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On properties of functions from Lizorkin–Triebel–Morrey type spaces

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Abstract. In this paper, we introduce a new function spaces of Lizorkin–Triebel–Morrey type and Sobolev type inequality prove is proved. Also, it is proved that the generalized derivatives of functions from this spaces satisfies the generalized Hölder condition.

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1. Introduction

In the paper we intoduce the Lizorkin–Triebel–Morrey-type space

$$F^{l}_{p,\theta,\varphi,\beta}(G_\varphi) \tag{1.1}$$

and by means of the method of integral representations we study both differential and difference-differential properties of functions from this space. Note that the spaces with parameters constructed on the basis of Sobolev's isotropic spaces, under some particular values of indices were first studied in Morrey's papers [6, 7]. Further, these results were developed and generalized in the papers of V. P. Il'in [4] A. S. Ross [15], Yu. V. Netrusov [14], A. Mazzucato [5], V. Kokilashvili, A. Meskhi, H. Rafeiro [4], V. S. Guliyev [2], Y. Sawano [17], E. Nakai [13] and [8–12] etc.

Let $G \subset R^n$; $l \in (0,\infty)^n$, $m_i \in N$, $k_i \in N_0$; $1 < p,\theta < \infty$; $\varphi(t) = (\varphi_1(t),\dots,\varphi_n(t))$, $\varphi_j(t) > 0 (t > 0)$ be continuously-differeniable functions, $\lim\limits_{t\to+0}\varphi_j(t) = 0$, $\lim\limits_{t\to+\infty}\varphi_j(t) = \infty$. We denote the set of such vector-functions by A. For any $x \in R^n$ we assume

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), (j = 1, 2, ..., n) \right\}.$$

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Definition 1.1. The normed linear space of functions $f \in L^{loc}(G)$ with the finite norm $(m_i > l_i - k_i \geq 0 \ (i = 1, ..., n))$:

$$\|f\|_{F^l_{p,\theta,\varphi,\beta}(G_\varphi)}$$
$$= \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^{n} \left\| \left\{ \int_0^{t_0} \left[\frac{\delta_i^{m_i}(\varphi_i(t)) D_i^{k_i} f}{(\varphi_i(t))^{(l_i-k_i)}} \right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)} \right\}^{\frac{1}{\theta}} \right\|_{p,\varphi,\beta}, \quad (1.2)$$

here

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t>0}} \left(|\varphi([t]_1)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right) \quad (1.3)$$

$$\delta_i^{m_i}(\varphi_i(t)) f(x) = \int_{-1}^{1} |\Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f(x)| dt,$$

$$\Delta_\iota^{m_i}\left(\varphi_\iota(t), G_{\varphi(t)}\right) f(x) = \begin{cases} \Delta_\iota^{m_i}(\varphi_\iota(t)) f(x) & [x, x + m_\iota \varphi_\iota(t) e_\iota] \subset G_{\varphi(t)}, \\ 0 & [x, x + m_\iota \varphi_\iota(t) e_\iota] \not\subseteq G_{\varphi(t)}, \end{cases}$$

$$\Delta_\iota^{m_i}(\varphi_\iota(t)) f(x) = \sum_{j=0}^{m_i} (-1)^{m_\iota - j} c^j_{m_\iota} f(x + j)\varphi_\iota(t) e_\iota, \ e_\iota = (0, ..., 0, 1, 0, ..., 0),$$

$|\varphi([t]_1)|^{-\beta} = \prod_{j=1}^{n} \varphi_j([t]_1)^{-\beta_j}$, $\beta_j \in [0, 1]$ and $[t]_1 = \min\{1, t\}$ and t_0-is a fixed positive number will be said the space with the parameters of the form $F^l_{p,\theta,\varphi,\beta}(G_\varphi)$.

The spaces $F^l_{p,\theta,\varphi,\beta}(G_\varphi)$ $\varphi_j(t) = t^{\varkappa_j}, \beta_j = \frac{a}{p}$ coincides with the space $F^l_{p,\theta,a,\varkappa}(G)$ studied in [8], in the case $\beta_j = 0 (j = 1, 2, \ldots, n)$ concides with the space $F^l_{p,\theta}(G)$. The spaces with such parameters with different norms were studied in the papers [2, 4, 13, 17].

