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Analytic in an unit ball functions of bounded L-index in joint variables

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(*Presented by V. Ya. Gutlyanskii*)

Abstract. A concept of boundedness of the **L**-index in joint variables (see in Bandura A. I., Bordulyak M. T., Skaskiv O. B. *Sufficient conditions of boundedness of L-index in joint variables,* Mat. Stud. **45** (2016), 12–26. dx.doi.org/10.15330/ms.45.1.12-26) is generalized for analytic in a ball function. It is proved criteria of boundedness of the **L**-index in joint variables which describe local behavior of partial derivatives on a skeleton of a polydisc.

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1. Introduction

A concept of entire function of bounded index appeared in a paper of B. Lepson [23]. An entire function f is said to be of bounded index if there exists an integer $N > 0$ that

$$(\forall z \in \mathbb{C})(\forall n \in \{0, 1, 2, \ldots\}) \colon \ \frac{|f^{(n)}(z)|}{n!} \leq \max\left\{\frac{|f^{(j)}(z)|}{j!} \colon 0 \leq j \leq N\right\}. \tag{1.1}$$

The least such integer N is called the index of f.

Note that the functions from this class have interesting properties. The concept is convenient to study the properties of entire solutions of differential equations. In particular, if an entire solution has bounded index then it immediately yields its growth estimates, an uniform in a some sense distribution of its zeros, a certain regular behavior of the solution etc.

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Afterwards, S. Shah [28] and W. Hayman [19] independently proved that every entire function of bounded index is a function of exponential type. Namely, its growth is at most the first order and normal type.

To study more general entire functions, A. D. Kuzyk and M. M. Sheremeta [21] introduced a boundedness of the l-index, replacing $\frac{|f^{(p)}(z)|}{p!}$ on $\frac{|f^{(p)}(z)|}{p!l^p(|z|)}$ in (1.1), where $l : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. It allows to consider an arbitrary entire function f with bounded multiplicity of zeros. Because for the function f there exists a positive continuous function $l(z)$ such that f is of bounded l-index [14]. Besides, there are papers where the definition of bounded l-index is generalizing for analytic function of one variable [22, 30].

In a multidimensional case a situation is more difficult and interesting. Recently we with N. V. Petrechko [12, 13] proposed approach to consider bounded $\mathbf{L}$-index in joint variables for analytic in a polydisc functions, where $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, $l_j : \mathbb{C}^n \to \mathbb{R}_+$ is a positive continuous functions, $j \in \{1, \ldots, n\}$. Although J. Gopala Krishna and S.M. Shah [20] introduced an analytic in a domain (a nonempty connected open set) $\Omega \subset \mathbb{C}^n$ $(n \in \mathbb{N})$ function of bounded index for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+$. But analytic in a domain function of bounded index by Krishna and Shah is an entire function. It follows from necessary condition of the l-index boundedness for analytic in the unit disc function ([29, Th.3.3,p.71]): $\int_0^r l(t)dt \to \infty$ as $r \to 1$ (we take $l(t) \equiv \alpha_1$). Thus, there arises necessity to introduce and to investigate bounded $\mathbf{L}$-index in joint variables for analytic in polydisc domain functions. Above-mentioned paper [12] is devoted analytic in a polydisc functions. Besides a polydisc, other example of polydisc domain in $\mathbb{C}^n$ is a ball. There are two known monographs [26, 32] about spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$: Bergman spaces, Hardy spaces, Besov spaces, Lipschitz spaces, the Bloch space, etc. It shows the relevance of research of properties of holomorphic function in the unit ball. In this paper we will introduce and study analytic in a ball functions of bounded $\mathbf{L}$-index in joint variables.

Of course, there are wide bibliography about entire functions of bounded $\mathbf{L}$-index in joint variables [9–11, 15–18, 24, 25].

Note that there exists other approach to consider bounded index in $\mathbb{C}^n$ — so-called functions of bounded L-index in direction (see [1–7]), where $L : \mathbb{C}^n \to \mathbb{R}_+$ is a positive continuous function.

