + \langle\bar{\Psi}_{xx}, \bar{\Psi}^*\rangle - \frac{1}{2}\langle\bar{\Psi}_x, \bar{\Psi}_x^*\rangle)_x + \frac{10}{3}(u\langle\bar{\Psi}, \bar{\Psi}^*\rangle)_x + \frac{10}{9}vv_x, \\
 v_{t_5} &= \frac{\partial}{\partial y}\left[\frac{2}{9}\left(-\frac{1}{2}u_{xxx} - 6uu_x + 2\langle\bar{\Psi}, \bar{\Psi}^*\rangle\right)_x - \frac{7}{12}u_x^2 - \frac{1}{3}uu_{xxx} - \frac{5}{3}u^3 + \right. \\
 &+ \left.\frac{2}{3}(\langle\bar{\Psi}, \bar{\Psi}_{xx}^*\rangle + \langle\bar{\Psi}_{xx}, \bar{\Psi}^*\rangle - \frac{1}{2}\langle\bar{\Psi}_x, \bar{\Psi}_x^*\rangle) + \frac{10}{3}u\langle\bar{\Psi}, \bar{\Psi}^*\rangle + \frac{5}{9}v^2\right], \\
 \bar{\Psi}_{t_5}^* &= \bar{\Psi}_{5x}^* + 5u\bar{\Psi}_{xx}^* + \left(\frac{15}{2}u_x - \frac{5}{3}v\right)\bar{\Psi}_{xx}^* + \left(5u^2 + \frac{35}{6}u_{xx} + \frac{5}{3}v_x + \frac{2}{3}\langle\bar{\Psi}, \bar{\Psi}^*\rangle\right)\bar{\Psi}_x^* + \\
 &+ \left(\frac{5}{3}u_{xxx} + 5uu_x + \frac{4}{3}\langle\bar{\Psi}, \bar{\Psi}^*\rangle_x - \frac{10}{9}v_{xx} - \frac{10}{3}uv + \frac{4}{3}(\langle\bar{\Psi}_x^*, \bar{\Psi}\rangle - \langle\bar{\Psi}^*, \bar{\Psi}_x\rangle)\right)\bar{\Psi}^*.
 \end{aligned} \tag{37}$$

The system (34) represents the Melnikov system [15] while systems (35) – (37) represent its first higher order symmetries.

If we impose the constraint (32) onto the conserved densities (19), then we obtain the following densities of the system (34) ($S_i \rightarrow \rho_i$; $i = 2, 3, \dots$):

$$\begin{aligned}
 \rho_2 &= u, \\
 \rho_3 &= v, \\
 \rho_4 &= \langle\bar{\Psi}, \bar{\Psi}\rangle, \\
 \rho_5 &= uv - \langle\bar{\Psi}, \bar{\Psi}^*\rangle, \\
 \rho_6 &= \langle\bar{\Psi}^*, \bar{\Psi}_{xx} + 2u\bar{\Psi}\rangle + \frac{1}{3}v^2 + \frac{1}{4}u_x^2 - u^3, \\
 \rho_7 &= \langle\bar{\Psi}_x, \bar{\Psi}_{xx}^*\rangle + \frac{3}{2}u(\langle\bar{\Psi}_x, \bar{\Psi}^*\rangle - \langle\bar{\Psi}, \bar{\Psi}_x^*\rangle) + v\langle\bar{\Psi}, \bar{\Psi}_x^*\rangle, \\
 \rho_8 &= \langle\bar{\Psi}_{xx}^*, \bar{\Psi}_x + 4u\bar{\Psi}\rangle + \left(4u_x - \frac{4}{3}v\right)\langle\bar{\Psi}, \bar{\Psi}_x^*\rangle + \\
 &+ \left(\frac{5}{3}u_{xx} + 2u^2 - \frac{2}{3}v_x + \frac{2}{3}v\langle\bar{\Psi}, \bar{\Psi}^*\rangle\right)\langle\bar{\Psi}, \bar{\Psi}^*\rangle + \\
 &+ \frac{1}{9}vv_{xx} + \frac{2}{3}uv^2 + \frac{1}{12}u_x u_{xxx} + \frac{3}{2}uu_x^2 - u^4.
 \end{aligned} \tag{38}$$

Let us now consider the Hamiltonian structure of the Melnikov system (34). We have the first Hamiltonian structure with the Hamiltonian operator

$$M_1 = \begin{pmatrix} \hat{0} & 0 & 0 & \hat{1} \\ \bar{0} & 0 & \partial_x & \bar{0} \\ \bar{0} & \partial_x & 0 & \bar{0} \\ -\hat{1} & 0 & 0 & \hat{0} \end{pmatrix}. \quad (39)$$