In the case when for any $t > 0$, there exists constant $C > 0$ such that $|\varphi([t]_1)| \leq C$, then it holds the embedding

$$L_{p,\varphi,\beta}(G) \hookrightarrow L_p(G) \qquad F^l_{p,\theta,\varphi,\beta}(G_\varphi) \hookrightarrow F^l_{p,\theta}(G_\varphi),$$

i.e.

$$\|f\|_{p,G} \leq c\|f\|_{p,\varphi,\beta;G}, \qquad \|f\|_{F^l_{p,\theta}(G_\varphi)} \leq c\|f\|_{F^l_{p,\theta,\varphi,\beta}(G_\varphi)}. \quad (1.4)$$

Fruthermore, in the case when $1 < \theta \leq r \leq s \leq \sigma < \infty$ and $\theta \leq p \leq \sigma$, then $B^l_{p,\theta,\varphi,\beta}(G_\varphi) \hookrightarrow F^l_{p,r,\varphi,\beta}(G_\varphi) \hookrightarrow F^l_{p,s,\varphi,\beta}(G_\varphi) \hookrightarrow B^l_{p,\sigma,\varphi,\beta}(G_\varphi)$, the space $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$ was determined and studied in [16].

Definition 1.2. A_n open set $G \subset R^n$ is said to satisfy condition of flexible $\varphi-$horn, if for some $\theta \in (0,1]^n, T \in (0,\infty)$ for any $x \in G$ there exists a

$$\rho(\varphi(t),x) = (\rho_1(\varphi_1(t),x),...,\rho_n(\varphi_n(t),x)), \quad 0 \leq t \leq T$$

with the following properties:

1. for all $j = 1,...,n, \rho_j(\varphi_j(t),x)$ is absolutely continuous on $[0,T]$; $\left|\rho'_j(\varphi_j(t),x)\right| \leq 1$ for almost $t \in [0,T]$;
2. $\rho_j(0,x) = 0; x + V(x,\theta) = x + \bigcup\limits_{0\leq t\leq T} [\rho(\varphi(t),x) + \varphi(t)\theta I] \subset G.$

Theorem 1.1. *Let* $1 < p < \infty$, $1 < \theta < \infty$, $G = \bigcup\limits_{\lambda=1}^{M} G^\lambda$ *and* $f \in F^l_{p,\theta}(G_\varphi)$. *Then one can construct the seguence* $h_s = h_s(x)$ $(s = 1,2,\ldots)$ *of infinitely differentiable finite in* R^n *functions such that*

$$\lim_{s\to\infty} \|f - h_s\|_{F^l_{p,\theta}(G_\varphi)} = 0. \tag{1.5}$$

Proof. For obtaining equality (1.5) we estimate the norm $\|f-h_s\|_{F^l_{p,\theta}(G_\varphi)}$.

$$\|f - h_s\|_{F^l_{p,\theta}(G_\varphi)} = \|f - h_s\|_{p,G}$$

$$+\sum_{i=1}^{n} \left\| \left\{ \int_0^{t_0} \left[\frac{\delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) D_i^{k_i}[f(\cdot) - h_s(\cdot)]}{(\varphi_i(t))^{l_i-k_i}} \right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)} \right\}^{\frac{1}{\theta}} \right\|_p. \tag{1.6}$$

The sequence $h_s(x)$ $(s = 1,2,\ldots)$ is determined by the equalities

$$h_s(x) = R(x,\varphi(t))|_{t=\frac{1}{s}} = \sum_{\lambda=1}^{M} \eta_\lambda(x) f_{\varphi^\lambda(t)}(x),$$

here the averaging functions are determined as:

$$f_{\varphi^\lambda(t)}(x) = \int_{R^n} f(x + \varphi^\lambda(t)y) K_\lambda(y) dy,$$

where $K_\lambda(y) \in C_0^\infty(R^n)$ $(\lambda = 1, 2, \dots M)$, $\sup p K_\lambda(\cdot) \subset [-1; 1]$

$$\int\limits_{R^n} K_\lambda(y)dy = 1,$$

the functions $\eta_\lambda = \eta_\lambda(x)$ $(\lambda = 1, 2, \dots, M)$ determine the expansion of a unit it the domain G, i.e.

1) $0 \le \eta_\lambda(x) \le 1$ in R^n;

2) $\eta_\lambda(x) = 0$ on $G \setminus G_\lambda$ for all $\lambda = 1, 2, \dots, M$;

3) $\sum\limits_{\lambda}^{M} \eta_\lambda(x) = 1$ on G

4) $|D^\alpha \eta_\lambda(x)| \le C_\alpha$, on R^n $|\alpha| \ge 0$.