2. Main definitions and notations

We need standard notations. Denote $\mathbb{R}_+ = [0, +\infty)$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n_+$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n_+$, $\mathbf{1}_j = (0, \ldots, 0, \underbrace{1}_{j\text{-th place}}, 0, \ldots, 0) \in \mathbb{R}^n_+$, $R = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $|z| = \sqrt{\sum_{j=1}^n |z_j|^2}$. For $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $B = (b_1, \ldots, b_n) \in \mathbb{R}^n$ we will use formal notations without violation of the existence of these expressions $AB = (a_1b_1, \cdots, a_nb_n)$, $A/B = (a_1/b_1, \ldots, a_n/b_n)$, $A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$, $\|A\| = a_1 + \cdots + a_n$, and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, \ldots, n\}$; the relation $A \leq B$ is defined similarly. For $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$ denote $K! = k_1! \cdot \ldots \cdot k_n!$. Addition, scalar multiplication, and conjugation are defined on $\mathbb{C}^n$ componentwise. For $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$ we define

$$\langle z, w \rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n,$$

where w_k is the complex conjugate of w_k. The polydisc $\{z \in \mathbb{C}^n : |z_j - z_j^0| < r_j, \ j = 1, \ldots, n\}$ is denoted by $\mathbb{D}^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : |z_j - z_j^0| = r_j, \ j = 1, \ldots, n\}$ is denoted by $\mathbb{T}^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : |z_j - z_j^0| \leq r_j, \ j = 1, \ldots, n\}$ is denoted by $\mathbb{D}^n[z^0, R]$, $\mathbb{D}^n = \mathbb{D}^n(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The open ball $\{z \in \mathbb{C}^n : |z - z^0| < r\}$ is denoted by $\mathbb{B}^n(z^0, r)$, its boundary is a sphere $\mathbb{S}^n(z^0, r) = \{z \in \mathbb{C}^n : |z - z^0| = r\}$, the closed ball $\{z \in \mathbb{C}^n : |z - z^0| \leq r\}$ is denoted by $\mathbb{B}^n[z^0, r]$, $\mathbb{B}^n = \mathbb{B}^n(\mathbf{0}, 1)$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : |z| < 1\}$.

For $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$ and the partial derivatives of an analytic in $\mathbb{B}^n$ function $F(z) = F(z_1, \ldots, z_n)$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \cdots + k_n} f}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, where $l_j(z) : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{B}^n) \colon \ l_j(z) > \beta/(1 - |z|), \ j \in \{1, \ldots, n\}, \tag{2.1}$$

where $\beta > \sqrt{n}$ is a some constant.

S. N. Strochyk, M. M. Sheremeta, V. O. Kushnir [29, 30] imposed a similar condition for a function $l : \mathbb{D} \to \mathbb{R}_+$ and $l : G \to \mathbb{R}_+$, where G is arbitrary domain in $\mathbb{C}$.

Remark 2.1. Note that if $R \in \mathbb{R}^n_+$, $|R| \leq \beta$, $z^0 \in \mathbb{B}^n$ and

$z \in \mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]$ then $z \in \mathbb{B}^n$. Indeed, we have

$$|z| \le |z-z^0| + |z^0| \le \sqrt{\sum_{j=1}^n \frac{r_j^2}{l_j^2(z^0)}} + |z^0| < \sqrt{\sum_{j=1}^n \frac{r_j^2}{\beta^2}(1-|z^0|)^2} + |z^0|$$

$$= \frac{(1-|z^0|)}{\beta}\sqrt{\sum_{j=1}^n r_j^2} + |z^0| \le \frac{(1-|z^0|)}{\beta}\beta + |z^0| = 1.$$

An analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ is said to be of *bounded* $\mathbf{L}$*-index (in joint variables),* if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{B}^n$ and for all $J \in \mathbb{Z}_+^n$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} :\ K \in \mathbb{Z}_+^n,\ \|K\| \le n_0\right\}. \tag{2.2}$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* F and is denoted by $N(F, \mathbf{L}, \mathbb{B}^n)$ (see [9]– [16]).

By $Q(\mathbb{B}^n)$ we denote the class of functions $\mathbf{L}$, which satisfy (2.1) and the following condition

$$(\forall R \in \mathbb{R}_+^n, |R| \le \beta,\ j \in \{1, \dots, n\})\colon\ 0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty, \tag{2.3}$$

where

$$\lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{B}^n} \inf\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}, \tag{2.4}$$

$$\lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{B}^n} \sup\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}. \tag{2.5}$$

$$\Lambda_1(R) = (\lambda_{1,1}(R), \dots, \lambda_{1,n}(R)),\ \Lambda_2(R) = (\lambda_{2,1}(R), \dots, \lambda_{2,n}(R)).$$

It is not difficult to verify that the class $Q(\mathbb{B}^n)$ can be defined as following: for every $j \in \{1, \dots, n\}$

$$\sup_{z,w \in \mathbb{B}^n}\left\{\frac{l_j(z)}{l_j(w)} : |z_k - w_k| \le \frac{r_k}{\min\{l_k(z), l_k(w)\}}, k \in \{1, \dots, n\}\right\} < \infty, \tag{2.6}$$

i. e. conditions (2.3) and (2.6) are equivalent.