The second Hamiltonian structure is too complicated for being written in explicit form. In the appendix this structure is described by a different method. Here, we shall confine ourselves to the scalar case $\bar{\Psi} = \Psi$, $\bar{\Psi}^* = \Psi^*$, where the Hamiltonian operator has the form

$$M_2 = \begin{pmatrix} -\frac{4}{3}\Psi\partial_x^{-1}\Psi & \frac{1}{3}\partial_x\Psi + \frac{1}{3}\Psi\partial_x & \frac{5}{4}\partial_x^2\Psi + \frac{5}{12}\Psi\partial_x^2 - & \frac{4}{3}\Psi\partial_x^{-1}\Psi^* + \partial_x^3 + \\ & & -\frac{5}{4}\Psi_{xx} + 2u\Psi & + \frac{3}{2}u\partial_x + \frac{3}{2}\partial_x u + v \\ \\ \frac{1}{3}\partial_x\Psi + \frac{1}{3}\Psi\partial_x & \frac{2}{9}\partial_x^3 + \frac{1}{3}\partial_x u + & v\partial_x + \frac{2}{3}v_x & \frac{1}{3}\partial_x\Psi^* + \frac{1}{3}\Psi^*\partial_x \\ & + \frac{1}{3}u\partial_x & & \\ \\ -\frac{5}{4}\Psi\partial_x^2 - \frac{5}{12}\partial_x^2\Psi + & v\partial_x + \frac{2}{3}v_x & -\frac{1}{6}\partial_x^5 - \frac{5}{4}\partial_x^3 u - \frac{5}{4}u\partial_x^3 + & \frac{5}{4}\Psi^*\partial_x^2 + \frac{5}{12}\partial_x^2\Psi^* - \\ + \frac{1}{4}\Psi_{xx} - 2u\Psi & & + \frac{3}{2}\partial_x^2 u_x - \frac{3}{4}u_x\partial_x^2 - & -\frac{1}{4}\Psi_{xx}^* + 2u\Psi^* \\ & & -3\partial_x u^2 - 3u^2\partial_x + & \\ & & + 2\Psi\partial_x\Psi^* + 2\Psi^*\partial_x\Psi & \\ \\ \frac{4}{3}\Psi^*\partial_x^{-1}\Psi + \partial_x^3 + & \frac{1}{3}\partial_x\Psi^* + \frac{1}{3}\Psi^*\partial_x & -\frac{5}{4}\partial_x^2\Psi^* - \frac{5}{12}\Psi^*\partial_x^2 + & -\frac{4}{3}\Psi^*\partial_x^{-1}\Psi^* \\ + \frac{3}{2}u\partial_x + \frac{3}{2}\partial_x u - v & & + \frac{1}{4}\Psi_{xx}^* - 2u\Psi^* & \end{pmatrix}. \quad (40)$$

From the operators (39) and (40) we obtain the recursion operator

$$\Lambda = M_1^{-1}M_2,$$

which generates the conserved densities and symmetries of the system (34) in the scalar case.

6. Some reductions in the multicomponent hierarchies. In this section, we first study the Yajima – Oikawa hierarchy under the trivial reduction

$$\bar{\Psi} = \bar{\Psi}^* = 0. \quad (41)$$

Under (41) the Hamiltonian structures given by (30) and (31) take the form

$$M_1^r = \frac{1}{2} \partial_x, \quad (42)$$

$$M_2^r = \frac{1}{8} \partial_x^3 + \frac{1}{2} \partial_x u + \frac{1}{2} u \partial_x, \quad (43)$$

and the conserved densities (29) become

$$\begin{aligned} \rho_2 &= u, \\ \rho_3 &= 0, \\ \rho_4 &= u^2, \\ \rho_5 &= 0, \\ \rho_6 &= u^3 - \frac{1}{4} u_x^2. \end{aligned} \quad (44)$$

Furthermore, the recursion operator becomes

$$\Lambda = (M_1^r)^{-1} M_2^r = \frac{1}{4} \partial_x^2 + 2u - \partial_x^{-1} u,$$

i.e., we have the KdV-hierarchy.