Obviously,

$$f(x) - h_s(x) = \sum_{\lambda}^{M} \eta_\lambda(x)(f(x) - f_{\varphi^\lambda(t)}(x)).$$

$$\|f(\cdot) - h_s(\cdot)\|_{F^l_{p,\theta}(G_\varphi)} \le \sum_{\lambda=1}^{M} \|\eta_\lambda(\cdot)(f(\cdot) - f_{\varphi^\lambda(t)}(\cdot))\|_{F^l_{p,\theta}(G^\lambda_\varphi)}$$

$$\le C \sum_{\lambda=1}^{M} \|(f(\cdot) - f_{\varphi^\lambda(t)}(\cdot))\|_{F^l_{p,\theta}(G^\lambda_\varphi)}, \tag{1.7}$$

$$\|(f(\cdot) - f_{\varphi^\lambda(t)}(\cdot))\|_{F^l_{p,\theta}(G^\lambda_\varphi)} = \|(f(\cdot) - f_{\varphi^\lambda(t)}(\cdot))\|_{p,G^\lambda}$$

$$+ \sum_{i=1}^{n} \left\| \left\{ \int\limits_0^{t_0} \left[\frac{\delta_i^{m_i}(\varphi_i(t), G^\lambda_{\varphi(t)}) D_i^{k_i}[f(\cdot) - f_{\varphi^\lambda(t)}(\cdot)]}{(\varphi_i(t))^{l_i - k_i}} \right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)} \right\}^{\frac{1}{\theta}} \right\|_p$$

$$\left\| \delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) D_i^{k_i}[f(\cdot) - f_{\varphi^\lambda(t)}(\cdot)] \right\|_p$$

$$\le C_1 \int\limits_{R^n} \left\| \Delta_i^{m_i}(\varphi_i(t), G^\lambda_{\varphi(t)}) D_i^{k_i}[f(\cdot) - f(\cdot + \varphi^\lambda(t)y)] \right\|_p |K_\lambda(y)|dy$$

$$\le \sup_{y \in R^n} \left\| \Delta_i^{m_i}(\varphi_i(t), G^\lambda_{\varphi(t)}) D_i^{k_i}[f(\cdot) - f(\cdot + \varphi^\lambda(t)y)] \right\|_p \tag{1.8}$$

The integral expansion in the right hand side of (1.8) is made arbitraryily small at rather small t, as consequence of continuity of L_p-average functions belonging to the space $L_p(G^\lambda_{\varphi(t)})$, i.e.

$$\sup_y \left\| \delta_i^{m_i}(\varphi_i(t), G^\lambda_{\varphi(t)}) D_i^{k_i}[f(\cdot) - f(\cdot + \varphi^\lambda(t)y)] \right\|_p$$

$$\leq C_1 \sup_y \left\| D_i^{k_i}[f(\cdot) - f(\cdot + \varphi^\lambda(t)y)] \right\|_{p, G^\lambda_{\varphi(t)}} < \varepsilon.$$

For rather small ε and t from inequality (1.7), in the case $\theta \leq p$ it follows equality (1.5). □

We can show that for $f \in F^l_{p,\theta}(G_\varphi)$, $1 < p, \theta < \infty$,$l \in (0, \infty)^n$,

$$\widetilde{A}^i_T = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{-\nu_j} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty,$$

then there exists $D^\nu f \in L_p(G)$ and the following indentity is valid for it

$$D^\nu f(x) = f^{(\nu)}_{\varphi(t)}(x)$$

$$+(-1)^{|\nu|} \sum_{i=1}^n \int_0^T \int_{R^n} L_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)} \right) f_i(x+y, t)$$

$$\times \prod_{j=1}^n (\varphi_j(t))^{-1-\nu_j} \frac{\varphi_i'(t)}{\varphi_i(t)} dt dy, \tag{1.9}$$

$$f^{(\nu)}_{\varphi(T)}(x) = \prod_{j=1}^n (\varphi_j(T))^{-1-\nu_j}$$

$$\times \int_{R^n} \Omega^{(\nu)} \left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{3\varphi(T)} \right) f(x+y) dy, \tag{1.10}$$

where

$$|f_i(x,t)| \leq \int_{-1}^1 |\delta_i^{m_i}(\varphi_i(t)) f(x + u\varphi_i(t))| du.$$

We note that a support of the integral representation (1.9) is

$$x + \bigcup_{0 \leq t \leq T} [\rho(\varphi(t), x) + \varphi(t)\theta I] \subset G.$$

At first we give an auxiliary lemma.