Example 2.1. The function $F(z) = \exp\{\frac{1}{(1-z_1)(1-z_2)}\}$ has bounded $\mathbf{L}$-index in joint variables with $\mathbf{L}(z) = \left(\frac{1}{(1-|z_1|)^2(1-|z|)}, \frac{1}{(1-|z|)(1-|z_2|)^2}\right)$ and $N(F, \mathbf{L}, \mathbb{B}^n) = 0$.

3. Local behavior of derivatives of function of bounded L-index in joint variables

The following theorem is basic in theory of functions of bounded index. It was necessary to prove more efficient criteria of index boundedness which describe a behavior of maximum modulus on a disc or a behavior of logarithmic derivative (see [1,6,22,29]).

Theorem 3.1. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. An analytic in $\mathbb{B}^n$ function F has bounded $\mathbf{L}$-index in joint variables if and only if for each $R \in \mathbb{R}^n_+$, $|R| \leq \beta$, there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for every $z^0 \in \mathbb{B}^n$ there exists $K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \leq n_0$, and*

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq n_0,\ z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\} \leq p_0 \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{3.1}$$

Proof. Let F be an analytic function of bounded $\mathbf{L}$-index in joint variables with $N = N(F, \mathbf{L}, \mathbb{B}^n) < \infty$. For every $R \in \mathbb{R}^n_+$, $|R| \leq \beta$, we put

$$q = q(R) = [2(N+1)\|R\| \prod_{j=1}^{n} (\lambda_{1,j}(R))^{-N} (\lambda_{2,j}(R))^{N+1}] + 1$$

where $[x]$ is the entire part of the real number x, i.e. it is a floor function. For $p \in \{0, \dots, q\}$ and $z^0 \in \mathbb{B}^n$ we denote

$$\begin{aligned} S_p(z^0, R) &= \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq N, z \in \mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\}, \\ S_p^*(z^0, R) &= \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)} : \|K\| \leq N, z \in \mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\}. \end{aligned}$$

Using (2.4) and $\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right] \subset \mathbb{D}^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]$, we have

$$\begin{aligned} S_p(z^0, R) &= \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z^0)} : \|K\| \leq N, z \in \mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\} \\ &\leq S_p^*(z^0, R) \max\left\{\prod_{j=1}^{n} \frac{l_j^N(z^0)}{l_j^N(z)} : z \in \mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\} \\ &\leq S_p^*(z^0, R) \prod_{j=1}^{n} (\lambda_{1,j}(R))^{-N}. \end{aligned}$$

and, using (2.5), we obtain

$$S_p^*(z^0,R)=\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z)}{\mathbf{L}^K(z^0)}:\|K\|\le N, z\in\mathbb{D}^n\left[z^0,\frac{pR}{q\mathbf{L}(z^0)}\right]\right\}$$

$$\le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}(\Lambda_2(R))^K:\|K\|\le N, z\in\mathbb{D}^n\left[z^0,\frac{pR}{q\mathbf{L}(z^0)}\right]\right\}$$

$$\le S_p(z^0,R)\prod_{j=1}^n(\lambda_{2,j}(R))^N. \qquad (3.2)$$

Let $K^{(p)}\in\mathbb{Z}_+^n$ with $\|K^{(p)}\|\le N$ and $z^{(p)}\in\mathbb{D}^n\left[z^0,\frac{pR}{q\mathbf{L}(z^0)}\right]$ be such that

$$S_p^*(z^0,R)=\frac{|F^{(K^{(p)})}(z^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} \qquad (3.3)$$