Let us consider the Melnikov hierarchy under the reduction (41). Under (41) the Hamiltonian structures given by (39) and (40) take the form

$$M_1^r = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \quad (45)$$

and

$$M_2^r = \begin{pmatrix} \frac{2}{5} \partial^3 + \frac{1}{3} \partial u + \frac{1}{3} u \partial & v \partial_x + \frac{2}{3} v_x \\ v \partial_x + \frac{1}{3} v_x & -\frac{1}{6} \partial^5 - \frac{5}{4} \partial^3 u - \frac{5}{4} u \partial^3 + \\ & + \frac{2}{3} \partial^2 u_x - \frac{3}{4} u_x \partial^2 - \\ & - 3 \partial u^2 - 3 u^2 \partial \end{pmatrix}. \quad (46)$$

The conserved densities (38) can be written as follows

$$\begin{aligned} \rho_2 &= u, \\ \rho_3 &= v, \\ \rho_4 &= 0, \\ \rho_5 &= uv, \\ \rho_6 &= \frac{1}{3} v^2 + \frac{1}{4} u_x^2 - u^3, \\ \rho_7 &= 0, \\ \rho_8 &= \frac{1}{9} v v_{xx} + \frac{2}{3} uv^2 + \frac{1}{12} u_x u_{xxx} + \frac{3}{2} u u_x^2 - u^4. \end{aligned} \quad (47)$$

Finally, by (45) and (46) we obtain the recursion operator

$$\Lambda = (M_1^r)^{-1} M_2^r$$

of the Boussinesq-hierarchy.

The Melnikov-hierarchy possesses also a nontrivial reduction, namely

$$\bar{\Psi} = \bar{\Psi}^* : v = 0. \tag{48}$$

Under (48) the system (35) becomes

$$\bar{\Psi}_t = \bar{\Psi}_{xxx} + 3u\bar{\Psi}_x + \frac{3}{2}u_x\bar{\Psi}, \tag{49}$$

$$u_t = \langle \bar{\Psi}, \bar{\Psi} \rangle_x,$$

while the system (37) takes the form

$$\begin{aligned} \bar{\Psi}_{t_5} = & \bar{\Psi}_{5x} + 5u\bar{\Psi}_{xxx} + \frac{15}{2}u_x\bar{\Psi}_{xx} + \left(5u^2 + \frac{35}{6}u_{xx} + \frac{2}{3}\langle \bar{\Psi}, \bar{\Psi} \rangle\right)\bar{\Psi}_x + \\ & + \left(\frac{5}{3}u_{xxx} + 5uu_x + \frac{4}{3}\langle \bar{\Psi}, \bar{\Psi} \rangle_x\right)\bar{\Psi}, \end{aligned} \tag{50}$$

$$\begin{aligned} u_{t_5} = & \frac{2}{9}\left[-\frac{1}{2}u_{xxx} - 6uu_x + 2\langle \bar{\Psi}, \bar{\Psi} \rangle_x\right] - \frac{3}{2}u_xu_{xx} - \frac{1}{3}uu_{xxx} - 5u^2u_x + \\ & + \frac{4}{3}\langle \bar{\Psi}, \bar{\Psi}_{xx} \rangle_x - \frac{1}{3}\langle \bar{\Psi}_x, \bar{\Psi}_x \rangle_x + \frac{10}{3}(u\langle \bar{\Psi}, \bar{\Psi} \rangle)_x. \end{aligned}$$

Furthermore, under (48) the densities (38) become

$$\begin{aligned} \rho_2 &= u, \\ \rho_3 &= 0, \\ \rho_4 &= \langle \bar{\Psi}, \bar{\Psi} \rangle, \\ \rho_5 &= 0, \\ \rho_6 &= \langle \bar{\Psi}, \bar{\Psi}_{xx} + 2u\bar{\Psi} \rangle + \frac{1}{4}u_x^2 - u^3, \\ \rho_7 &= 0, \end{aligned} \tag{51}$$

$$\begin{aligned} \rho_8 &= \langle \bar{\Psi}_{xx}, \bar{\Psi}_{xx} + 4u\bar{\Psi} \rangle + \left(4u_x - \frac{4}{3}v\right)\langle \bar{\Psi}, \bar{\Psi}_x^* \rangle_x \\ & \times \left(\frac{5}{3}u_{xx} + 2u^2 + \frac{2}{3}\langle \bar{\Psi}, \bar{\Psi} \rangle\right)\langle \bar{\Psi}, \bar{\Psi} \rangle + \frac{1}{12}u_xu_{xxx} + \frac{3}{2}uu_x^2 - u^4. \end{aligned}$$