Let $\Omega(\cdot, y)$ and $L_i(\cdot, y, z) \in C_0^\infty(R^n)$, be such that

$$S(L_i) \subset I_{\varphi(t)} = \left\{ x : |x_j| < \frac{1}{2}\varphi_j(t),\ j = 1, 2, ..., n \right\}.$$

Let T be a positive value, $0 < T \le 1$ and assume

$$V = \bigcup_{0<t\le T} \left\{ y : \left(\frac{y}{\varphi_j(t)} \right) \in S\left(M_i\right) \right\}$$

and $U \subset G$, $V \subset I_{\varphi(t)}$ then $U + V \subset G$.

Lemma 1.1 *Let* $1 \le p \le q \le r \le \infty$, $0 < \eta, t \le T \le 1$, $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_j \ge 0$ *be entire* $(j = 1, 2, \ldots, n)$;$\delta_i^{m_i}(t) f \in L_{p,\varphi,\beta}(G)$ *and let*

$$E(x) = \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-1-\nu_j} \int\limits_{R^n} \Omega^{\nu} \left(\frac{y}{\varphi(t)}, \frac{\rho\left(\varphi(t), x\right)}{3\varphi(t)} \right) f(x+y) dy, \quad (1.11)$$

$$E_{\eta}^{i}\left(x\right) = \int\limits_{0}^{\eta} R_i\left(x, t\right) \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-1-\nu_j} \frac{\varphi_i'(t)}{\varphi_i(t)} dt \quad (1.12)$$

$$E_{\eta T}^{i}\left(x\right) = \int\limits_{\eta}^{T} R_i\left(x, t\right) \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-1-\nu_j} \frac{\varphi_i'(t)}{\varphi_i(t)} dt \quad (1.13)$$

$$A_T^i = \int\limits_{0}^{T} \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-\nu_j - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right)} \frac{\varphi_i'(t)}{\left(\varphi_i(t)\right)^{1-l_i}} dt < \infty,$$

where

$$R_i\left(x, y, t\right) = \int\limits_{R^n} L_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho\left(\varphi(t), x\right)}{\varphi(t)} \right) f\left(x + y, t\right) dy \quad (1.14)$$

$$|f_i(x,t)| \le C \int\limits_{-1}^{1} |\delta_i^{m_i}(\varphi_i(t)) f(x + u\varphi_i(t))| du.$$

Then for any $\overline{x} \in U$ *the following inequalities one valid*

$$\sup_{\overline{x}\in U} \|E\|_{q,U_{\psi(\xi)}(\overline{x})} \le \|f\|_{p,\varphi,\beta;G} \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-\nu_j - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right)}$$

$$\times \prod_{j=1}^{n} \left(\psi_j[\xi]_1\right)^{\beta_j \frac{p}{q}}, \quad (1.15)$$

$$\sup_{\overline{x}\in U} \left\|E_{\eta}^{i}\right\|_{q,U_{\psi(\xi)}(\overline{x})} \le C_1 \left\| \left(\varphi_i(t)\right)^{-l_i} \delta_i^{m_i}\left(\varphi_i(t)\right) f \right\|_{p,\varphi,\beta}$$

$$\times \left|A_{\eta}^{i}\right| \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p}{q}}, \qquad (1.16)$$

$$\sup_{\overline{x}\in U} \left\|E_{\eta T}^{i}\right\|_{q,U_{\psi(\xi)}(\overline{x})} \le C_2 \left\|(\varphi_i(t))^{-l_i}\, \delta_i^{m_i}\left(\varphi_i(t)\right) f\right\|_{p,\varphi,\beta}$$

$$\times \left|A_{\eta T}^{i}\right| \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p}{q}}, \qquad (1.17)$$

here $U_{\psi(\xi)}\left(\overline{x}\right) = \left\{x : |x_j - \overline{x}_j| < \frac{1}{2}\psi_j\left(\xi\right), j = 1, 2, ..., n\right\}$ *and* $\psi \in A$, C_1, C_2 *-are the constants independent of* φ, ξ, η *and* T.

Proof. Applying sequentially the Minkowsky generalized inequality for any $\overline{x} \in U$

$$\left\|E_{\eta}^{i}\right\|_{q,U_{\psi(\xi)}(\overline{x})} \le \int_0^{\eta} \left\|R_i\left(\cdot, t\right)\right\|_{p,U_{\psi(\xi)}(\overline{x})}$$

$$\times \prod_{j=1}^{n} \left(\varphi_j\left(T\right)\right)^{-\nu_j - 2} \varphi_j'\left(t\right) dt. \qquad (1.18)$$

From the Hölder inequality $(q \le r)$ we have

$$\left\|R_i\left(\cdot, t\right)\right\|_{q,U_{\psi(\xi)}(\overline{x})} \le \left\|R_i\left(\cdot, t\right)\right\|_{r,U_{\psi(\xi)}(\overline{x})} \prod_{j=1}^{n} \left(\psi_j\left(\xi\right)\right)^{\frac{1}{q} - \frac{1}{r}}. \qquad (1.19)$$

Now estimate the norm $\left\|R_i\left(\cdot, t\right)\right\|_{r,U_{\psi(\xi)}(\overline{x})}$.

