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On the nonperiodic groups, whose subgroups of infinite special rank are transitively normal

Presented by Corresponding Member of the NAS of Ukraine V.P. Motornyi

This paper devoted to the nonperiodic locally generalized radical groups, whose subgroups of infinite special rank are transitively normal. We proved that if such a group G includes an ascendant locally nilpotent subgroup of infinite special rank, then G is Abelian.

Keywords: *finite special rank, soluble group, periodic group, locally nilpotent radical, locally nilpotent residual, transitively normal subgroups.*

The groups with certain prescribed properties of subgroups form one of the central subjects of research in group theory. Their investigation introduced many important concepts such as the finiteness conditions, local nilpotence, local solubility, subnormality, permutability, some important numerical invariants on groups, in particular, distinct group ranks, etc. Choosing specific prescribed properties and particular families of subgroups which possess these properties, we come to distinct classes of groups. There is an enormous array of papers devoted to this theme. In the present paper, we will consider the influence of the following two families of subgroups on the structure of a group. These are the family of subgroups of finite special rank and the family of transitively normal subgroups.

We say that a subgroup H of a group G is *transitively normal*, if H is normal in every subgroup $K \geqslant H$, in which H is subnormal [1].

It is well known that the relation “to be a normal subgroup” is not transitive.

A group G is said to be a T-*group*, if this relation is transitive in G.

A group G is said to be a $\overline{T}$ -*group*, if every subgroup of G is a T-group.

It is not hard to see that every subgroup of G is transitively normal, if and only if G is a $\overline{T}$ -group.

A group G is said to be a group of *a finite special rank* r, if every finitely generated subgroup of G can be generated by at most r elements, and there exists a finitely generated subgroup H, which has exactly r generators [2]. The theory of groups of finite special rank is one of the best developed parts of group theory (see, e.g., surveys [3-5] and book [6]). In [7], M.R. Dixon, M.J. Evans, and H. Smith initiated the investigation of the groups, whose subgroups of infinite special rank have some fixed property P. This research has been prolonged by many authors for distinct natural properties P (see, e.g., survey [5]). In [8], the study of the groups, whose subgroups of infinite special rank are transitively normal was initiated. More precisely, the structure of periodic soluble groups with this property has been described there. In the current paper, we begin to study of the nonperiodic groups, whose subgroups of infinite special rank are transitively normal. The main result of this paper is the following

Theorem. Let G be a nonperiodic locally generalized radical group, whose subgroups of infinite special rank are transitively normal. If G includes an ascendant locally nilpotent subgroup of infinite special rank, then G is Abelian.

Corollary 1. *Let G be a nonperiodic locally generalized radical group, whose subgroups of infinite special rank are transitively normal. If G includes an ascendant Abelian subgroup of infinite special rank, then G is Abelian.*

We derive the corollaries of this theorem for some types of subgroups that are transitively normal.

Recall that a subgroup H is said to be *contranormal* in a group G, if its normal closure H^G coincides with G.

A subgroup H of a group G is said to be *polynormal* in a group G, if, for each subgroup S including H, the subgroup H is contranormal in H^S [9].

It is not hard to see that every polynormal subgroup is transitively normal; so, we can obtain

Corollary 2. *Let G be a nonperiodic locally generalized radical group, whose subgroups of infinite special rank are polynormal. If G includes an ascendant locally nilpotent subgroup of infinite special rank, then G is Abelian.*

A subgroup H is said to be *paranormal* in a group G, if H is contranormal in the subgroup $\langle H, H^g \rangle$ for all elements g of a group G [9].

Every paranormal subgroup of G is polynormal in G [9]; so we obtain

Corollary 3. *Let G be a nonperiodic locally generalized radical group, whose subgroups of infinite special rank are paranormal. If G includes an ascendant locally nilpotent subgroup of infinite special rank, then G is Abelian.*

A subgroup H of a group G is called *weakly normal* in G, if the inclusion $H^g \leqslant N_G(H)$ always implies $g \in N_G(H)$ [10] $\leqslant NG(H)$.

We note that every weakly normal subgroup is transitively normal; so we obtain

Corollary 4. *Let G be a nonperiodic locally generalized radical group, whose subgroups of infinite special rank are weakly normal. If G includes an ascendant locally nilpotent subgroup of infinite special rank, then G is Abelian.*

A subgroup H of a group G is called *weakly pronormal* in G, if the subgroups H and H^g are conjugate in $H^{<g>}$ for each element g of the group G [9].

We say that a subgroup H of a group G has the *Frattini property*, if, for every subgroups K, L such that $H \leqslant K$ and K is normal in L, we have $L = N_L(K)K$.

We note that a subgroup H is weakly pronormal on G, if and only if H has the Frattini property [9]. Clearly, every subgroup having the Frattini property is transitively normal; so we obtain

Corollary 5. *Let G be a nonperiodic locally generalized radical group, whose subgroups of infinite special rank are weakly pronormal. If G includes an ascendant locally nilpotent subgroup of infinite special rank, then G is Abelian.*

A subgroup H of a group G is called *pronormal* in G, if the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$ for each element g of the group G. Clearly, every pronormal subgroup is transitively normal; so we obtain

Corollary 6. *Let G be a nonperiodic locally generalized radical group, whose subgroups of infinite special rank are pronormal. If G includes an ascendant locally nilpotent subgroup of infinite special rank, then G is Abelian.*

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НЕПЕРІОДИЧНІ ГРУПИ, В ЯКИХ ПІДГРУПИ НЕСКІНЧЕННОГО СПЕЦІАЛЬНОГО РАНГУ Є ТРАНЗИТИВНО НОРМАЛЬНИМИ

Досліджено неперіодичні локально узагальнені радикальні групи, в яких підгрупи нескінченного спеціального рангу є транзитивно нормальними. Доведено, що якщо така група G містить у собі висхідну локально нільпотентну підгрупу нескінченного спеціального рангу, то G абелева.

Ключові слова: *нескінченний спеціальний ранг, розв'язна група, періодична група, локально нільпотентний радикал, локально нільпотентний резидуал, транзитивно нормальні підгрупи.*

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НЕПЕРИОДИЧЕСКИЕ ГРУППЫ, У КОТОРЫХ ПОДГРУППЫ БЕСКОНЕЧНОГО СПЕЦИАЛЬНОГО РАНГА ТРАНЗИТИВНО НОРМАЛЬНЫ

Исследованы непериодические локально обобщенные радикальные группы, у которых подгруппы бесконечного специального ранга транзитивно нормальны. Доказано, что если в такую группу G входит восходящая локально нильпотентная подгруппа бесконечного специального ранга, то G абелева.

Ключевые слова: *конечный специальный ранг, разрешимая группа, периодическая группа, локально нильпотентный радикал, локально нильпотентный резидуал, транзитивно нормальные подгруппы.*