The first Hamiltonian structure of the Melnikov system given by (39) can not be reduced under (48) while the second structure given by (40) gives rise to the following Hamiltonian structure for the system (49) in the scalar case $\bar{\Psi} = \Psi$:

$$M_2^r = \begin{pmatrix} \frac{1}{3}\partial_x^3 + \frac{3}{4}u\partial_x + \frac{3}{4}\partial_x u & \frac{1}{3}\partial_x \Psi + \frac{1}{3}\Psi\partial_x \\ \frac{1}{3}\partial_x \Psi + \frac{1}{3}\Psi\partial_x & \frac{2}{9}\partial_x^3 + \frac{1}{3}u\partial_x + \frac{1}{3}\partial_x u \end{pmatrix}. \tag{52}$$

In the scalar case, the system (49) can be written in the Hamiltonian form as $(\Psi_t, u)_t = M_2^r \nabla \Psi^2$. In the general case, the system (49) represents a multicomponent analogue of the Drinfeld – Sokolov equation [23] and the system (50) represents its first higher order symmetry. The Hamiltonian structure of the multicomponent system (49) is described in the appendix.

7. Conclusion. In this paper, we have studied the KP-hierarchy under constraints of the type $S_k = \langle \bar{\Psi}^*, \bar{\Psi} \rangle$. In the first three cases, $k = 2, 3, 4$, we have shown that multicomponent integrable systems arise from the t_2 -flow of the KP-hierarchy. The case $k = 2$ is well-known [11–13] and was presented as an introductory example. In the cases $k = 3, 4$, we have found new bi-Hamiltonian structures of the reduced systems. Through the bi-Hamiltonian structure the Yajima – Oikawa system (27) and the Melnikov system (34) possess a hierarchy of commuting flows. We can show by straightforward calculation that the first few members of those hierarchies coincide with the flows obtained by the reduction of the corresponding higher order KP-flows. Furthermore, we can show that the first few members on the hierarchy of conserved densities of (27) and (34) coincide with the densities which we obtain from the densities (19) of the KP-hierarchy.

The second interesting aspect of reduction by symmetry constraints is the possibility of decomposition of the KP-equation $u_t = \partial_x S_4$ into 2 commuting systems in $1 + 1$ dimension [13, 14]. Examining our previous results, we obtain the following possibilities of solving of the KP-equation. If $\bar{\Psi}, \bar{\Psi}^*$ is a solution to (21), (22), then $u(x, y, t) = \langle \bar{\Psi}, \bar{\Psi}^* \rangle$ is a solution to the KP-equation. If $\bar{\Psi}, u, \bar{\Psi}^*$ is a solution to (27), (28), then $u(x, y, t) = \partial_y^{-1} \langle \bar{\Psi}, \bar{\Psi}^* \rangle_x$ is a solution to the KP-equation. If $\bar{\Psi}, u, v, \bar{\Psi}^*$ is a solution to (34), (35), then $u(x, y, t)$ given by (36) is a solution to the KP-equation.

8. Appendix. Let us now discuss the second Hamiltonian structure of the multicomponent Melnikov system (34) and the Hamiltonian structure of the multicomponent Drinfeld – Sokolov system (49). To do this, we first consider the Poisson brackets

$$\{F, G\} = \int \partial_x (\nabla F) M (\nabla G), \quad (53)$$

where M is a Hamiltonian operator and ∇F denotes the gradient of F with respect to the corresponding variables.

Instead of using a Hamiltonian operator M , we can express the Poisson brackets (53) in a more compact form. Following [24], we first introduce $\bar{\zeta} = (\bar{\Psi}, u, v, \bar{\Psi}^*)$ or $\bar{\zeta} = (\bar{\Psi}, u, \bar{\Psi}^*)$, respectively, and express (53) as

$$\{F, G\} = \iint \partial_x \partial_y (\nabla F(x)) (\nabla G(y)) \{\zeta(x), \zeta(y)\}. \quad (54)$$

If $\{\zeta(x), \zeta(y)\}$ is defined properly, then working out (54) gives us the Hamiltonian operator in (53).