Let χ be a characteristic function of the set $S\left(L_i\right)$. Again applying the Hölder inequality for representing the function in the form (1.14) in the case $1 \le p \le r \le \infty$, $s \le r$ as $\left(\frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r}\right)$, we get

$$\left\|R_i\left(\cdot, t\right)\right\|_{r,U_{\psi(\xi)}(\overline{x})} \le \sup_{x\in U_{\psi(\xi)}(\overline{x})} \left(\int_{R^n} \left|f_i\left(x+y, t\right)\right|^p \chi\left(\frac{y}{\varphi\left(t\right)}\right) dy\right)^{\frac{1}{p} - \frac{1}{r}}$$

$$\times \sup_{y\in V} \left(\int_{U_{\psi(\xi)}(\overline{x})} \left|f_i\left(x+y, t\right)\right|^p dx\right)^{\frac{1}{r}} \left(\int_{R^n} \left|\widetilde{L}_i\left(\frac{y}{\varphi\left(t\right)}\right)\right|^s dy\right)^{\frac{1}{s}}. \qquad (1.20)$$

It is assumed that $|L_i(x, y, z)| \le |\widetilde{L}_i(x)|$ and $L_i^{1/2} \in C_0^{\infty}(R^n)$.

For any $x \in U$ we have

$$\int_{R^n} \left|f_i\left(x+y, t\right)\right|^p \chi\left(\frac{y}{\varphi\left(t\right)}\right) dy$$

$$\leq \int\limits_{(U+V)_{\varphi(t)}(\overline{x})} |f_i\,(x+y,t)|^p\,dy \leq \int\limits_{G_{\varphi(t)}(\overline{x})} |f_i(y,t)|^p dy$$

$$\leq (\varphi_i(t))^{pl_i} \left\|(\varphi_i(t))^{-l_i}\delta_i^{m_i}(\varphi_i(t))f\right\|_{p,\varphi,\beta}^p \prod_{j=1}^n (\varphi_j\,(t))^{\beta_j p}\,. \qquad (1.21)$$

For $y \in V$

$$\int\limits_{U_{\psi(\xi)}(\overline{x})} |f_i\,(x+y,t)|^p\,dx \leq \int\limits_{(U+V)_{\psi_\xi}(\overline{x}+y)(\overline{x})} \left|f_{i)}\,(x,t)\right|^p dx$$

$$\leq (\varphi_i(t))^{pl_i} \left\|(\varphi_i(t))^{-l_i}\delta_i^{m_i}(\varphi_i(t))f\right\|_{p,\varphi,\beta} \prod_{j=1}^n (\psi_j\,([\xi]_1))^{\beta_j p}$$

$$\varphi([t]_1) \leq \psi([t]_1)). \qquad (1.22)$$

$$\int\limits_{R^n} \left|\widetilde{L}_i\left(\frac{y}{\varphi\,(t)}\right)\right|^s dy = \left\|\widetilde{L}_i\right\|_s^s \prod_{j=1}^n (\varphi_j\,(t))\,. \qquad (1.23)$$

From inequalities (1.20)–(1.23) it follows that

$$\|R\,(\cdot,t)\|_{r,U_{\psi(\xi)}(\overline{x})} \leq \left\|\widetilde{L}_i\right\|_s \cdot \left\|(\varphi_i(t))^{-l_i}\delta_i^{m_i}(\varphi_i(t))f\right\|_{p,\varphi,\beta} (\varphi_i(t))^{l_i}$$

$$\times \prod_{j=1}^n (\varphi_j\,(t))^{\frac{1}{s}+\beta_j p^i\left(\frac{1}{p^i}-\frac{1}{r}\right)} \prod_{j=1}^n (\psi_j\,([\xi_j]_1))^{\frac{\beta_j p^i}{r}}\,. \qquad (1.24)$$

and by the inequality (1.19) we have

$$\|R\,(\cdot,t)\|_{q,U_{\psi(\xi)}(\overline{x})} \leq C_1 \left\|(\varphi_i(t))^{-l_i}\delta_i^{m_i}(\varphi_i(t))f\right\|_{p,\varphi,\beta} (\varphi_i(t))^{l_i}$$

$$\times \prod_{j=1}^n (\varphi_j\,(t))^{\frac{1}{s}+\beta_j p\left(\frac{1}{p}-\frac{1}{r}\right)} \prod_{j=1}^n (\psi_j\,([\xi_j]_1))^{\frac{\beta_j p}{r}}\,(\psi_j\,(\xi))^{\frac{1}{q}-\frac{1}{r}}\,.$$