Since by the maximum principle $z^{(p)}\in\mathbb{T}^n(z^0,\frac{pR}{q\mathbf{L}(z^0)})$, we have $z^{(p)}\ne z^0$. We choose $\widetilde{z}_j^{(p)}=z_j^0+\frac{p-1}{p}(z_j^{(p)}-z_j^0)$. Then for every $j\in\{1,\dots,n\}$ we have

$$|\widetilde{z}_j^{(p)}-z_j^0| = \frac{p-1}{p}|z_j^{(p)}-z_j^0|=\frac{p-1}{p}\frac{pr_j}{ql_j(z^0)}, \qquad (3.4)$$

$$\begin{aligned}|\widetilde{z}_j^{(p)}-z_j^{(p)}| &= |z_j^0+\frac{p-1}{p}(z_j^{(p)}-z_j^0)-z_j^{(p)}|=\frac{1}{p}|z_j^0-z_j^{(p)}|\\ &= \frac{1}{p}\frac{pr_j}{ql_j(z^0)}=\frac{r_j}{ql_j(z^0)}. \qquad (3.5)\end{aligned}$$

From (3.4) we obtain $\widetilde{z}^{(p)}\in\mathbb{D}^n\left[z^0,\frac{(p-1)R}{q(R)\mathbf{L}(z^0)}\right]$ and

$$S_{p-1}^*(z^0,R)\ge\frac{|F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}.$$

From (3.3) it follows that

$$\begin{aligned}0 &\le S_p^*(z^0,R)-S_{p-1}^*(z^0,R)\le\frac{|F^{(K^{(p)})}(z^{(p)})|-|F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\\ &= \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\int_0^1\frac{d}{dt}|F^{(K^{(p)})}(\widetilde{z}^{(p)}+t(z^{(p)}-\widetilde{z}^{(p)}))|dt\\ &\le \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\\ &\times\int_0^1\sum_{j=1}^n|z_j^{(p)}-\widetilde{z}_j^{(p)}|\left|F^{(K^{(p)}+\mathbf{1}_j)}(\widetilde{z}^{(p)}+t(z^{(p)}-\widetilde{z}^{(p)}))\right|dt\end{aligned}$$

$$= \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} \times \sum_{j=1}^{n} |z_j^{(p)} - \widetilde{z}_j^{(p)}| \left| F^{(K^{(p)}+\mathbf{1}_j)}(\widetilde{z}^{(p)} + t^*(z^{(p)} - \widetilde{z}^{(p)}))\right|, \qquad (3.6)$$

where $0 \le t^* \le 1, \widetilde{z}^{(p)} + t^*(z^{(p)} - \widetilde{z}^{(p)}) \in \mathbb{D}^n(z^0, \frac{pR}{q\mathbf{L}(z^0)})$. For $z \in \mathbb{D}^n(z^0, \frac{pR}{q\mathbf{L}(z^0)})$ and $J \in \mathbb{Z}^n_+$, $\|J\| \le N+1$ we have

$$\frac{|F^{(J)}(z)|\mathbf{L}^J(z)}{J!\mathbf{L}^J(z^0)\mathbf{L}^J(z)} \le (\Lambda_2(R))^J \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le N\right\}$$
$$\le \prod_{j=1}^{n}(\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N} \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)} : \|K\| \le N\right\}$$
$$\le \prod_{j=1}^{n}(\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N} S_p^*(z^0, R).$$

From (3.6) and (3.5) we obtain

$$0 \le S_p^*(z^0,R) - S_{p-1}^*(z^0,R) \le \prod_{j=1}^{n}(\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N}$$
$$\times S_p^*(z^0,R)\sum_{j=1}^{n}(k_j^{(p)}+1)l_j(z^0)|z_j^{(p)} - \widetilde{z}_j^{(p)}|$$
$$= \prod_{j=1}^{n}(\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N}\frac{S_p^*(z^0,R)}{q(R)}\sum_{j=1}^{n}(k_j^{(p)}+1)r_j$$
$$\le \prod_{j=1}^{n}(\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N}\frac{S_p^*(z^0,R)}{q(R)}(N+1)\|R\| \le \frac{1}{2}S_p^*(z^0,R).$$

This inequality implies $S_p^*(z^0,R) \le 2S_{p-1}^*(z^0,R)$, and in view of inequality (3.2) we have

$$S_p(z^0,R) \le 2\prod_{j=1}^{n}(\lambda_{1,j}(R))^{-N}S_{p-1}^*(z^0,R)$$
$$\le 2\prod_{j=1}^{n}(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N}S_{p-1}(z^0,R)$$

Therefore,

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le N, z \in \mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\} = S_q(z^0,R)$$

$$\begin{aligned}
&\leq \ 2\prod_{j=1}^{n}(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N}S_{q-1}(z^0,R)\leq\dots\\
&\leq \ (2\prod_{j=1}^{n}(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N})^{q}S_{0}(z^0,R)\\
&= \ (2\prod_{j=1}^{n}(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N})^{q}\\
&\times \ \max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^{K}(z^0)}:\|K\|\leq N\right\}. \qquad (3.7)
\end{aligned}$$

From (3.7) we obtain inequality (3.1) with

$$p_0=(2\prod_{j=1}^{n}(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N})^{q}$$

and some K^0 with $\|K^0\|\leq N$. The necessity of condition (3.1) is proved.