For instance, defining $\{\zeta(x), \zeta(y)\}_1$ by

$$\begin{aligned} \{u(x), u(y)\}_1 &= \frac{1}{2} \delta'(x-y), \\ \{u(x), \Psi_i(y)\}_1 &= 0, \\ \{u(x), \Psi_i^*(y)\}_1 &= 0, \\ \{\Psi_i(x), \Psi_j(y)\}_1 &= 0, \\ \{\Psi_i^*(x), \Psi_j^*(y)\}_1 &= 0, \\ \{\Psi_i(x), \Psi_j^*(y)\}_1 &= \delta_i^j \delta(x-y), \end{aligned} \quad (55)$$

where $\delta(z)$ denotes the Dirac-function and δ_i^j is a Kronecker-symbol.

$$\delta_i^j = \begin{cases} 1, & i=j, \\ 0, & i \neq j, \end{cases}$$

gives us the first Hamiltonian operator (30) of the Yajima – Oikawa system (27).

Let

$$e(z) = \begin{cases} -1, & z < 0, \\ 1, & z > 0, \end{cases}$$

then the second Hamiltonian structure for the system (34) is given by

$$\{\Psi_i(x), \Psi_j(y)\}_2 = -\frac{4}{3}\Psi_i(x)\Psi_j(y)e(x-y),$$

$$\{\Psi_i(x), u(y)\}_2 = \frac{2}{3}\Psi_i(x)\delta'(x-y) - \frac{1}{3}\Psi_{ix}(x)\delta(x-y),$$

$$\begin{aligned} \{\Psi_i(x), v(y)\}_2 &= \frac{5}{3}\Psi_i(x)\delta''(x-y) - \frac{5}{6}\Psi_{ix}(x)\delta''(x-y) + \\ &+ \left(\frac{1}{6}\Psi_{ix}(x) + 2u(x)\Psi_i(x)\right)\delta(x-y), \end{aligned}$$

$$\{u(x), \Psi_i^*(y)\}_2 = \frac{2}{3}\Psi_i^*(x)\delta'(x-y) - \frac{1}{3}\Psi_{ix}^*(x)\delta(x-y),$$

$$\begin{aligned} \{v(x), \Psi_i^*(y)\}_2 &= \frac{5}{3}\Psi_i^*(x)\delta''(x-y) - \frac{5}{6}\Psi_{ix}^*(x)\delta''(x-y) + \\ &+ \left(\frac{1}{6}\Psi_{ix}^*(x) + 2u(x)\Psi_i^*(x)\right)\delta(x-y), \end{aligned} \quad (56)$$

$$\begin{aligned} \{\Psi_i(x), \Psi_j^*(y)\}_2 &= \frac{4}{3}\Psi_i(x)\Psi_j^*e(x-y) + \delta_i^j(\delta'''(x-y) + 3u(x)\delta'(x-y) - \\ &- \frac{3}{2}u_x(x)\delta(x-y) + v(x)\delta(x-y)), \end{aligned}$$

$$\{\Psi_i^*(x), \Psi_j^*(y)\}_2 = -\frac{4}{3}\Psi_i^*(x)\Psi_j^*e(x-y),$$

$$\{u(x), u(y)\}_2 = \frac{2}{9}\delta'''(x-y) + \frac{2}{3}u(x)\delta'(x-y) - \frac{1}{3}u_x(x)\delta(x-y),$$

$$\{u(x), v(y)\}_2 = v(x)\delta'(x-y) - \frac{1}{3}v_x(x)\delta(x-y),$$

$$\{v(x), v(y)\}_2 = -\frac{1}{6}\delta^{(5)}(x-y) - \frac{5}{2}u(x)\delta'''(x-y) + \frac{9}{2}u_x(x)\delta''(x-y) -$$

$$- \left(\frac{9}{4}u_{xx}(x) + 6u^2 - 4\langle \bar{\Psi}, \bar{\Psi}^* \rangle\right)\delta'(x-y) +$$

$$+ \left(\frac{1}{2}u_{xxx}(x) + 6u(x)u_x(x) - 2\langle \bar{\Psi}, \bar{\Psi}^* \rangle_x(x)\right)\delta(x-y).$$

Furthermore, for the system (49), we obtain the following Hamiltonian structure:

$$\{u(x), u(y)\}'_2 = \frac{2}{9} \delta'''(x-y) + \frac{2}{3} u(x) \delta'(x-y) - \frac{1}{3} u_x(x) \delta(x-y),$$

$$\{u(x), \Psi_i(y)\}'_2 = \frac{2}{3} \Psi_i(x) \delta'(x-y) - \frac{1}{3} \Psi_{ix}(x) \delta(x-y), \quad (57)$$

$$\{\Psi_i(x), \Psi_j(y)\}'_2 = \frac{1}{2} \delta_i^j \left(\delta'''(x-y) + 3u(x) \delta'(x-y) - \frac{3}{2} u_x(x) \delta(x-y) \right).$$

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