From inequalities (1.18) for $(r=q)$ and for any $\overline{x} \in U$ reduce to the estimation

$$\sup_{\overline{x}\in U} \left\|E_\eta^i\right\|_{p,U_{\psi(\xi)}(\overline{x})} \leq C_2 \left\|(\varphi_i(t))^{-l_i}\delta_i^{m_i}(\varphi_i(t))f\right\|_{p,\varphi,\beta} \prod_{j=1}^n (\psi_j\,([\xi]_1))^{\beta_j \frac{p}{q}}\,|A_\eta^i|.$$

□

Corollary 1. *From inequalities* (1.15)–(1.17) *we get the following inequality:*

$$\|E\|_{q,\psi,\beta_1;U} \leq \|f\|_{p,\varphi,\beta;G}\,, \tag{1.25}$$

$$\left\|E^i_\eta\right\|_{q,\psi,\beta_1;U} \leq \left\|(\varphi_i(t))^{-l_i}\delta_i^{m_i}(\varphi_i(t))f\right\|_{p,\varphi,\beta}, \tag{1.26}$$

$$\left\|E^i_{\eta,T}\right\|_{q,\psi,\beta_1;U} \leq \left\|(\varphi_i(t))^{-l_i}\delta_i^{m_i}(\varphi_i(t))f\right\|_{p,\varphi,\beta}. \tag{1.27}$$

2. Main results

We prove two theorems on the properties of functions form the space $F^l_{p,\theta,\varphi,\beta}\,(G_\varphi)$.

Theorem 2.1. *Let the domain* $G \subset R^n$ *satisfy the condition of flexible* φ*-horn* [12], $1 < p \leq q \leq \infty$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \geq 0$ *be entire* $j = 1, 2, ..., n$, $1 < \theta_1 \leq \theta_2 < \infty$; $A^i_T < \infty$ $(i = 1, 2, ..., n)$ *and let* $f \in F^l_{p,\theta,\varphi,\beta}(G_\varphi)$. *It holds the imbedding*

$$D^\nu : F^l_{p,\theta_1,\varphi,\beta}(G_\varphi) \hookrightarrow L_{q,\psi,\beta^1}(G)$$

i.e. for $f \in F^l_{p,\theta,\varphi,\beta}\,(G_\varphi)$ *in the domain* G *there exist the generalized derivatives* $D^\nu f$ *and the following inequalities are valid for it.*

$$\|D^\nu f\|_{q,G} \leq C_1 \left(F(T)\|f\|_{p,\varphi,\beta;G}\right.$$

$$\left. + \sum_{i=1}^n \left|A^i_T\right| \left\| \left\{ \int_0^{t_0} \left[\frac{\delta_i^{m_i}\,(\varphi_i(t))\,f(\cdot)}{(\varphi_i(t))^{l_i}} \right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)} \right\}^{\frac{1}{\theta}} \right\|_{p,\varphi,\beta} \right), \tag{2.1}$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \leq C_2\, \|f\|_{F^l_{p,\theta,\varphi,\beta}(G)}\,,\ p \leq q < \infty, \tag{2.2}$$

In particular, if

$$A^i_{T,0} = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{-\nu_j-(1-\beta_j p)\frac{1}{p}}\, \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty, (i = 1, 2, \ldots, n)\,,$$

then $D^\nu f\,(x)$ *is continuous in the domain* G, *and*

$$\sup_{x\in G} |D^\nu f(x)| \leq C_1 \left(F_0(t)\|f\|_{p,\varphi,\beta;G}\right.$$

$$+\sum_{i=1}^{n}|A^{i}_{T,0}|\left\|\left\{\int\limits_{0}^{t_0}\left[\frac{\delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f}{\left(\varphi_i(t)\right)^{l_i}}\right]^{\theta}\frac{d\varphi_i(t)}{\varphi_i(t)}\right\}^{\frac{1}{\theta}}\right\|_{p,\varphi,\beta}\Bigg) \tag{2.3}$$

$0<T\le\min\{1,t_0\}$, C_1, C_2, *are the constants independt of* f, C_1 *independent of* T.