Now we prove the sufficiency. Suppose that for every $R\in\mathbb{R}^n_+$, $|R|\leq\beta$, there exist $n_0\in\mathbb{Z}_+$, $p_0>1$ such that for all $z_0\in\mathbb{B}^n$ and some $K^0\in\mathbb{Z}^n_+$, $\|K^0\|\leq n_0$, the inequality (3.1) holds.

We write Cauchy's formula as following $\forall z^0\in\mathbb{B}^n$ $\forall k\in\mathbb{Z}^n_+$ $\forall S\in\mathbb{Z}^n_+$

$$\frac{F^{(K+S)}(z^0)}{S!}=\frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)}\frac{F^{(K)}(z)}{(z-z^0)^{S+\mathbf{1}}}dz.$$

Therefore, applying (3.1), we have

$$\begin{aligned}
\frac{|F^{(K+S)}(z^0)|}{S!} &\leq \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)}\frac{|F^{(K)}(z)|}{|z-z^0|^{S+\mathbf{1}}}|dz|\\
&\leq \int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)}|F^{(K)}(z)|\frac{\mathbf{L}^{S+\mathbf{1}}(z^0)}{(2\pi)^nR^{S+\mathbf{1}}}|dz|\\
&\leq \int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)}|F^{(K^0)}(z^0)|\\
&\times \frac{K!p_0\prod_{j=1}^{n}\lambda_{2,j}^{n_0}(R)\mathbf{L}^{S+K+\mathbf{1}}(z^0)}{(2\pi)^nK^0!R^{S+\mathbf{1}}\mathbf{L}^{K^0}(z^0)}|dz|\\
&= |F^{(K^0)}(z^0)|\frac{K!p_0\prod_{j=1}^{n}\lambda_{2,j}^{n_0}(R)\mathbf{L}^{S+K}(z^0)}{K^0!R^{S}\mathbf{L}^{K^0}(z^0)}.
\end{aligned}$$

This implies

$$\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{S+K}(z^0)}\leq\frac{\prod_{j=1}^{n}\lambda_{2,j}^{n_0}(R)p_0K!S!}{(K+S)!R^{S}}\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \qquad (3.8)$$

Obviously, that

$$\frac{K!S!}{(K+S)!} = \frac{s_1!}{(k_1+1)\cdot\ldots\cdot(k_1+s_1)}\cdots\frac{s_n!}{(k_n+1)\cdot\ldots\cdot(k_n+s_n)} \le 1.$$

We choose $r_j \in (1, \beta/\sqrt{n}]$, $j \in \{1,\ldots,n\}$. Then $|R| = \sqrt{\sum_{j=1}^n r_j^2} \le \beta$. Hence, $\frac{p_0\prod_{j=1}^n \lambda_{2,j}^{n_0}(R)}{R^S} \to 0$ as $\|S\| \to +\infty$. Thus, there exists s_0 such that for all $S \in \mathbb{Z}_+^n$ with $\|S\| \ge s_0$ the inequality holds

$$\frac{p_0 K!S!\prod_{j=1}^n \lambda_{2,j}^{n_0}(R)}{(K+S)!R^S} \le 1.$$

Inequality (3.8) yields $\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{K+S}(z^0)} \le \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}$. This means that for every $j \in \mathbb{Z}_+^n$

$$\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)} \le \max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(z^0)} : \|K\| \le s_0 + n_0\right\}$$

where s_0 and n_0 are independent of z_0. Therefore, the analytic in $\mathbb{B}^n$ function F has bounded $\mathbf{L}$-index in joint variables with $N(F,\mathbf{L},\mathbb{B}^n) \le s_0 + n_0$. □