Proof. Under the conditions of our theorem, there exist generalized derivatives $D^{\nu}f$. Indeed, if $A^i_T<\infty$, $\{i=1,2,...,n\}$, and then for $f\in F^l_{p,\theta,\varphi,\beta}(G_\varphi)\to F^l_{p,\theta}(G_\varphi)$ there exist generalized derivatives $D^\nu f\in L_p(G)$ and for almost each point $x\in G$ the integral representations (1.9) and (1.10) are valid.

Based on the Minkowski inequality we have

$$\|D^\nu f\|_{q,G}\le\left\|f^{(\nu)}_{\varphi(T)}\right\|_{q,G}+\sum_{i=1}^{n}\left\|E^i_T\right\|_{q,G}. \tag{2.4}$$

By means of inequality (1.15) for $U=G$, $t=T$ we get

$$\begin{aligned}\left\|f^{(\nu)}_{\varphi(t)}\right\|_{q,G}&\le\|f\|_{p,\varphi,\beta;G}\prod_{j=1}^{n}\left(\varphi_j(T)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j=1}^{n}\left(\psi_j([\xi]_1)\right)^{\beta_j\frac{p}{q}}\\&\le C_1F(T)\|f\|_{p,\varphi,\beta;G},\end{aligned} \tag{2.5}$$

and by the inequality (1.18) for $U=G$, $M_i=K^{(\nu)}_i$ $\eta=T$ we get

$$\left\|E^i_T\right\|_{q,G}\le C_2|A^i_T|\left\|\left(\varphi_i(t)\right)^{-l_i}\delta_i^{m_i}\left(\varphi_i(t)\right)f\right\|_{p,\varphi,\beta}. \tag{2.6}$$

From inequalities (2.4)–(2.6), on condition $1<\theta<\infty$ and $p\le\theta$ we get inequlity (2.1). By means of inequalities (1.25) and (1.26) we get inequality (2.2).

Let $A^i_{T,0}<\infty$ $(i=1,2,\dots,n)$ and we show that $D^\nu f$ is continuous on G. Based on inequality (2.1) for $q=\infty, p\le\theta$ we have

$$\begin{aligned}\left\|D^\nu f-f^{(\nu)}_{\varphi(T)}\right\|_{\infty,G}&\le\sum_{i=1}^{n}|A^i_{T,0}|\\&\times\left\|\left\{\int\limits_{0}^{t_0}\left[\frac{\delta_i^{m_i}(\varphi_i(t))f(\cdot)}{(\varphi_i(t))^{l_i}}\right]^{\theta}\frac{d\varphi_i(t)}{\varphi_i(t)}\right\}^{\frac{1}{\theta}}\right\|_{p,\varphi,\beta}.\end{aligned}$$

For $T\to 0$ $\left\|D^\nu f-f^{(\nu)}_{\varphi(T)}\right\|_{\infty,G}=0$. Since $f^{(\nu)}_{\varphi(T)}$ is continuous on G, the convergence in $L_\infty(G)$ coincides in this case with uniform one, and consequentey $D^\nu f(x)$ is continuous on G. The theorem is proved. □

Let γ be n-dimensional vector.

Theorem 2.2. *Let the conditions of theorem 2.1 be fulfilled. Then for* $A_T^i < \infty$ $(i = 1, 2, ..., n)$ *the derivative* $D^\nu f$ *satisfies on* G *the generalized Hölder condition, more exactly,*

$$\left\|\Delta(\gamma, G) D^\nu f\right\|_{q,G} \le C\|f\|_{F^l_{p,\theta,\varphi,\beta}(G_\varphi)} \cdot \left|h\left(|\gamma|, \varphi; T\right)\right|, \qquad (2.7)$$

here C *-is a constant independent of* f, $|\gamma|$ *and* T.

In particular, $A^i_{T,0} < \infty$, $(i = 1, 2, \ldots, n)$, *then*

$$\sup_{x\in G}\left|\Delta(\gamma, G) D^\nu f(x)\right| \le C\|f\|_{F^l_{p,\theta,\varphi,\beta}(G_\varphi)} \cdot \left|h_0\left(|\gamma|, \varphi, T\right)\right|. \qquad (2.8)$$

Here $h\left(|\gamma|, \varphi, T\right) = \max_i \left\{|\gamma|, A^i_{|\gamma|}, A^i_{|\gamma|,T}\right\}$

$$\left(h_0\left(|\gamma|, \varphi, T\right) = \max_i \left\{|\gamma|, A^i_{|\gamma|,0}, A^i_{|\gamma|,T,0}\right\}\right).$$