Theorem 3.2. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. In order that an analytic in $\mathbb{B}^n$ function F be of bounded $\mathbf{L}$-index in joint variables it is necessary that for every $R \in \mathbb{R}_+^n$, $|R| \le \beta$, $\exists n_0 \in \mathbb{Z}_+$ $\exists p \ge 1$ $\forall z^0 \in \mathbb{B}^n$ $\exists K^0 \in \mathbb{Z}_+^n$, $\|K^0\| \le n_0$, and*

$$\max\left\{|F^{(K^0)}(z)| : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\} \le p|F^{(K^0)}(z^0)| \qquad (3.9)$$

and it is sufficient that for every $R \in \mathbb{R}_+^n$, $|R| \le \beta$, $\exists n_0 \in \mathbb{Z}_+$ $\exists p \ge 1$ $\forall z^0 \in \mathbb{B}^n$ $\forall j \in \{1,\ldots,n\}$ $\exists K_j^0 = (0,\ldots,0,\underbrace{k_j^0}_{j\text{-th place}},0,\ldots,0)$ such that $k_j^0 \le n_0$ and

$$\max\left\{|F^{(K_j^0)}(z)| : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\} \le p|F^{(K_j^0)}(z^0)|. \qquad (3.10)$$

Proof. Proof of Theorem 3.1 implies that the inequality (3.1) is true for some K^0. Therefore, we have

$$\frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} \ge \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z)} : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}$$

$$= \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!}\frac{\mathbf{L}^{K^0}(z^0)}{\mathbf{L}^{K^0}(z^0)\mathbf{L}^{K^0}(z)} : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}$$

$$\geq \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!}\frac{\prod_{j=1}^n(\lambda_{2,j}(R))^{-n_0}}{\mathbf{L}^{K^0}(z^0)} : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}.$$

This inequality implies

$$\frac{p_0\prod_{j=1}^n(\lambda_{2,j}(R))^{n_0}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} \geq \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z^0)} : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}. \tag{3.11}$$

From (3.11) we obtain inequality (3.9) with $p = p_0\prod_{j=1}^n(\lambda_{2,j}(R))^{n_0}$. The necessity of condition (3.9) is proved.

Now we prove the sufficiency of (3.10). Suppose that for every $R \in \mathbb{R}^n_+$, $|R| \leq \beta$, $\exists n_0 \in \mathbb{Z}_+, p > 1$ such that $\forall z_0 \in \mathbb{B}^n$ and some $K^0_J \in \mathbb{Z}^n_+$ with $k^0_j \leq n_0$ the inequality (3.10) holds.

We write Cauchy's formula as following $\forall z^0 \in \mathbb{B}^n$ $\forall S \in \mathbb{Z}^n_+$

$$\frac{F^{(K^0_J+S)}(z^0)}{S!} = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))}\frac{F^{(K^0_J)}(z)}{(z-z^0)^{S+\mathbf{1}}}dz.$$

This yields

$$\frac{|F^{(K^0_j+S)}(z^0)|}{S!} \leq \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))}\frac{|F^{(K^0_j)}(z)|}{|z-z^0|^{S+\mathbf{1}}}|dz|$$

$$\leq \frac{1}{(2\pi)^n}\max\{|F^{(K^0_j)}(z)| : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\}\frac{\mathbf{L}^{S+\mathbf{1}}(z^0)}{R^{S+\mathbf{1}}}$$

$$\times \int_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))}|dz|$$

$$= \max\{|F^{(K^0_j)}(z)| : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\}\frac{\mathbf{L}^S(z^0)}{R^S}.$$

Now we put $R = (\frac{\beta}{\sqrt{n}}, \ldots, \frac{\beta}{\sqrt{n}})$ and use (3.10)

$$\frac{|F^{(K^0_j+S)}(z^0)|}{S!} \leq \frac{\mathbf{L}^S(z^0)}{(\beta/\sqrt{n})^{\|S\|}}\max\{|F^{(K^0_j)}(z)| : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\}$$

$$\leq \frac{p\mathbf{L}^S(z^0)}{(\beta/\sqrt{n})^{\|S\|}}|F^{(K^0_j)}(z^0)|. \tag{3.12}$$

We choose $S\in\mathbb{Z}^n_+$ such that $\|S\|\geq s_0$, where $\frac{p}{(\beta/\sqrt{n})^{s_0}}\leq 1$. Therefore, (3.12) implies that for all $j\in\{1,\ldots,n\}$ and $k_j^0\leq n_0$

$$\frac{|F^{(K_j^0+S)}(z^0)|}{\mathbf{L}^{K_j^0+S}(z^0)(K_j^0+S)!}\leq\frac{p}{(\beta/\sqrt{n})^{\|S\|}}\frac{S!K_j^0!}{(S+K_j^0)!}\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}\leq\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}.$$

Consequently, $N(F,\mathbf{L},\mathbb{B}^n)\leq n_0+s_0$. □

Remark 3.1. Inequality (3.9) is necessary and sufficient condition of boundedness of l-index for functions of one variable [22, 29, 31]. But it is unknown whether this condition is sufficient condition of boundedness of $\mathbf{L}$-index in joint variables. Our restrictions (3.10) are corresponding multidimensional sufficient conditions.

Lemma 3.1. *Let $\mathbf{L}_1$, $\mathbf{L}_2\in Q(\mathbb{B}^n)$ and for every $z\in\mathbb{B}^n$ $\mathbf{L}_1(z)\leq\mathbf{L}_2(z)$. If analytic in $\mathbb{B}^n$ function F has bounded $\mathbf{L}_1$-index in joint variables then F is of bounded $\mathbf{L}_2$-index in joint variables and $N(F,\mathbf{L}_2,\mathbb{B}^n)\leq nN(F,\mathbf{L}_1,\mathbb{B}^n)$.*

Proof. Let $N(F,\mathbf{L}_1,\mathbb{B}^n)=n_0$. Using (2.2) we deduce

$$\begin{aligned}\frac{|F^{(J)}(z)|}{J!\mathbf{L}_2^J(z)}&=\frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\frac{|F^{(J)}(z)|}{J!\mathbf{L}_1^J(z)}\\ &\leq\frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_1^K(z)}:\ K\in\mathbb{Z}^n_+,\ \|K\|\leq n_0\right\}\\ &\leq\frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\max\left\{\frac{\mathbf{L}_2^K(z)}{\mathbf{L}_1^K(z)}\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}^n_+,\ \|K\|\leq n_0\right\}\\ &\leq\max_{\|K\|\leq n_0}\left(\frac{\mathbf{L}_1(z)}{\mathbf{L}_2(z)}\right)^{J-K}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}^n_+,\ \|K\|\leq n_0\right\}.\end{aligned}$$

Since $\mathbf{L}_1(z)\leq\mathbf{L}_2(z)$ it means that for all $\|J\|\geq nn_0$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}_2^J(z)}\leq\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}^n_+,\ \|K\|\leq n_0\right\}.$$

Thus, F has bounded $\mathbf{L}_2$-index in joint variables and $N(F,\mathbf{L}_2,\mathbb{B}^n)\leq nN(F,\mathbf{L}_1,\mathbb{B}^n)$. □

Denote $\widetilde{\mathbf{L}}(z)=(\widetilde{l}_1(z),\ldots,\widetilde{l}_n(z))$. The notation $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ means that there exist $\Theta_1=(\theta_{1,j},\ldots,\theta_{1,n})\in\mathbb{R}^n_+$, $\Theta_2=(\theta_{2,j},\ldots,\theta_{2,n})\in\mathbb{R}^n_+$ such that $\forall z\in\mathbb{B}^n$ $\theta_{1,j}\widetilde{l}_j(z)\leq l_j(z)\leq\theta_{2,j}\widetilde{l}_j(z)$ for each $j\in\{1,\ldots,n\}$.

Theorem 3.3. *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$, $\mathbf{L} \asymp \widetilde{\mathbf{L}}$, $\beta|\Theta_1| > \sqrt{n}$. *An analytic in* $\mathbb{B}^n$ *function* F *has bounded* $\widetilde{\mathbf{L}}$*-index in joint variables if and only if it has bounded* $\mathbf{L}$*-index.*

Proof. It is easy to prove that if $\mathbf{L} \in Q(\mathbb{B}^n)$ and $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ then $\widetilde{\mathbf{L}} \in Q(\mathbb{B}^n)$.

Let $N(F, \widetilde{\mathbf{L}}, \mathbb{B}^n) = \widetilde{n}_0 < +\infty$. Then by Theorem 3.1 for every $\widetilde{R} = (\widetilde{r}_1, \ldots, \widetilde{r}_n) \in \mathbb{R}^n_+$, $|R| \le \beta$, there exists $\widetilde{p} \ge 1$ such that for each $z^0 \in \mathbb{B}^n$ and some K^0 with $\|K^0\| \le \widetilde{n}_0$, the inequality (3.1) holds with $\widetilde{\mathbf{L}}$ and $\widetilde{R}$ instead of $\mathbf{L}$ and R. Hence