Proof. According to the definition of the domain satisfying the flexible horn condition (lemma 8.6 in [1]) there exists the domain

$$G_\omega \subset G\left(\omega = \rho r(x),\ \rho > 0,\ r(x) = d(x, \partial G),\ x \in G\right).$$

Assume that, $|\gamma| < \omega$, then for any $x \in G_\omega$ the segment connecting the point $x, x+\gamma$ is contained in G. Then for all points of this segment the indentities (1.9) and (1.10) with the same kernels are valid. After some tharnsformations, from (1.9) and (1.10) we have

$$\left|\Delta(\gamma, G) D^\nu f(x)\right| \le \prod_{j=1}^n (\varphi_j(t))^{-1-\nu_j}$$

$$\times \int_{R^n} |f(x+y)| \left|\Omega^{(\nu)}\left(\frac{y-\gamma}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{3\varphi(t)}\right) - \Omega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{3\varphi(T)}\right)\right| dydz$$

$$+ \sum_{i=1}^n \left\{ \int_0^{|\gamma|}\int_{R^n} \left|L_i^\nu\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t,x))}{\varphi(t)}\right)\right| |f(x+y,t)| \right.$$

$$\times \left|\prod_{j=1}^n (\varphi_j(t))^{-1-\nu_j} \frac{\varphi_i'(t)}{\varphi_i(t)}\right| dtdy +$$

$$+ \int_{|\gamma|}^T \int_{R^n} \left|L_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t,x))}{\varphi(t)}\right)\right| \left|\prod_{j=1}^n (\varphi_j(t))^{-1-\nu_j} \frac{\varphi_i'(t)}{\varphi_i(t)}\right|$$

$$\times \int_0^1 |f(x+y+v,t\gamma)|\,dvdydt \Bigg\}$$

$$= E_1(x,\gamma) + \sum_{i=1}^{n} \left(E_2(x,\gamma) + E_3(x,\gamma)\right), \qquad (2.9)$$

where $0 < T \le \min\{1, T_0\}$. Assume that $|\gamma| < T$. Consequently, $|\gamma| < \min(\omega, T)$. If $x \in G \setminus G_\omega$, then

$$\Delta(\gamma, G)\, D^\nu f(x) = 0.$$

From inequality (2.9) we get

$$\|\Delta(\gamma,G)\, D^\nu f\|_{q,G} \le \|E(\cdot,\gamma)\|_{q,G_\omega}$$

$$+ \sum_{i=1}^{n} \left(\|E_1(\cdot,\gamma)\|_{q,G_\omega} + \|E_2(\cdot,\gamma)\|_{q,G_\omega}\right), \qquad (2.10)$$

$$E(x,\gamma) \le \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - 2} \int_0^{|\gamma|} d\zeta \int_{R^n}\int_{R^n} |f(x+\zeta e_\gamma + y)|$$

$$\times \left| D_j \Omega^{(\nu)} \left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) \right| dy.$$

Assuming $\xi e_\gamma + G_\omega \subset G$ from inequality (2.10) , by means of the inequality (1.15), we get

$$\|E(\cdot,\gamma)\|_{q,G_\omega} \le C_1 |\gamma|\, \|f\|_{p,\varphi,\beta}. \qquad (2.11)$$

Based on inequality (1.16), for $U = G$, $\eta = |\gamma|$ we have

$$\|E_2(\cdot,\gamma)\|_{q,G_\omega} \le C_2 \left|A^i_{|\gamma|}\right| \left\|(\varphi_i(t))^{-l_i}\, \delta_i^{m_i}(\varphi_i(t))\, f\right\|_{p,\varphi,\beta} \qquad (2.12)$$

and based on inequality (1.17) for $U = G$, $\eta = |\gamma|$ we get

$$\|E_3(\cdot,\gamma)\|_{q,G_\omega} \le C_3 \left|A^i_{|\gamma|,T}\right| \left\|(\varphi_i(t))^{-l_i}\, \delta_i^{m_i}(\varphi_i(t))\, f\right\|_{p,\varphi,\beta;G}. \qquad (2.13)$$

By means of inequalities (2.10)–(2.13) provided $p \le \theta$ we get inequality (2.7). Now assume that $|\gamma| \ge \min(\omega, T)$. Then we get

$$\|\Delta(\gamma,G)\, D^\nu f\|_{q,G} \le 2\, \|D^\nu f\|_{q,G} \le C(\omega T)\, \|D^\nu f\|_{q,G}\, |h(|\gamma|, \varphi; T)|.$$

Estimating $\|D^\nu f\|_{q,G}$ by means of the inequality (2.1), we again get inequality (2.7). □

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