$$\begin{aligned}
&\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} = \frac{\widetilde{p}}{K^0!}\frac{\Theta_2^{K^0}|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\mathbf{L}^{K^0}(z^0)} \ge \frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\widetilde{\mathbf{L}}^{K^0}(z^0)} \\
\ge\ & \frac{1}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)} : \|K\| \le \widetilde{n}_0, z \in \mathbb{D}^n\left[z^0, \widetilde{R}/\widetilde{\mathbf{L}}(z)\right]\right\} \\
\ge\ & \frac{1}{\Theta_2^{K^0}}\max\left\{\frac{\Theta_1^K|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le \widetilde{n}_0, z \in \mathbb{D}^n\left[z^0, \Theta_1\widetilde{R}/\mathbf{L}(z)\right]\right\} \\
\ge\ & \frac{\min_{0\le\|K\|\le n_0}\{\Theta_1^K\}}{\Theta_2^{K^0}} \\
\times\ & \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le \widetilde{n}_0, z \in \mathbb{D}^n\left[z^0, \Theta_1\widetilde{R}/\widetilde{\mathbf{L}}(z)\right]\right\}.
\end{aligned}$$

In view of Theorem 3.1 we obtain that function F has bounded $\mathbf{L}$-index in joint variables. □

Theorem 3.4. *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$. *An analytic in* $\mathbb{B}^n$ *function* F *has bounded* $\mathbf{L}$*-index in joint variables if and only if there exist* $R \in \mathbb{R}^n_+$, *with* $|R| \le \beta$, $n_0 \in \mathbb{Z}_+$, $p_0 > 1$ *such that for each* $z^0 \in \mathbb{B}^n$ *and for some* $K^0 \in \mathbb{Z}^n_+$ *with* $\|K^0\| \le n_0$ *the inequality* (3.1) *holds.*

Proof. The necessity of this theorem follows from the necessity of Theorem 3.1. We prove the sufficiency. The proof of Theorem 3.1 with $R = (\frac{\beta}{\sqrt{n}}, \ldots, \frac{\beta}{\sqrt{n}})$ implies that $N(F, \mathbf{L}, \mathbb{B}^n) < +\infty$.

Let $\mathbf{L}^*(z) = \frac{R_0\mathbf{L}(z)}{R}$, $R^0 = (\frac{\beta}{\sqrt{n}}, \ldots, \frac{\beta}{\sqrt{n}})$. In general case from validity of (3.1) for F, $\mathbf{L}$ and $R = (r_1, \ldots, r_n)$ with $|R| \le \beta$, $R \ne R^0$, we obtain

$$\begin{aligned}
&\max\left\{\frac{|F^{(K)}(z)|}{K!(\mathbf{L}^*(z^0))^K} : \|K\| \le n_0, z \in \mathbb{D}^n\left[z^0, R_0/\mathbf{L}^*(z^0)\right]\right\} \\
=\ & \max\left\{\frac{|F^{(K)}(z)|}{K!(R_0\mathbf{L}(z)/R)^K} : \|K\| \le n_0, z \in \mathbb{D}^n\left[z^0, R_0/(R_0\mathbf{L}(z)/R)\right]\right\}
\end{aligned}$$

$$\leq \max\left\{\frac{n^{\|K\|/2}|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq n_0, z\in\mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}$$
$$\leq \frac{p_0}{K^0!}\frac{n^{n_0/2}|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} = \frac{n^{n_0/2}(\beta/\sqrt{n})^{\|K^0\|}p_0}{R^{K^0}K^0!}\frac{|F^{(K^0)}(z^0)|}{(R_0\mathbf{L}(z)/R)^{K^0}}$$
$$< n^{n_0/2}p_0\max_{\|K^0\|\leq n_0}\frac{(\beta/\sqrt{n})^{\|K^0\|}}{R^{K^0}}\frac{|F^{(K^0)}(z^0)|}{K^0!(\mathbf{L}^*(z))^{K^0}}.$$

i. e. (3.1) holds for F, $\mathbf{L}^*$ and $R_0 = (\frac{\beta}{\sqrt{n}}, \dots, \frac{\beta}{\sqrt{n}})$. As above we apply Theorem 3.1 to the function $F(z)$ and $\mathbf{L}^*(z) = R_0\mathbf{L}(z)/R$. This implies that F is of bounded $\mathbf{L}^*$-index in joint variables. Therefore, by Theorem 3.3 the function F has bounded $\mathbf{L}$-index in joint variables. □

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